Journal of Statistics Applications & Probability
An International Journal

http://dx.doi.org/10.18576/jsap/070213

Exponentiated Mukherjee-Islam Distribution

Aafaq Ahmad Rather and C. Subramanian*

Department of Statistics, Annamalai University, Chidambaram, Tamil Nadu, India

Received: 13 Mar. 2018, Revised: 2 Apr. 2018, Accepted: 7 Apr. 2018.

Published online: 1 Jul. 2018.

Abstract: In this paper, we study the family of distributions termed as exponentiated mukherjee-islam distribution. The distribution has three parameters (one scale and two shape). The survival function, failure rate and moments of the distributions have been derived. The maximum likelihood estimators of the parameters and their asymptotics have been discussed and also the order statistics have been derived.

Keywords: Exponentiated mukherjee-islam distribution, moments, parameter estimation and fisher information matrix, order statistics.

1 Introduction

A new family of distributions, namely the exponentiated exponential distribution was introduced by Gupta *et al.* (1998). The family has two parameters (scale and shape) similar to the Weibull or gamma family. Some properties of the distribution was studied by Gupta and Kundu (2001). They observed that many properties of the new family are similar to those of weibull or gamma family. They (2001a, 2002) also examined the estimation and inference aspects of the distribution. A class of goodness-of-fit tests for the distribution with estimated parameters has been proposed by Hassan (2005).

Mukherjee-Islam distribution was introduced by Mukherjee and Islam (1983). It is finite range distribution, which is one of the most important property of it in recent time, in reliability analysis. Its mathematical form is simple and can be handled easily, that is why, it is preferred to use over more complex distribution such as normal, weibull, beta etc.

A random variable X is said to have an exponentiated distribution if its cumulative distribution function (cdf) is given by

$$F_\alpha(x) = [G(x)]^\alpha \ , \ x \epsilon R^1 \ , \alpha > 0. \qquad (1)$$

and pdf of X is given by

$$f_\alpha(x) = \alpha[G(x)]^{\alpha-1} g(x) \qquad (2)$$

2 Exponentiated Mukherjee-Islam Distribution

Let us consider the mukherjee-Islam distribution with probability density function (pdf)

$$g(x) = \frac{p}{\theta^p} x^{p-1} \ , \ 0 < x < \theta, \ \ p > 0, \theta > 0 \qquad (3)$$

and the cumulative distribution function (cdf)

$$G(x) = \left(\frac{x}{\theta}\right)^p \qquad 0 < x < \theta, \ \ p > 0, \theta > 0 \qquad (4)$$

where θ is the scale parameter and p is the shape parameter.
substituting (4) in (1),we will get the cdf of exponentiated mukherjee-islam distribution (EMID)

$$F_\alpha(x) = \left[\left(\frac{x}{\theta}\right)^p\right]^\alpha \ , \quad p > 0, \theta > 0$$

and its probability density function (pdf) is

*Corresponding author e-mail: aafaq7741@gmail.com

$$f_{\alpha}(x) = \frac{\alpha p\, x^{\alpha p-1}}{\theta^{\alpha p}}$$

The corresponding survival function is

$$S_{\alpha}(x) = 1 - \left[\left(\frac{x}{\theta}\right)^{p}\right]^{\alpha}$$

and the failure rate is

$$h_{\alpha}(x) = \frac{\alpha p\, x^{\alpha p-1}}{\theta^{\alpha p} - x^{\alpha p}}$$

Here (α , p) denote the shape parameters and θ is the scale parameter.

Fig. 1, Fig. 2, Fig. 3 and Fig. 4 shows the possible shapes of cdf, pdf, survival function and failure rate of the EMID for different values of parameters α, p and θ.

