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Effective Estimation of Ratio and Product of Two Population Means in Presence of Random Non-Response in Successive Sampling

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Abstract: The present article intends to develop efficient estimation strategy to reduce the negative impact of random non - response at both occasions in two occasion successive sampling. Utilizing the information on an auxiliary variable effective imputation strategy was developed to cope with the non -response situation. Estimators for the current occasion are also derived as a particular case when there is non-response either on the first occasion or on the second occasion. To study the efficacy of the suggested imputation method, performances of the proposed estimators are performed in two different situations: with and without non-response. The pre-eminence of the suggested estimator has been established through empirical studies carried over some natural population dataset and artificially generated population dataset, which presents the soundness and usefulness of the suggested estimator in practice. Suitable recommendations to the survey statistician are also made.

Keywords: Product estimation, ratio estimation, successive sampling, study variable, auxiliary variable, bias, mean square error, percent relative loss

1 Introduction

A variety of practical problems can fall in the arena of applied and environmental sciences where various characters opt to change with respect to different parameters; such changes are inherent behavior of the nature. Some type of changes directly or indirectly affects the quality of living and surrounding of the human beings. This requires the continuous monitoring of the real life situation in hand. The theory and practice of surveying the same population at different points of time technically called repetitive sampling or sampling over successive occasions or rotation sampling and have been given considerable attention by the survey statisticians. Successive (rotation) sampling provides a strong tool for generating the reliable estimates at different occasions. For example, monthly data on the prices of goods are collected to determine the consumer price index, political opinion surveys are conducted at regular intervals to know the voters preference, etc. Theory of successive sampling appears to have started with the work of [6]. He pioneered using the entire information collected in the previous investigations (occasions). Further the theory of successive sampling was extended by [10,11,4, 3,2] and many others.
It is worth to be mentioned that most of the recent works of successive sampling are based on the problem of estimation of population mean. However, in many practical situations an estimate of the population ratio or product of two characters for the most recent occasion may be of considerable interest, such as, the ratio of corn acres to wheat acres, the ratio of expenditure on labour to total expenditure, the product of cultivated area and yield rate, product of mortality rate and area of a locality. For instance, if data on yield of corn and wheat from certain agricultural plot is available for previous few seasons then one may estimate their ratio for the current season and may decide accordingly to cultivate suitable crops for more income and if the data on income and expenditure are available for previous few financial years, then one may estimate their yearly ratio and plan for suitable investment for the current year. Similarly, if data on the product

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of concentration and volume of a liquid are available from previous few experiments, then one may easily study the characteristic and session wise variations of the liquid. The work of [14] and [15] are the few directions available in this context. However, it may be noted that the above estimation strategies are based on single phase sampling ignoring the changes of pattern over successive occasions as well as based on the assumptions that complete response available from the units selected for the sample and if non - response occurs in the sampled units then the observations are taken only from the responding units of corresponding sample ignoring completely the non-responding ones. Thus, no mechanisms are developed there to reduce the negative impact of non-response in such realistic situations. Non - response is an unavoidable phenomenon in nature and estimates based on only the responding units may lead to biased estimate. For example, the estimate based on the voters responding for exit polls may lead to biased estimates as non - responding ones may indicate their trend to some different political party about which they do not want to reveal in public for security reasons. It may be found that in some practical situations, data cannot always be collected from all the units selected in the sample. As many respondents do not reply, available sample returns is incomplete. The resulting incompleteness is called non-response. Repeated surveys are more prone to this problem than single occasion surveys. For examples, in case of milk yield surveys the animal may be sold or may die during the survey period; in agricultural surveys, the yield of some pickings may be damaged or lost or the enumerators may fail to record them. [12] advocated three concepts: missing at random (MAR), observed at random (OAR) and parameter distribution (PD). Rubin Defined: "The data are MAR if the probability of the observed missingness pattern, given the observed and unobserved data, does not depend on the value of the unobserved data". [5] have distinguished the meaning of missing at random (MAR) and missing completely at random (MCAR) in a very nice way. Statisticians have long known that failure to account for the stochastic nature of incompleteness can damage the actual conclusion. Therefore, we need to discover suitable mechanism to reduce the negative impact of non-response in sample survey. Many methods are used to deal with non-response in sample surveys. Imputation, the practice of "filling up" missing data with plausible values, is one of them. It is typically used when needed to substitute missing item value with certain fabricated values in the sample surveys. To deal with missing values effectively, [13] and [7] suggested imputation methods that make incomplete datasets structurally complete and its analysis simple. Imputation may also be carried out with the aid of an auxiliary variate, if such is available. For example, ([8,9]) used the information on an available auxiliary variate for imputation purpose. Later, [19,1,16] and [17] suggested several new imputation methods using auxiliary information.

