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Solutions For Some Conformable Differential Equations

Ndolane Sene

Département de mathématiques de la décision, Université Cheikh Anta Diop de Dakar, Laboratoire Lmdan, BP 5683 Dakar Fann, Sénégal.

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Abstract: In recent work, the operator method and other methods have been introduced to solve the sequential linear conformable differential equations with constant coefficients. This paper deals with the analytic candidate solutions of the conformable differential equations. The conformable integral operator is the main key of this paper and is an additional method to get the candidate solutions of the conformable differential equations. Several examples are given to illustrate our main results of this paper.

Keywords: Conformable derivative,conformable integral, conformable differential equations.

1 Introduction

During the last years, much work gave some important results about the role of fractional calculus in physics, control engineering and signal processing [1,2,3,4,5,6]. There exist various fractional derivative operators in fractional calculus as: Riemann-Liouville fractional derivative [7], Caputo fractional derivative [7], Atangana-Baleanu fractional derivative in Caputo sense and Riemann-Liouville sense [8], Atangana Koca fractional derivative, Caputo-Fabrizio fractional derivative [9], and other.

Recently conformable derivative operator was introduced in the literature by Khalil [10]. Many problems appear with this conformable derivative operator. Cauchy problem is known very important in many field of science and engineering. This paper contribute also to give solution for Cauchy problem in the context of conformable differential equations.

Finding the solution of the conformable differential equations with or without perturbation terms play important role in the stability analysis. Many results to get the candidate solutions of the conformable differential equations can be found in [5,11]. But there are many other classes of the differential equations of which we do not have the explicit solutions. There are many methods to get the analytic candidate solutions of the conformable differential equations, we can cite: the Lie symmetry method, the Invariant subspace method [12], the Wronskianand [13], Abel's formula [14], D'Alambert approach [14] and others.

This paper deals with the analytic candidate solutions of the conformable differential equations. We give the candidate solutions of the particular class of the conformable differential equations. Here we use conformable integral operator to get the analytic candidate solutions of the conformable differential equations.

The paper is organized as follows : in section 2, after recalling some necessary definitions, we describe the classes of the conformable differential equations, and provide our main results. In section 4, we give some numerical examples to illustrate our main results.

2 Preliminary Concepts and Main Results

2.1 Preliminary definitions

In this section, we introduce some definitions of the fractional calculus and several lemmas. We use them to establish the main results in this paper. We begin by recalling the conformable derivative operator introduced by Khalil and al. in [10].

* Corresponding author e-mail: ndolanesene@yahoo.fr

We prove conformable derivative operator is not in general monotone.
Definition 1 Given a function $f : [0,+\infty[\longrightarrow \mathbb{R}$. Then the conformable derivative of f of order α is defined by

$$T_\alpha(f)(t) = \lim_{\varepsilon\to 0} \frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon} \tag{1}$$

all $t > 0$, $\alpha \in (0,1)$. If f is α-differentiable in $(0,a)$, $a > 0$, and $\lim_{\varepsilon\to 0^+} f^{(\alpha)}(t)$ exists, then define

$$f^{(\alpha)}(0) = \lim_{\varepsilon\to 0^+} f^{(\alpha)}(t).$$

Definition 2 We denote by $C_\infty((0,+\infty),\mathbb{R}^n)$ the set of function $y \in C_\infty((0,+\infty),\mathbb{R}^n)$ such that $T_\alpha(y)(t)$ exists and is continuous on $(0,+\infty)$.
Lemma 1 Let $\alpha \in (0,1)$ and f is α-differentiable at point $t > 0$. If f is differentiable, then

$$T_\alpha(f)(t) = t^{1-\alpha}\frac{df}{dt}. \tag{2}$$

From expression given by (2), it is clear if $\alpha = 1$, we recover the classical derivative. Khalil's definition of conformable derivative satisfies the following properties (see [10] for details):
Lemma 2 Let $\alpha \in (0,1)$ and f, g be α-differentiable at point $t > 0$. Then

1. $T_\alpha(af + bg) = aT_\alpha(f) + bT_\alpha(g)$ for all constant $a, b \in \mathbb{R}$.
2. $T_\alpha(\lambda) = 0$, for all constant function $f(t) = \lambda$.
3. $T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f)$.
4. $T_\alpha(\frac{f}{g}) = \frac{fT_\alpha(g)-gT_\alpha(f)}{g^2}$.
5. The following triangular inequality:

$$T_\alpha(|f+g|) \le T_\alpha(|f|) + T_\alpha(|g|) \tag{3}$$

does not in general hold.

