Sohag Journal of Mathematics
An International Journal

http://dx.doi.org/10.18576/sjm/070202

Note on Euler-Bernoulli Equation

Anvar Hasanov[1] *and Junesang Choi*[2,*]

[1]Institute of Mathematics, National University of Uzbekistan, Durmon yuli str. 29, Tashkent 100125, Uzbekistan
[2]Department of Mathematics, Dongguk University, Gyeongju 38066, Republic of Korea

Received: 14 Feb. 2018, Revised: 2 Nov. 2019, Accepted: 23 Dec. 2019
Published online: 1 May 2020

Abstract: The Euler-Bernoulli equation, which is a fourth-order partial differential equation, and its related ones have been investigated in diverse ways. Here, by suitably choosing the transverse displacement function, the fourth-order partial differential equation reduces to a fourth-order ordinary differential equation. Then we solve the fourth-order ordinary differential equation using the theory of generalized hypergeometric functions.

Keywords: Euler-Bernoulli equation, generalized hypergeometric functions

1 Introduction

Since Daniel Bernoulli and Leonard Euler developed the theory of the Euler-Bernoulli beam problem, its related ones have been investigated in diverse ways (see, e.g. [1], [2], [3], [5], [6], [10], [11] and the references cited therein). Here we consider a rectangular rod length l $(0 \leq x \leq l)$, height h and width b. Let $u(t,x)$ be the transverse displacement at time t and position x from one end of the rod (or beam) taken as the origin. Then the $u(t,x)$ satisfies the following fourth-order partial differential equation (see, e.g., [11, p. 333])

$$12\,S\,\rho\, u_{tt} + E\,b\,h^3\, u_{xxxx} = 0, \tag{1}$$

where ρ is rod density, S is cross sectional area, E is modulus of elasticity of the rod material. Letting

$$12\,S\,\rho = x^\eta \ \ (\eta = \text{constant}) \quad \text{and} \quad E\,b\,h^3 = a^2 = \text{constant}, \tag{2}$$

equation (1) takes in the following form

$$x^\eta\, u_{tt} + a^2\, u_{xxxx} = 0 \quad \left(x \in \mathbb{R}^+;\, \eta \in \mathbb{R}_0^+\right). \tag{3}$$

Here and in the following, let $\mathbb{C}$, $\mathbb{R}^+$, and $\mathbb{Z}_0^-$ be the sets of complex numbers, positive real numbers, and non-positive integers, respectively, and let $\mathbb{R}_0^+ := \mathbb{R}^+ \cup \{0\}$.

In this paper, by choosing $u(t,x)$ in (3) as in (16), the fourth-order partial differential equation (3) reduces to a fourth-order ordinary differential equation. Then we present a general solution of the equation (16) using the theory of generalized hypergeometric functions.

2 Generalized hypergeometric function and its associated differential equation

Consider the following generalized hypergeometric function (see, e.g. [9, Section 1.5])

$$ {}_2F_3\,(a_1, a_2\,;\, c_1, c_2, c_3\,;\, x) = \sum_{m=0}^{\infty} \frac{(a_1)_m\,(a_2)_m}{(c_1)_m\,(c_2)_m\,(c_3)_m\, m!}\, x^m \tag{4}$$

$$\left(c_j \in \mathbb{C} \setminus \mathbb{Z}_0^- \ (j = 1, 2, 3)\right),$$

where $(\lambda)_m$ is the Pochhammer symbol (see, e.g. [9, Section 1.1]). The function (4) satisfies the following fourth order ordinary differential equation (see, e.g. [7, pp. 74-80])

$$\begin{aligned} x^3\,\frac{d^4u}{dx^4} &+ (c_1 + c_2 + c_3 + 3)\, x^2\, \frac{d^3u}{dx^3} \\ &+ (c_1c_2 + c_2c_3 + c_3c_1 + c_1 + c_2 + c_3 + 1 - x)\, x\, \frac{d^2u}{dx^2} \\ &+ [c_1\,c_2\,c_3 - (a_1 + a_2 + 1)x]\,\frac{du}{dx} - a_1\,a_2\,u = 0. \end{aligned} \tag{5}$$