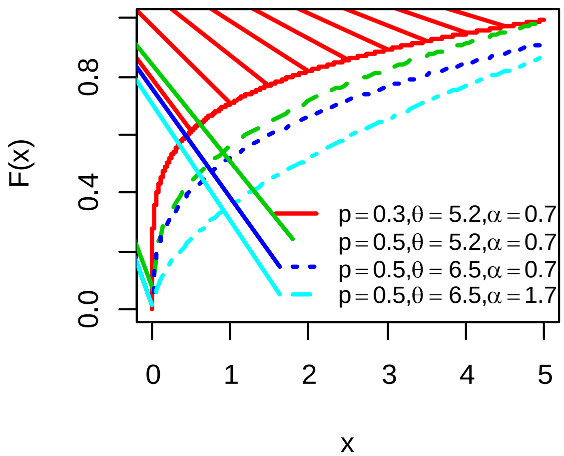

Fig. 1: CDF plot of EMID

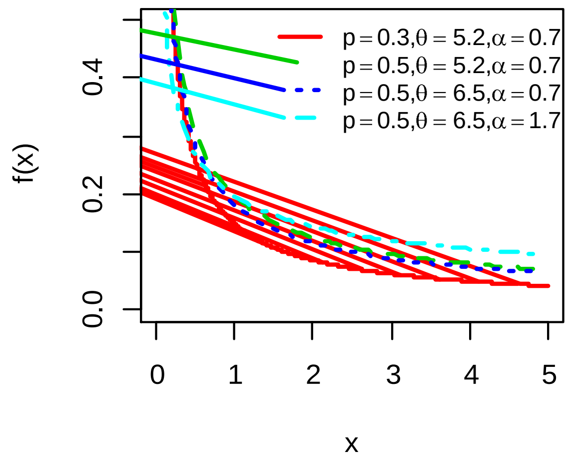

Fig. 2: pdf plot of EMID

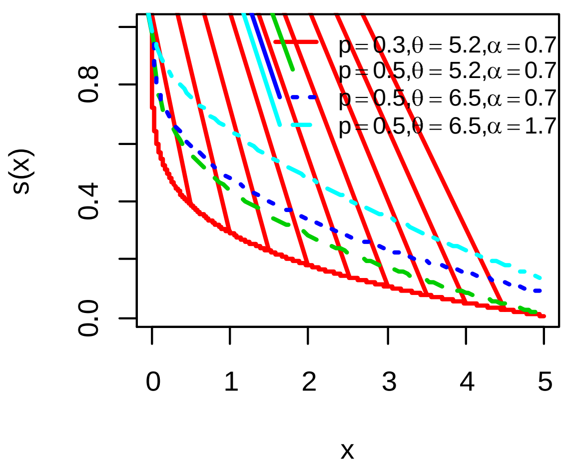

Fig. 3: Survival Function plot of EMID

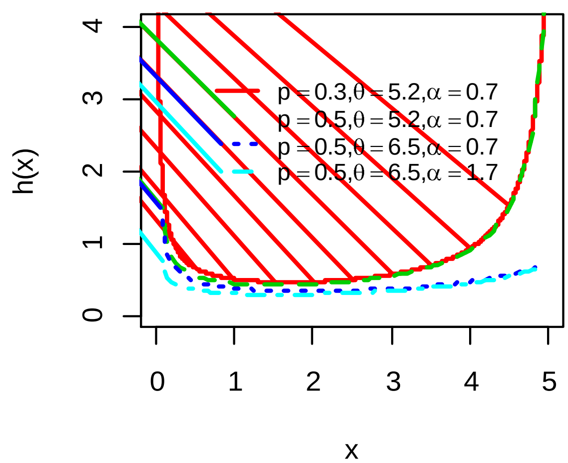

Fig. 4: Failure Rate Function plot of EMID

3 Moments

The r^th^order moment $E(x^r)$ of exponentiated mukherjee-islam distribution are computed as follows

$$E(x^r) = \int_0^\theta x^r \ f(x)dx$$
$$E(x^r) = \int_0^\theta x^r \ \frac{\alpha p \, x^{\alpha p-1}}{\theta^{\alpha p}} \ dx$$
$$E(x^r) = \frac{\alpha p \, \theta^r}{r+\alpha p} \tag{5}$$