Motivated by the arguments and discussions, the objective of our present work is to reduce the negative impact of non-response while estimating the ratio and product of two population character at current occasion in two-occasion successive sampling. Following the MAR response mechanism, one efficient imputation strategy has been suggested to cope with the problem of non-response situation. Estimators for the current occasion are derived when non-response occurs on both the occasion or on the either of the occasion. The performances of our proposed estimators have been demonstrated empirically and suitable recommendations are made.

2 Notations and Sample Structures on Two Occasions

Consider a finite population $U = (U_1, U_2, U_3, ..., U_N)$ of N units and y and x are the variables under study with population mean $\overline{Y}$ and $\overline{X}$ and z is the auxiliary variable which is stable over both the occasions with population mean $\overline{Z}$. Let y_k, x_k and z_k be the values of y, x and z respectively for the $k^{th}(k = 1, 2, ..., N)$ unit in the population. We wish to estimate the parameter $R_{(\alpha)} = \frac{\overline{Y}}{\overline{X}^{\alpha}}$ on the current (second) where, α is a scalar which takes values 1 and -1.

It is to be noted that:

i. For $\alpha = 1$, $R_{(\alpha)} \rightarrow R_{(1)} = \frac{\overline{Y}}{\overline{X}}$ (Ratio of two population means)

ii. For $\alpha = -1$, $R_{(\alpha)} \rightarrow R_{(-1)} = \overline{YX}$ (Product of two population means)

We assume that there is random non-response at both the occasions. A simple random sample (without replacement) S_n of n units is drawn on the first occasion. Let the number of non-responding units out of n units, which are drawn on the first occasion, be denoted by r_1, the set of non-responding units in S_n by Ar_1 and that of the responding units by Ar_1^c. A random sub-sample S_m of $m = n\lambda$ units is retained (matched) for its use on the current (second) occasion from the units which responded on the first occasion and it is assumed these matched units are completely responding at the current (second) occasion as well. A fresh simple random sample (without replacement) S_u of $u = (n-m) = n\mu$ units is drawn on the second occasion from the entire population so that the sample size on the second (current) occasion is also n. Let the number of non-responding units out of u units, which are drawn afresh on the current occasion, be denoted by r_2, the set of non-responding units in S_u by Ar_2 and that of responding units by Ar_2^c. Here λ and μ $(\lambda + \mu = 1)$ are the fractions of the matched and fresh sample, respectively, at the current occasion. For every unit belonging to the responding unit sets, the values on the study variables are observed, however, if they are belonging to the non-responding unit sets, the values on the study variables are assumed to be missing and therefore, the imputed values are derived for such units which are based on the responding units of the sample. We have considered that the occurrences of random non-response situation follow the discrete probability distribution as presented below.