Let $u(x)$ be a solution of (5). We find the other linearly independent solutions of (5) in a neighborhood of $x = 0$. To do this, let

$$u(x) := x^\gamma\, w(x) \quad (\gamma \text{ is a constant}). \tag{6}$$

* Corresponding author e-mail: junesang@dongguk.ac.kr

Applying (6) to equation (5), we get

$$
\begin{aligned}
&x^3\frac{d^4w}{dx^4}+(4\gamma+c_1+c_2+c_3+3)\,x^2\frac{d^3w}{dx^3}\\
&+\big[6\gamma(\gamma-1)+3\gamma(c_1+c_2+c_3+3)\\
&+(c_1c_2+c_2c_3+c_3c_1+c_1+c_2+c_3+1-x)\big]x\frac{d^2w}{dx^2}\\
&+\big[4\gamma(\gamma-1)(\gamma-2)+3\gamma(\gamma-1)(c_1+c_2+c_3+3)\\
&+2\gamma(c_1c_2+c_2c_3+c_3c_1+c_1+c_2+c_3+1-x)\\
&+c_1c_2c_3-(a_1+a_2+1)x\big]\frac{dw}{dx}\\
&+\big[\gamma\{(\gamma-1)(\gamma-2)(\gamma-3)\\
&+(\gamma-1)(\gamma-2)(c_1+c_2+c_3+3)\\
&+(\gamma-1)(c_1c_2+c_2c_3+c_3c_1+c_1+c_2+c_3+1)\\
&+c_1c_2c_3\}x^{-1}\\
&-\gamma(\gamma-1)-\gamma(a_1+a_2+1)-a_1a_2\big]\,w=0.
\end{aligned}\tag{7}
$$

Using the following factorization in (7)

$$
\begin{aligned}
&(\gamma-1)(\gamma-2)(\gamma-3)+(\gamma-1)(\gamma-2)(c_1+c_2+c_3+3)\\
&+(\gamma-1)(c_1c_2+c_2c_3+c_3c_1+c_1+c_2+c_3+1)+c_1c_2c_3\\
&=(\gamma+c_1-1)(\gamma+c_2-1)(\gamma+c_3-1),
\end{aligned}\tag{8}
$$

we obtain

$$
\begin{aligned}
&x^3\frac{d^4w}{dx^4}+(4\gamma+c_1+c_2+c_3+3)\,x^2\frac{d^3w}{dx^3}\\
&+\big[6\gamma(\gamma-1)+3\gamma(c_1+c_2+c_3+3)\\
&+(c_1c_2+c_2c_3+c_3c_1+c_1+c_2+c_3+1-x)\big]x\frac{d^2w}{dx^2}\\
&+\big[4\gamma(\gamma-1)(\gamma-2)+3\gamma(\gamma-1)(c_1+c_2+c_3+3)\\
&+2\gamma(c_1c_2+c_2c_3+c_3c_1+c_1+c_2+c_3+1-x)\\
&+c_1c_2c_3-(a_1+a_2+1)x\big]\frac{dw}{dx}\\
&+\big[\gamma(\gamma+c_1-1)(\gamma+c_2-1)(\gamma+c_3-1)x^{-1}\\
&-(\gamma+a_1)(\gamma+a_2)\big]\,w=0.
\end{aligned}\tag{9}
$$

To vanish the term x^{-1} (see, e.g. [8, Section 18.2]), we should have

$$\gamma(\gamma+c_1-1)(\gamma+c_2-1)(\gamma+c_3-1)=0, \tag{10}$$

which gives the following four solutions

$$\gamma=0,\ \gamma=1-c_1,\ \gamma=1-c_2,\ \gamma=1-c_3. \tag{11}$$

Applying each of the four solutions in (11) to (9), we obtain the following four linearly independent solutions of (5)

$$u_1={}_2F_3\,(a_1,a_2\,;c_1,c_2,c_3\,;x)\,, \tag{12}$$

$$
\begin{aligned}
u_2=&x^{1-c_1}\,{}_2F_3\big(1-c_1+a_1,\,1-c_1+a_2\,;\\
&2-c_1,\,1+c_2-c_1,\,1+c_3-c_1\,;x\big),
\end{aligned}\tag{13}
$$

$$
\begin{aligned}
u_3=&x^{1-c_2}\,{}_2F_3\big(1-c_2+a_1,\,1-c_2+a_2\,;\\
&2-c_2,\,1+c_1-c_2,\,1+c_3-c_2\,;x\big)
\end{aligned}\tag{14}
$$

and

$$
\begin{aligned}
u_4=&x^{1-c_3}\,{}_2F_3\big(1-c_3+a_1,\,1-c_3+a_2\,;\\
&2-c_3,\,1+c_1-c_3,\,1+c_2-c_3\,;x\big).
\end{aligned}\tag{15}
$$