We give the following counterexample to illustrate the items (5). The proof of the other items can be found in [10].
Counterexample: To see that, let the function $f(t) = t^2$ and $g(t) = t$ on interval $[0,1]$, we have that $|f| = f \le g \le |g|$. But $T_\alpha(|f|)(1) = 2$ and $T_\alpha(|g|)(1) = 1$. And remark that $T_\alpha(|g|)(1) \le T_\alpha(|f|)(1)$. Then T_α is not a monotone operator. In this condition the triangular inequality is not satisfied.
Definition 3 The conformable integral starting from a of a function f of order $\alpha \in (0,1]$ is defined by

$$I_\alpha^a f(t) = \int_a^t x^{\alpha-1} f(x)dx. \tag{4}$$

Lemma 3 Let $\alpha \in (0,1]$ and f is any continuous function in a domain of I_α, for $t > a$ we have

$$T_\alpha I_\alpha^a(f)(t) = f(t). \tag{5}$$

Consequently, it follows that
Lemma 4 Let $\alpha \in (0,1]$ and f is any continuous function in a domain of I_α, for $t > a$ we have

$$\frac{d}{dt}\left[I_\alpha^a(f)(t)\right] = \frac{f(t)}{t^{1-\alpha}}. \tag{6}$$

This lemma is fundamental to get the analytic solution of the conformable differential equations.
Definition 4 The conformable exponential function is defined for every $s \ge 0$ by

$$E_\alpha(\lambda, s) = \exp\left(\lambda\frac{s^\alpha}{\alpha}\right), \tag{7}$$

where $\alpha \in (0,1)$ and $\lambda \in \mathbb{R}$.

2.2 Main results

The similar conformable differential equations were solved in [15,16,17,18], and the methods used to solve these equations use some transformations. We can cite, for example the operator method. In this paper, we solve the conformable differential equations using the conformable integral operator (an additional method) rather than transforming it into an ordinary one with a singularity.

2.2.1 Conformable differential equations of order α

We are now ready to state the main results of this paper on the conformable differential equations called here ”the conformable differential equations of order α”. Generally the conformable differential equations of order α which we consider in this paper is mathematically represented by the following form

$$T_\alpha(y) + f(t)y = g(t), \tag{8}$$

where $0 < \alpha < 1$, $y \in \mathbb{R}^n$, $T_\alpha(y)$ denotes the conformable derivative of y and $f, g : \mathbb{R} \to \mathbb{R}$ are α-differentiable functions. If $\alpha = 1$ we recover the classical differential equations of first order expressed as $y' + f(t)y = g(t)$. We first take the case in which $g = 0$. The particular conformable differential equation which we obtain is

$$T_\alpha(y) + f(t)y = 0. \tag{9}$$

The conformable differential equation defined by (9) is called the homogeneous differential equation. We have the following result.

Theorem 1 The homogeneous solution of the conformable differential equation (9) is given by

$$y_h(t) = y_0 e^{-I_\alpha^0(f)(t)}.$$

where f is any continuous function in the domain of I_α^0.

Proof: To prove Theorem 1, we have just verified that the equation (9) is satisfied by getting the function $y(t) = y_0 e^{-I_\alpha^0(f)(t)}$. By replacing above candidate solution into the conformable differential equation (9), and using the Lemma 4, we get :

$$\begin{aligned}
T_\alpha(y_h) + f(t)y_h &= y_0 t^{1-\alpha} \frac{d}{dt}\left[e^{-I_\alpha^0(f)(t)}\right] + y_0 e^{-I_\alpha^0(f)(t)} \\
&= -y_0 t^{1-\alpha} \frac{d}{dt}\left[I_\alpha^0(f)(t)\right] e^{-I_\alpha^0(f)(t)} + y_0 f(t) e^{-I_\alpha^0(f)(t)} \\
&= -y_0 t^{1-\alpha} \frac{f(t)}{t^{1-\alpha}} e^{-I_\alpha^0(f)(t)} + y_0 f(t) e^{-I_\alpha^0(f)(t)} \\
&= 0.
\end{aligned}$$

Then we conclude that the homogeneous solution of the conformable differential equation (9) is given by $y_h(t) = y_0 e^{-I_\alpha^0(f)(t)}$. That ends the proof of Theorem 1.