Put r = 1 and 2 in (5), we get

$$E(x) = \frac{\alpha p\theta}{1+\alpha p} \quad \text{and} \quad E(x^2) = \frac{\alpha p \, \theta^2}{2+\alpha p}$$

Therefore, $V(x) = \alpha p\theta \left[\frac{\theta}{\alpha p+2} - \frac{\alpha p\theta}{(\alpha p+1)^2}\right]$

Also the different moments of the exponentiated mukherjee-islam distribution can be obtained using the moment generating function. Let X have exponentiated mukherjee-islam distribution, then the moment generating function of X is obtained by

$$M_X(t) = E(e^{tx}) = \int_0^\theta e^{tx} f(x)dx$$
$$= \int_0^\theta \left(1 + tx + \frac{(tx)^2}{2!} + \cdots\right) f(x)dx$$
$$= \int_0^\theta \sum_{r=0}^\theta \frac{t^r}{r!} x^r \ f(x)dx$$
$$= \sum_{r=0}^\theta \frac{t^r}{r!} E(x^r)$$
$$= \sum_{r=0}^\theta \frac{(t\theta)^r}{r!} \frac{\alpha p}{(r+\alpha p)}$$

4 Maximum Likelihood Estimators and Fisher's Information Matrix

In this section, we will discuss the maximum likelihood estimators of the parameters of the exponentiated mukherjee-islam distribution and their asymptotic properties. Let $X_1, X_2, X_3, \ldots, X_n$ be a random sample of size n from the exponentiated mukherjee-islam distribution, then the likelihood function is given by

$$L(\alpha, p, \theta) = \frac{(\alpha p)^n}{\theta^{n\alpha p}} \sum_{i=1}^n {x_i}^{\alpha p-1}$$

The log likelihood function is given by

$$\ln L(\alpha, p, \theta) = n \ln \alpha + n \ln p - n\alpha p \ln \theta + (\alpha p-1) \sum_{i=1}^n \ln x_i \tag{6}$$

Therefore the MLE's of α, p, θ which maximize (6) must satisfy the normal equation

$$\frac{\partial}{\partial\alpha} \ln L(\alpha, p, \theta) = \frac{n}{\alpha} - np \ln \theta + p\sum_{i=1}^n \ln x_i = 0 \tag{7}$$
$$\frac{\partial}{\partial p} \ln L(\alpha, p, \theta) = \frac{n}{p} - n\alpha \ln \theta + \alpha\sum_{i=1}^n \ln x_i = 0 \tag{8}$$
$$\frac{\partial}{\partial\theta} \ln L(\alpha, p, \theta) = \frac{n\alpha p}{\theta} = 0 \tag{9}$$

From (7), (8) and (9), we obtain the MLE's of α, p and θ are given by

$$\hat{\alpha} = \frac{n}{p(n \ln \theta - \sum_{i=1}^n \ln x_i)}$$
$$\hat{p} = \frac{n}{\alpha(n \ln \theta - \sum_{i=1}^n \ln x_i)}$$

and $\hat{\theta} = \infty$, which is an absured result.

Now we apply inspection method. Let us consider n-ordered samples $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$, then

$$0 \le X_{(1)} \le X_{(2)} \le \ldots \le X_{(n)} \le \theta$$
$$\Rightarrow \quad \theta \ge X_{(n)}$$

Therefore, MLE of $\theta = X_{(n)}$ = the largest sample observation.