3 A solution of Euler-Bernoulli equation (3)

In (3), letting

$$
\begin{aligned}
&u=p(t;a)\,\omega(\sigma),\ \text{where}\ p(t;a):=\left(-a^2t^2\right)^{-1}\\
&\text{and}\ \sigma:=-\frac{4}{a^2t^2\,(\eta+4)^4}\,x^{\eta+4},
\end{aligned}\tag{16}
$$

where $x\in\mathbb{R}^+$, $\eta\in\mathbb{R}_0^+$, and a^2 is the same as in (2), we find the fourth-order *partial* differential equation (3) which reduces to a fourth-order *ordinary* differential equation (25) or (26). Then we solve the fourth-order ordinary differential equation (26) using the differential equation satisfied by ${}_2F_3$ in Section 2.

Theorem 1. *The four linearly independent solutions of the Euler-Bernoulli equation in the form* (26) *are given as follows*:

$$u_1=p(t;a)\,{}_2F_3\left(1,\frac{3}{2};\frac{\alpha+3}{4},\frac{\alpha+1}{2},\frac{3\alpha+1}{4};\sigma\right), \tag{17}$$

$$u_2=p(t;a)\,\sigma^{\frac{1-\alpha}{4}}\,{}_1F_2\left(\frac{7-\alpha}{4};\frac{\alpha+3}{4},\frac{\alpha+1}{2};\sigma\right), \tag{18}$$

$$u_3=p(t;a)\,\sigma^{\frac{1-\alpha}{2}}\,{}_1F_2\left(\frac{4-\alpha}{2};\frac{5-\alpha}{4},\frac{\alpha+3}{4};\sigma\right), \tag{19}$$

and

$$u_4=p(t;a)\,\sigma^{\frac{3}{4}(1-\alpha)}\,{}_1F_2\left(\frac{9-3\alpha}{4};\frac{3-\alpha}{2},\frac{5-\alpha}{4};\sigma\right), \tag{20}$$

where $\alpha:=\frac{\eta}{\eta+4}$.

Proof. For simplicity, let $p:=p(t;a)$. We find

$$u_{tt}=p_{tt}\,\omega+2\,p_t\,\omega_\sigma\,\sigma_t+p\,\omega_{\sigma\sigma}\,\sigma_t^2+p\,\omega_\sigma\,\sigma_{tt} \tag{21}$$

and

$$
\begin{aligned}
u_{xxxx}=&p_{xxxx}\,\omega+6\,p_{xx}\,\omega_\sigma\,\sigma_{xx}+4\,p_{xxx}\,\omega_\sigma\,\sigma_x+6\,p_{xx}\,\omega_{\sigma\sigma}\,\sigma_x^2\\
&+4\,p_x\,\omega_{\sigma\sigma\sigma}\,\sigma_x^3+12\,p_x\,\omega_{\sigma\sigma}\,\sigma_x\,\sigma_{xx}+4\,p_x\,\omega_\sigma\,\sigma_{xxx}\\
&+p\,\omega_{\sigma\sigma\sigma\sigma}\,\sigma_x^4+6\,p\,\omega_{\sigma\sigma\sigma}\,\sigma_x^2\,\sigma_{xx}\\
&+3\,p\,\omega_{\sigma\sigma}\,\sigma_{xx}^2+4\,p\,\omega_{\sigma\sigma}\,\sigma_x\,\sigma_{xxx}+p\,\omega_\sigma\,\sigma_{xxx}.
\end{aligned}\tag{22}
$$