To get the general solution of the conformable differential equation defined by (9), we calculate particular solution y_p obtained by the following theorem.

Theorem 2 The particular solution of the conformable differential equation (9) is given by

$$y_p(t) = \lambda(t) e^{-I_\alpha^0(f)(t)},$$

where f is any continuous function in the domain of I_α^0 and the function $\lambda : \mathbb{R} \to \mathbb{R}$ is obtained by the following condition

$$\lambda(t) = I_\alpha^0(g(t) e^{I_\alpha^0(f)(t)}).$$

Proof: To prove this theorem, we have just verified that the equation (9) is satisfied by getting the function $y_p(t) = \lambda(t) e^{-I_\alpha^0(f)(t)}$. Replacing above candidate solution into the conformable differential equation (9), and using the Lemma 4, we have that:

$$\begin{aligned}
T_\alpha(y_p) + f(t)y_p &= T_\alpha(I_\alpha^0(g(t) e^{-I_\alpha^0(f)(t)}) e^{-I_\alpha^0(f)(t)})) + f(t) I_\alpha^0(g(t) e^{I_\alpha^0(f)(t)}) e^{-I_\alpha^0(f)(t)} \\
&= g(t) e^0 - I_\alpha^0(g(t) e^{-I_\alpha^0(f)(t)}) f(t) e^{-I_\alpha^0(f)(t)} + f(t) I_\alpha^0(g(t) e^{-I_\alpha^0(f)(t)}) e^{-I_\alpha^0(f)(t)} \\
&= g(t).
\end{aligned}$$

Then we conclude that the particular solution of the conformable differential equation (9) is given by $y_p(t) = \lambda(t) e^{-I_\alpha(f)(t)}$. That ends the proof of Theorem 2.

The general candidate solution of the conformable differential equations defined by (9) is given by the following

$$y(t) = y_h(t) + y_p(t).$$

2.2.2 Conformable differential equations of order 2α

In this section, we are ready to state the main results on the differential equations called here ”the conformable differential equations of order 2α”. Generally the conformable differential equations of order 2α which we consider in this paper are mathematically represented by the following form

$$T_{\alpha}^{(2)}(y)+aT_{\alpha}(y)+by=0, \tag{10}$$

where $a,b\in\mathbb{R}$, $0<\alpha\leq 1$, $T_{\alpha}(y)$ denotes the conformable derivative of the α-differentiable function y, $T_{\alpha}^{(2)}(y)=T_{\alpha}(T_{\alpha}(y))$. We make the following theorems.

Theorem 3 If $a+1=0$ and $b=0$ then the candidate solution of the conformable differential equation (10) is given by

$$y(t)=y_0e^{-a\frac{t^{\alpha}}{\alpha}}.$$

Proof: Let $T_{\alpha}(y)=w$, equation (10) becomes $T_{\alpha}(w)+aw=0$ whose solution is given by applying Theorem 1. Then $w(t)=w_0e^{-a\frac{t^{\alpha}}{\alpha}}$. To get the solution y we apply the inverse operator I_{α}^{0} and we obtain

$$y(t)=I_{\alpha}^{0}(w_0e^{-a\frac{t^{\alpha}}{\alpha}})=y_0e^{-a\frac{t^{\alpha}}{\alpha}}.$$

Theorem 4 If the characteristic equation of (10) defined by $r^2+ar+b=0$ have two distinct solutions r_1 and r_2 then the candidate solution of the conformable differential equation (10) is given by

$$y(t)=Ae^{r_1\frac{t^{\alpha}}{\alpha}}+Be^{r_2\frac{t^{\alpha}}{\alpha}}.$$

Proof: Let $r_1\in\mathbb{C}$ and $r_2\in\mathbb{C}$ be two solutions of the characteristic equation of (10) defined by $r^2+ar+b=0$. Let $y=ze^{r_1\frac{t^{\alpha}}{\alpha}}$, then $T_{\alpha}(y)=[T_{\alpha}(z)+r_1z]e^{r_1\frac{t^{\alpha}}{\alpha}}$ and $T_{\alpha}(T_{\alpha}(y))=\left[T_{\alpha}(T_{\alpha}(z))+2r_1T_{\alpha}(z)+r_1^2z\right]e^{r_1\frac{t^{\alpha}}{\alpha}}$. Replacing into the conformable differential equation (10), we have that

$$T_{\alpha}^{(2)}(y)+aT_{\alpha}(y)+by=[T_{\alpha}(T_{\alpha}(z))+(2r_1+a)T_{\alpha}(z)]e^{r_1\frac{t^{\alpha}}{\alpha}}=0.$$