2.1 Non-Response Probability Model

Suppose random non-response situations occur at the first occasion on the sample S_n of size n. Then $r_1[r_1 = 0, 1, ..., (n-2)]$ is the number of sampling units on which information could not be collected due to random non-response and the observations of the respective variables on which random non-response occur can be taken from the remaining $(n-r_1)$ units of S_n. Similarly $r_2[r_2 = 0, 1, ..., (u-2)]$ is the number of sampling units on which information could not be collected due to non-response on the sample S_u of u units, drawn afresh on the second (current) occasion. If p_1 and p_2 be the probabilities of a non-response on first and second (current) occasion respectively, then r_1 and r_2 has the following discrete distribution:

$$P(r_1) = \frac{n-r_1}{nq_1+2p_1}\binom{n-2}{r_1} p_1^{r_1} q_1^{n-2-r_1}; r_1 = 0, 1, ..., (n-2) \tag{1}$$

$$P(r_2) = \frac{u-r_2}{uq_2+2p_2}\binom{u-2}{r_2} p_2^{r_2} q_2^{u-2-r_2}; r_2 = 0, 1, ..., (u-2) \tag{2}$$

where $q_i = 1-p_i (i = 1, 2)$ and $\binom{n-2}{r_1}$, $\binom{u-2}{r_2}$ denote the total number of ways of obtaining r_1 and r_2 non-responses out of (n-2) and (u-2) total possible non-responses, respectively, for instance, see [20].

The following notations are used hereafter:
$\overline{Y}_h$, $\overline{X}_h$: Population means of the study variables y and x on the h^{th}(h = 1, 2) occasion respectively.
$\overline{Z}$: Population mean of the auxiliary variable z, stable over both the occasions.
$R_{h(\alpha)}$=$\frac{\overline{Y}_2}{\overline{X}_2{}^{\alpha}}$; $(\alpha = 1, -1)$:Population parameter under study on the h^{th} (h = 1, 2) occasion.
$\overline{y}_n$, $\overline{x}_n$: Sample means of the variables y and x based on the sample S_n of size n on 1^{st} occasion.
$\overline{y}_{(n-r_1)}$, $\overline{x}_{(n-r_1)}$: Sample means of the variables y and x based on the sample Ar_1^c of responding units from sample S_n on 1^{st} occasion.
$\overline{y}_u$, $\overline{x}_u$: Sample means of the variables y and x based on the sample S_u of size u drawn afresh on the 2^{nd} occasion.
$\overline{y}_{(u-r_2)}$, $\overline{x}_{(u-r_2)}$: Sample means of the variables y and x based on the sample Ar_2^c of responding units from sample S_u on 2^{nd} occasion.
$\overline{y}_{hm}$, $\overline{x}_{hm}$: Sample means of the variables y and x based on the sample S_m of size m on h^{th} (h = 1, 2) occasion.
$\overline{z}_n$, $\overline{z}_m$, $\overline{z}_u$: Sample mean of the auxiliary variable z based on sample sizes shown in suffices.
$\overline{z}_{r_1}$, $\overline{z}_{r_2}$: Sample mean of z based on the samples Ar_1 and Ar_2 respectively.
C_{y_1}, C_{y_2}, C_{x_1}, C_{x_2}, C_z: Coefficients of variation of the respective variables shown in the suffices.
$\rho_{y_1x_1}$, $\rho_{y_2x_2}$, ρ_{y_2z}, ρ_{x_2z}: Correlation coefficients between the variables shown in the subscripts.
$R_{n(\alpha)} = \frac{\overline{y}_n}{\overline{x}_n{}^{\alpha}}$; $R_{(n-r_1)(\alpha)} = \frac{\overline{y}_{(n-r_1)}}{\overline{x}_{(n-r_1)}{}^{\alpha}}$; $R_{u(\alpha)} = \frac{\overline{y}_u}{\overline{x}_u{}^{\alpha}}$; $R_{(u-r_2)(\alpha)} = \frac{\overline{y}_{(u-r_2)}}{\overline{x}_{(u-r_2)}{}^{\alpha}}$; $R_{hm(\alpha)} = \frac{\overline{y}_{hm}}{\overline{x}_{hm}{}^{\alpha}}$, (h = 1, 2)
$f_m = \frac{1}{m}$, $f_u = \frac{1}{u}$, $f_n = \frac{1}{n}$, $f^* = \frac{1}{uq_2+2p_2}$, $f^{**} = \frac{1}{nq_1+2p_1}$, $f_m - f_n = f_1$, $f^{**} - f_m = f_2$.