Substituting (21) and (22) into (3), we obtain

$$\begin{aligned}&a^2 p \sigma_x^4 \omega_{\sigma\sigma\sigma\sigma} + \left(4a^2 p_x \sigma_x^3 + 6a^2 p \sigma_x^2 \sigma_{xx}\right) \omega_{\sigma\sigma\sigma} \\ &+ \left(6a^2 p_{xx} \sigma_x^2 + 12a^2 p_x \sigma_x \sigma_{xx} + 3a^2 p \sigma_{xx}^2 + x^\eta p \sigma_t^2 \right.\\ &\left.+ 4a^2 p \sigma_x \sigma_{xxx}\right) \omega_{\sigma\sigma} \\ &+ \left(2x^\eta p_t \sigma_t + x^\eta p \sigma_{tt} + 6a^2 p_{xx} \sigma_{xx} + 4a^2 p_{xxx} \sigma_x \right.\\ &\left.+ 4a^2 p_x \sigma_{xxx} + a^2 p \sigma_{xxx}\right) \omega_\sigma + \left(a^2 p_{xxx} + x^\eta p_{tt}\right) \omega = 0. \end{aligned} \tag{23}$$

Considering $p_x = 0$ in (23), we get

$$\begin{aligned}&a^2 p \sigma_x^4 \omega_{\sigma\sigma\sigma\sigma} + 6a^2 p \sigma_x^2 \sigma_{xx} \omega_{\sigma\sigma\sigma} \\ &+ p\left(3a^2 \sigma_{xx}^2 + x^\eta \sigma_t^2 + 4a^2 \sigma_x \sigma_{xxx}\right) \omega_{\sigma\sigma} \\ &\quad + \left(2x^\eta p_t \sigma_t + x^\eta p \sigma_{tt} + a^2 p \sigma_{xxxx}\right) \omega_\sigma + x^\eta p_{tt} \omega = 0. \end{aligned} \tag{24}$$

Using

$$\sigma_x = -\frac{4}{a^2 t^2 (\eta+4)^3} x^{\eta+3}, \ldots, p_{tt} = -6a^2 p^2$$

in (24), we have

$$\begin{aligned}&\sigma^3 \omega_{\sigma\sigma\sigma\sigma} + 6\frac{\eta+3}{\eta+4} \sigma^2 \omega_{\sigma\sigma\sigma} \\ &+ \left(\frac{\eta+3}{\eta+4} \cdot \frac{7\eta+17}{\eta+4} - \sigma\right) \sigma \omega_{\sigma\sigma} \\ &+ \left(\frac{\eta+3}{\eta+4} \cdot \frac{\eta+2}{\eta+4} \cdot \frac{\eta+1}{\eta+4} - \frac{7}{2}\sigma\right) \omega_\sigma - \frac{3}{2}\omega = 0. \end{aligned} \tag{25}$$

Setting $\alpha := \frac{\eta}{\eta+4}$ in (25), we obtain

$$\begin{aligned}&\sigma^3 \omega_{\sigma\sigma\sigma\sigma} + \left(\frac{\alpha+3}{4} + \frac{\alpha+1}{2} + \frac{3\alpha+1}{4} + 3\right) \sigma^2 \omega_{\sigma\sigma\sigma} \\ &+ \left(\frac{\alpha+3}{4} \cdot \frac{\alpha+1}{2} + \frac{\alpha+1}{2} \cdot \frac{3\alpha+1}{4} + \frac{3\alpha+1}{4} \cdot \frac{\alpha+3}{4} \right.\\ &\left.+ \frac{\alpha+3}{4} + \frac{\alpha+1}{2} + \frac{3\alpha+1}{4} + 1 - \sigma\right) \sigma \omega_{\sigma\sigma} \\ &+ \left[\frac{\alpha+3}{4} \cdot \frac{\alpha+1}{2} \cdot \frac{3\alpha+1}{4} - \left(1 + \frac{3}{2} + 1\right)\sigma\right] \omega_\sigma - \frac{3}{2}\omega = 0. \end{aligned} \tag{26}$$

Comparing (9) $(\gamma = 0)$ with (26) and setting

$$a_1 = 1,\ a_2 = \frac{3}{2},\ c_1 = \frac{\alpha+3}{4},\ c_2 = \frac{\alpha+1}{2},\ c_3 = \frac{3\alpha+1}{4},\ x = \sigma$$

in (12), we obtain the following four linearly independent solutions of (26):

$$\omega_1 = {}_2F_3\left(1, \frac{3}{2}; \frac{\alpha+3}{4}, \frac{\alpha+1}{2}, \frac{3\alpha+1}{4}; \sigma\right), \tag{27}$$

$$\omega_2 = \sigma^{\frac{1-\alpha}{4}}\, {}_1F_2\left(\frac{7-\alpha}{4}; \frac{\alpha+3}{4}, \frac{\alpha+1}{2}; \sigma\right), \tag{28}$$

$$\omega_3 = \sigma^{\frac{1-\alpha}{2}}\, {}_1F_2\left(\frac{4-\alpha}{2}; \frac{5-\alpha}{4}, \frac{\alpha+3}{4}; \sigma\right), \tag{29}$$

and

$$\omega_4 = \sigma^{\frac{3}{4}(1-\alpha)}\, {}_1F_2\left(\frac{9-3\alpha}{4}; \frac{3-\alpha}{2}, \frac{5-\alpha}{4}; \sigma\right). \tag{30}$$