We can observe if y is the solution of (10) then $T(z)$ is also the solution of the conformable differential equation of order α defined by $T_{\alpha}(w)+(2r_1+a)w=0$. Using the fact that $2r_1+a=r_1-r_2$, we obtain

$$T_{\alpha}(w)+(r_1-r_2)w=0,$$

whose solution is obtained using the conformable integral operator

$$\begin{aligned} w(t) &= w_0e^{-I_{\alpha}^{0}(r_1-r_2)} \\ &= w_0e^{(r_2-r_1)\frac{t^{\alpha}}{\alpha}}. \end{aligned}$$

Using the operator I_{α} starting at non zero constant d, it follows that

$$\begin{aligned} z(t) &= I_{\alpha}^{d}\left[w_0e^{(r_2-r_1)\frac{t^{\alpha}}{\alpha}}\right] \\ &= \frac{w_0}{r_2-r_1}e^{(r_2-r_1)\frac{t^{\alpha}}{\alpha}}+\beta, \end{aligned}$$

where β is a constant. Finally replacing in $y=ze^{r_1\frac{t^{\alpha}}{\alpha}}$, we have that

$$y(t)=\frac{w_0}{r_2-r_1}e^{r_2\frac{t^{\alpha}}{\alpha}}+\beta e^{r_1\frac{t^{\alpha}}{\alpha}}.$$

Letting $B=\frac{w_0}{r_2-r_1}$ and $A=\beta$, we get $y(t)=Ae^{r_2\frac{t^{\alpha}}{\alpha}}+Be^{r_2\frac{t^{\alpha}}{\alpha}}$. That ends the proof of the theorem.

Theorem 5 If the characteristic equation of (10) defined by $r^2+ar+b=0$ admits a double solution r which satisfies $2r+a=0$ then the candidate solution of the conformable differential equation (10) is given by

$$y(t)=(A+B\frac{t^{\alpha}}{\alpha})e^{r\frac{t^{\alpha}}{\alpha}}.$$

Proof: Let $r \in \mathbb{C}$ be the double solution of the characteristic equation of (10) defined by $r^2 + ar + b = 0$. Let $y = ze^{r_1 \frac{t^\alpha}{\alpha}}$. We have that $2r + a = 0$. We do the same work as in Theorem 4, it follows that if y is solution of (10) then $T_\alpha(z)$ is also the solution of the conformable differential equation of order α defined by $T_\alpha(w) = 0$, whose solution is got using the conformable integral operator

$$\begin{aligned} w(t) &= w_0 e^{-I_\alpha^0(0)} \\ &= w_0 \end{aligned}$$

Using the operator I_α starting at non zero constant d, it follows that

$$\begin{aligned} z(t) &= I_\alpha^d [w_0] \\ &= w_0 \frac{t^\alpha}{\alpha} + \beta, \end{aligned}$$

where β is a constant. Finally replacing in $y = ze^{r_1 \frac{t^\alpha}{\alpha}}$, we obtain that

$$y(t) = (w_0 \frac{t^\alpha}{\alpha} + \beta) e^{r \frac{t^\alpha}{\alpha}}.$$

Letting $A = \beta$ and $B = w_0$, we have $y(t) = (A + B\frac{t^\alpha}{\alpha}) e^{r \frac{t^\alpha}{\alpha}}$. That ends the proof of the theorem.

3 Numerical Examples

In this section, many examples are given to illustrate the proposed theorems.

• For illustration of Theorem 1, let Cauchy conformable differential equation defined as

$$T_\alpha(y) = \lambda y. \tag{11}$$

Applying Theorem 1, the analytic candidate solution is given by

$$\begin{aligned} y(t) &= y_0 e^{I_\alpha^0(\lambda)(t)} \\ &= y_0 E_\alpha(\lambda, t) \end{aligned}$$

• For illustration of Theorem 1, let Cauchy conformable differential equation defined as

$$T_{1/2}(y) + y = 0. \tag{12}$$

Applying Theorem 1, the analytic solution is given by

$$\begin{aligned} y(t) &= y_0 e^{-I_{1/2}^0(1)} \\ &= y_0 e^{-t} = y_0 E_{1/2}(-1, t). \end{aligned}$$