To obtain confidence interval we use the asymptotic normality results. We have that, if $\hat{\lambda} = (\hat{\alpha}, \hat{p}, \hat{\theta})$ denotes the MLE of $\hat{\lambda} = (\alpha, p, \theta)'$, then

$$\sqrt{n}(\hat{\lambda} - \lambda) \rightarrow N_3(0, I^{-1}(\lambda))$$

Where $I^{-1}(\lambda)$ is Fisher's Information Matrix given by

$$I(\lambda) = -\frac{1}{n}\begin{bmatrix} E\left(\frac{\partial^2 L}{\partial \alpha^2}\right) & E\left(\frac{\partial^2 L}{\partial \alpha\,\partial p}\right) & E\left(\frac{\partial^2 L}{\partial \alpha\,\partial \theta}\right) \\ E\left(\frac{\partial^2 L}{\partial \alpha\,\partial p}\right) & E\left(\frac{\partial^2 L}{\partial P^2}\right) & E\left(\frac{\partial^2 L}{\partial P\,\partial \theta}\right) \\ E\left(\frac{\partial^2 L}{\partial \alpha\,\partial \theta}\right) & E\left(\frac{\partial^2 L}{\partial P\,\partial \theta}\right) & E\left(\frac{\partial^2 L}{\partial \theta^2}\right) \end{bmatrix}$$

Where $E\left(\frac{\partial^2 L}{\partial \alpha^2}\right) = \frac{np}{\theta}$, $E\left(\frac{\partial^2 L}{\partial P^2}\right) = -\frac{n}{p^2}$, $E\left(\frac{\partial^2 L}{\partial \theta^2}\right) = \frac{n\alpha p}{\theta^2}$

$E\left(\frac{\partial^2 L}{\partial \alpha\,\partial \theta}\right) = -\frac{np}{\theta}$, $E\left(\frac{\partial^2 L}{\partial P\,\partial \theta}\right) = -\frac{n\alpha}{\theta}$, $E\left(\frac{\partial^2 L}{\partial \alpha\,\partial p}\right) = -n \ln\theta + \sum_{i=1}^{n} \ln x_i$

λ being unknown, we estimate $I^{-1}(\lambda)$ by $I^{-1}(\hat{\lambda})$ and can use this to obtain asymptotic confidence intervals for α, p and θ.

5 Order Statistics

Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ denotes the order statistics of a random sample $X_1, X_2, \ldots, X_n$ from a contineous population with cumulative density function $F_X(x)$ and probability density function $f_X(x)$, then the pdf of rth order statistics $X_{(r)}$ is given by

$$f_{x(r)}(x) = \frac{n!}{(r-1)!\,(n-r)!}\, f_X(x)(FX(x))^{r-1}\,(1 - FX(x))^{n-r} \quad (10)$$

Therefore, the pdf of the rth order statistics $X_{(r)}$ for exponentiated mukherjee-islam distribution is given by

$$f_{x(r)}(x) = \frac{n!}{(r-1)!\,(n-r)!}\, \frac{\alpha p}{\theta^{\alpha p}}\, x^{\alpha p-1}\left[\left(\frac{x}{\theta}\right)^{\alpha p}\right]^{r-1}\left[1 - \left(\frac{x}{\theta}\right)^{\alpha p}\right]^{n-r} \quad (11)$$

Using the equation (11), the density function of the largest order statistics $X_{(n)}$ is

$$f_{x(n)}(x) = \frac{n\alpha p}{\theta^{\alpha p}}\, x^{\alpha p-1}\left[\left(\frac{x}{\theta}\right)^{\alpha p}\right]^{n-1}$$
$$= \frac{n\alpha p}{x^{\alpha p}}\left[\left(\frac{x}{\theta}\right)^{\alpha p}\right]^{n}$$

and the pdf of the 1st order statistics $X_{(1)}$ is given by

$$f_{x(1)}(x) = \frac{n\alpha p}{\theta^{\alpha p}}\, x^{\alpha p-1}\left[1 - \left(\frac{x}{\theta}\right)^{\alpha p}\right]^{n-1}$$

6 Conclusion

In this paper, we have proposed a new family of distributions called the exponentiated mukherjee-islam distribution. The distribution has three parameters (one scale and two shape). The survival function, failure rate and moments of the distribution have been derived. The parameters have been obtained using maximum likelihood technique. Also Fisher information matrix and order statistics have been derived.