Considering (16), we obtain the desired solutions.

4 Conclusion remark

In this paper, by suitably choosing the transverse displacement function, the Euler-Bernoulli equation; a partial differential equation, reduces to a fourth-order ordinary differential equation, which is shown to be solved using the theory of generalized hypergeometric functions. Certain similar partial differential equations are believed to be solved by applying the method used here.

Acknowledgement

The authors are grateful to the anonymous referee for the constructive comments which improved this paper.

References

[1] A.E. Baroudi and F. Razafimahery. Transverse vibration analysis of Euler-Bernoulli beam carrying point masse submerged in fluid media, *Internat. J. Eng. Tech.*, **4(2)**, 369–380, 2015.
[2] L. Chalah-Rezguia, F. Chalaha, K. Faleka, A. Balib and A. Nechnecha. Transverse vibration analysis of uniform beams under various ends restraints, *APCBEE Procedia*, **9**, 328–333, 2014.
[3] S.M. Han, H. Benaroya and T. Wei. Dynamics of Transversely vibrating beams using four engineering theories, *J. Sound and Vibration*, **225(5)**, 935–988, 1999.
[4] H.P.W. Gottlieb. Isospectral Euler–Bernoulli beam with continuous density and rigidity functions, *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, **413**, 235—250, 1987.
[5] A. Mirzabeigy and R. Madoliat. Large amplitude free vibration of axially loaded beams resting on variable elastic foundation, *Alexandria Eng. J.*, **55**, 1107–1114, 2016.
[6] R. Naz and F. M. Mahomed. Dynamic Euler-Bernoulli beam equation: classification and reductions, *Math. Probl. Eng.*, **2015**, Article ID 520491, 2015. http://dx.doi.org/10.1155/2015/520491
[7] E.D. Rainville. *Special Functions*, Macmillan Company, New York, 1960; Reprinted by Chelsea Publishing Company, Bronx, New York, 74–80, 1971.
[8] E.D. Rainville, P.E. Bedient and R.E. Bedient. *Elementary Differential Equations*, eighth edi., Prentice-Hall, Inc., 360–362, 1997.
[9] H.M. Srivastava and J. Choi. *Zeta and q-Zeta Functions and Associated Series and Integrals*, Elsevier Science Publishers, Amsterdam, London and New York, 1–12, 63–73, 2012.

[10] C.W. Soh. Euler–Bernoulli beams from a symmetry standpoint-characterization, *J. Math. Anal. Appl.*, **345**, 387—395, 2008.
[11] S.P. Timoshenko. *Course of Stability Theory*, Kiev, 333, 1972.

Anvar Hasanov received his M.S. degree from Tashkent State University (Uzbekistan) in 1975 and his Ph.D. degree from the Institute of Mathematics of the Uzbek Academy of Sciences (Uzbekistan) in 1982. He stayed at Dongguk University in Gyeongju in the Republic of Korea for the year 2010 as a Research Professor. He is a research professor at the University of Gent (Belgium) and, currently works at the Institute of Mathematics of the Academy of Sciences of the Republic of Uzbekistan. He has published many papers in applied mathematics and special functions.

Junesang Choi received his B.S. and M.S. from Gyeongsang National University (Republic of Korea) in 1981 and 1983, respectively, and his Ph.D. from the Florida State University in 1991. He has taught Mathematics at Dongguk University (Gyeongju) in the Republic of Korea from September in 1991 to February in 2019. Now he, as an emeritus professor there, continues to make mathematical researches. He has written more than 300 papers, whose research subjects include diverse special functions, analytic number theory, fractional calculus, complex analysis, and so on, and has collaborated with over 50 co-authors. His other publications include (for example) 4 books, one edited volume, and a chapter in a monograph.