• For illustration of Theorem 1 and 2, let the following conformable differential equation defined as

$$T_{1/2}(y) + \sqrt{x} y = xe^{-x}. \tag{13}$$

Applying Theorem 1, the homogeneous solution is given by

$$\begin{aligned} y_h(x) &= y_0 e^{-I_{1/2}^0(\sqrt{x})} \\ &= y_0 e^{-x} = y_0 E_1(-1, x). \end{aligned}$$

To get the particular solution, applying Theorem 2, we obtain

$$y_p(x) = \lambda(x) e^{-x},$$

where

$$\lambda(t) = I_\alpha^0(xe^{-x} e^x) = \frac{2}{3} x^{\frac{3}{2}}$$

and then the particular solution is that $y_p(x) = \frac{2}{3}x^{\frac{3}{2}}e^{-x}$. Finally the general candidate solution is given by

$$y(x) = y_h(x) + y_p(x) = y_0 e^{-x} + \frac{2}{3}x^{\frac{3}{2}}e^{-x}.$$

• Let's the following particular conformable differential equation defined as

$$T_\alpha(y) + t^{1-\alpha}y = 0. \tag{14}$$

This conformable differential equation is equivalent to the ordinary differential equation $y' + y = 0$ and we easily prove its solution is independent to α. Applying Theorem 1, the candidate solution is given by

$$\begin{aligned} y(x) &= y_0 e^{-I_\alpha^0(t^{1-\alpha})} \\ &= y_0 e^{-x}. \end{aligned}$$

• For illustration of Theorem 3, let the following conformable differential equation defined as

$$T_{1/2}(T_{1/2}(y)) - T_{1/2}(y) = 0. \tag{15}$$

Applying Theorem 3, the candidate solution is given by

$$y(t) = ce^{2\sqrt{t}}$$

where c is an constant.

• For illustration of Theorem 4, let the following conformable differential equation defined as

$$T_{1/2}(T_{1/2}(y)) - 3T_{1/2}(y) + 2y = 0. \tag{16}$$

The characteristic equation is $r^2 - 3r + 2 = 0$, and have two distinct solutions $r_1 = -1$ and $r_2 = -2$. Applying Theorem 4, the candidate solution is given by

$$y(t) = Ae^{-\sqrt{t}} + Be^{-2\sqrt{t}}$$

where A and B are constants.

• For illustration of Theorem 5, let the following conformable differential equation defined as

$$T_{1/2}(T_{1/2}(y)) + 2T_{1/2}(y) + y = 0. \tag{17}$$

The characteristic equation is $r^2 + 2r + 1 = 0$, and has double solution $r = -1$ which satisfies the condition $2r + a = -2 + 2 = 0$. Applying Theorem 5, the candidate solution is given by

$$y(t) = (A + B\sqrt{t})e^{-\sqrt{t}},$$

where A and B are constants.

4 Cauchy Conformable Differential Equations

In this section we investigate to find solution of the conformable differential equation defined by

$$T_\alpha x = Ax + g(t,x), \tag{18}$$

where $x \in \mathbb{R}^n$ is state variable, A is an matrix in $\mathbb{R}^{n\times n}$ and $g : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ continuous function and satisfying $g(t,0) = 0$.

Let now get the solutions of the perturbed conformable fractional differential equations (18). The solutions of (18) are fundamental in many problems with conformable fractional derivative.

Proposition 1 Let $\alpha \in (0,1)$. Then the unique solution of the following initial value problem $T_\alpha x = Ax + g(t,x)$ with initial condition $x(t_0) = \eta \in \mathbb{R}^n$ is given by

$$x(t) = \eta E_\alpha(A, t - t_0) + \int_{t_0}^{t} (t-s)^{\alpha-1} E_\alpha(-A, s - t_0) E_\alpha(A, t - t_0) g(s, x(s)) ds. \tag{19}$$

Proof: We first get the homogeneous solution of the conformable differential equation defined by

$$T_\alpha x = Ax. \tag{20}$$

Applying integral operator described in Theorem 1, the homogeneous solution is given by

$$x_h(t) = \eta E_\alpha(A, t - t_0)$$

We now get the particular solution of the equation (18) obtained by the method described in Theorem 2, and is given by

$$x_p(t) = \int_{t_0}^{t} (t-s)^{\alpha-1} E_\alpha(-A, s - t_0) E_\alpha(A, t - t_0) g(s, x(s)) ds.$$

The general solution obtained with $x(t) = x_h(t) + x_p(t)$ is expressed as

$$x(t) = \eta E_\alpha(A, t - t_0) + \int_{t_0}^{t} (t-s)^{\alpha-1} E_\alpha(-A, s - t_0) E_\alpha(A, t - t_0) g(s, x(s)) ds.$$

The exponential function $E_\alpha(A, t - t_0)$ is calculated using the propositions described later. We distinguish two cases: when the matrix A is diagonalizable and when the matrix A is not diagonalizable.