Acknowledgement

The authors are highly thankful to the referees for their constructive suggestions, which have immensely helped the presentation of the paper and the editor for their valuable suggestions.

References

[1] B. Abdul-Moniem and H. F. Abdel-Hameed, "Exponentiated lomax distribution", International Journal of Mathematical Archive-3(5), pp.2144-2150, 2012.

[2] Gupta, R. D. and D. Kundu, "Exponentiated exponential family: an alternative to gamma and Weibull", Biometrical Journal, 43(1), pp. 117-130, 2001.

[3] Gupta, R. D. and D. Kundu, "Generalized exponential distribution: different method of estimations", Journal of Statistical Computation and Simulation, 69, 315-337, 2001a.

[4] Gupta, R. D. and D. Kundu, "Generalized exponential distribution: statistical inferences", Journal of Statistical Theory and Applications, 1,101-118, 2002.

[5] Hassan, A. S. (2005), "Goodness-of-fit for the generalized exponential distribution", InterStat, July, No.1.

[6] M. Masoom Ali, Manisha Pal and Jungsoo Woo, "Some exponentiated distributions", The Korean Communications in Statistics, vol. 14 No.1, pp 93-109, 2007.

[7] M. Pal, M. M. Ali and J. Woo, "Exponentiated weibull distribution", Statistica, auno LXVI, n. 2, pp 139-147, 2006.

[8] Mudholkar, G. S. and Srivastava, D. K., "Exponentiated weibull family for analyzing bathtub failure data", IEEE Transactions of Reliability, vol. 42, 299-302, 1993.

[9] Shashi Saxena, Shazia Zarrin, Mustafa Kamal and Arif-ul-islam, "Computation of reliability and bayesian analysis of system reliability for Mukherjee Islam Failure Model", American Journal of Mathematics and Statistics,2(2): 1-4, 2012.

[10] S. P. Mukherjee, and A. Islam, "A finite-range distribution of finite times", Naval Research Logistics Quarterly: pp. 487-491, 1983.

[11] Zahoor Ahmad, Adil Rashid and T. R. Jan, "A New Discrete Compound Distribution with Application", J. Stat. Appl. Pro. 6, No.1, 233-241, 2017.

[12] Zahoor Ahmad, Adil Rashid and T. R. Jan, "Generalized Version of Complementary Lindley Power Series Distribution with Applications", Pak.j.stat.oper.res. vol.XIV No.1 2018, pp139-155, 2018.

[13] Zahoor Ahmad and Adil Rashid, "Maximum Entropy Formalism for Zero Truncated Poission and Bionomial Distribution", J. Stat. Appl. Pro. 6, No. 2, 441-444, 2017.

Aafaq Ahmad Rather, is a Research Scientist at the Department of Statistics, Annamalai University, Tamil Nadu, India. He did his M.Sc in Statistics from University of Kashmir, Srinagar with Distinction and M.Phil from Agra University, Agra , India and topped the academic honors and has been a Gold Medalist. His research interests are in the areas of Generalized Probability distributions, Bio-Statistics and Stochastic Modeling. He has published research articles in reputed both international and national journals.

C. Subramanian, is an Assistant Professor in the Department of Statistics, Annamalai University, Chidambaram, India. He did his M.Sc and M.Phil degrees in Statistics from Department of Statistics, Bharathiar University, Coimbatore, India in 1989 and 1990 respectively. He received his Ph.D degree in Statistics from Department of Statistics, Annamalai University, Chidambaram, India in 2003. He has published work in international and national journals of repute. His current research areas are in the Stochastic Modeling, Bio-Statistics and Generalised Distributions.