Proposition 2 If the matrix A is diagonalizable the solution of equation (18) with $g(t, x) = 0$ is obtained doing the following classical procedure:

- Determine the matrix $P \in \mathbb{R}^{n\times n}$ and the matrix $D \in \mathbb{R}^{n\times n}$ such that $D = P^{-1}AP$, where P is a invertible matrix of eigenvectors and D is diagonal matrix.
- Resolve using integral operator the conformable differential equation defined by $T_\alpha(y) = Dy$, where $y \in \mathbb{R}^n$.
- The solution is obtained using the transformation $x = Py$.

Example: We give example to illustrate of the Proposition 2. Let the particular conformable differential equation of (18) defined by

$$\begin{aligned} T_\alpha x &= -x + y + z, \\ T_\alpha y &= \ \ x - y + z, \\ T_\alpha z &= \ \ x + y - z, \end{aligned}$$

where $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$. Calculating the eigenvalues of the matrix A we obtain $\lambda_1 = 1$ and $\lambda_2 = -2$. The eigenvectors related to the eigenvalues λ_1 and λ_2 are respectively given by

$$u = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad v = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \qquad w = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

Thus in the first step of the resolution, the matrix P and D are given by

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix} \qquad P = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

In the second step of the resolution, we resolve the conformable differential equation $T_\alpha(Y) = DY$, where $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$, using integral operator. We have the following equation

$$\begin{aligned} T_\alpha y_1 &= \ \ y_1 \\ T_\alpha y_2 &= -2y_2 \\ T_\alpha y_3 &= -2y_3 \end{aligned}$$

Applying Theorem 1, we obtain the following solutions

$$\begin{aligned} y_1 &= \ \ y_{10} \exp\left(\frac{t^\alpha}{\alpha}\right) = y_{10} E_\alpha(1, t) \\ y_2 &= y_{20} \exp\left(\frac{-2t^\alpha}{\alpha}\right) = y_{20} E_\alpha(-2, t) \\ y_3 &= y_{30} \exp\left(\frac{-2t^\alpha}{\alpha}\right) = y_{30} E_\alpha(-2, t) \end{aligned}$$

where $y_{10} \in \mathbb{R}$, $y_{20} \in \mathbb{R}$ and $y_{30} \in \mathbb{R}$ are constants. To obtain the solutions of the conformable fractional differential equation defined by (18), we use the transformation $X = PY$, where $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$. Finally, we obtain

$$\begin{array}{ll} x = & y_{10}E_\alpha(1,t) - y_{20}E_\alpha(-2,t) - y_{30}E_\alpha(-2,t) \\ y = & y_{10}E_\alpha(1,t) + y_{20}E_\alpha(-2,t) \\ z = & y_{10}E_\alpha(1,t) + y_{30}E_\alpha(-2,t) \end{array}$$

The second case is obtained when the matrix A is not diagonalizable, we use the Jordan matrix. The solution of the conformable differential equation (18) with $g(t,x) = 0$ is described in the following proposition.
Proposition 2 If the matrix A is not diagonalizable the solution of (18) with $g(t,x) = 0$ is obtained as follows:
• Determine the matrix $P \in \mathbb{R}^{n\times n}$ and the matrix $J \in \mathbb{R}^{n\times n}$ such that $J = P^{-1}AP$, where P is a invertible matrix and J is Jordan matrix.
• Resolve using integral operator the conformable differential equation defined by $T_\alpha(y) = Dy$, where $y \in \mathbb{R}^n$.
• The solution is obtained using the transformation $x = Py$.

5 Conclusions

It is not easy to find the form of the general solution of the conformable differential equations. We know that the conformable differential equation has many possible solutions. This paper contributes to give the explicit form of the first-candidate solution of the conformable differential equation. We have also discuss on Cauchy problem.

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