3 Formulation of Estimation Strategy

To estimate the population parameter $R_{2(\alpha)}$ on the second (current) occasion, two different estimators are considered. One estimator T_u is based on the sample S_u of size $u(= n\mu)$ drawn afresh on the second occasion and the second estimator T_m is based on the sample S_m of size $m(= n\lambda)$ common to both the occasions. Estimators T_u and T_m are structured to cope with the problem of non-response which occurs on both the occasions.
Since, the information on the auxiliary variable z is readily available for the sample S_u, therefore, we propose the following imputation method based on responding and non-responding units of the sample S_u to estimate the population parameter under study $R_{2(\alpha)}$ as:

$$R_{u(\alpha).i} = \begin{cases} R_{(u-r_2)(\alpha)} \exp\left(\frac{\overline{Z}-\overline{z}_u}{\overline{Z}+\overline{z}_u}\right) & if \;\; i \in Ar_2^c \\ R_{(u-r_2)(\alpha)} \frac{\overline{Z}-z_i}{\overline{Z}-\overline{z}_{r_2}} \exp\left(\frac{\overline{Z}-\overline{z}_u}{\overline{Z}+\overline{z}_u}\right) & if \;\; i \in Ar_2 \end{cases} \tag{3}$$

Under the above imputation method, the estimator T_u for estimating $R_{2(\alpha)}$ can be derived as
$T_u = \frac{1}{u}\sum_{i\in S_u} R_{u(\alpha).i}$
$=\frac{1}{u}[\sum_{i\in Ar_2} R_{u(\alpha).i} + \sum_{i\in Ar_2^c} R_{u(\alpha).i}]$

$$\therefore T_u = R_{(u-r_2)(\alpha)} \exp\left(\frac{\overline{Z}-\overline{z}_u}{\overline{Z}+\overline{z}_u}\right) \tag{4}$$

The estimator T_m is based on the sample S_m, which utilize the information on an auxiliary variable z as well as information on the first occasion. Since there is non-response on the first occasion as well, therefore, we replace the missing values on the first occasion by the derived imputed values and the estimator based on responding and non-responding units of the sample S_m to estimate $R_{1(\alpha)}$ is proposed as:

$$R_{n(\alpha).i} = \begin{cases} R_{(n-r_1)(\alpha)} \exp\left(\frac{\overline{Z}-\overline{z}_n}{\overline{Z}+\overline{z}_n}\right) & if \ \ i \in Ar_1^c \\ R_{(n-r_1)(\alpha)} \frac{\overline{Z}-z_i}{\overline{Z}-\overline{z}_{r_1}} \exp\left(\frac{\overline{Z}-\overline{z}_n}{\overline{Z}+\overline{z}_n}\right) & if \ \ i \in Ar_1 \end{cases} \tag{5}$$

Therefore, following the imputation method, the estimator of $R_{1(\alpha)}$ based on the sample of size n is derived as
$R^*_{n(\alpha)} = \frac{1}{n}\sum_{i\in S_n} R_{n(\alpha).i}$
$=\frac{1}{n}[\sum_{i\in Ar_1} R_{n(\alpha).i} + \sum_{i\in Ar_1^c} R_{n(\alpha).i}]$

$$\therefore R^*_{n(\alpha)} = R_{(n-r_1)(\alpha)} \exp\left(\frac{\overline{Z}-\overline{z}_n}{\overline{Z}+\overline{z}_n}\right) \tag{6}$$

In follow up the above discussion, we suggest the following ratio type estimator for the parameter under study $R_{2(\alpha)}$ on the second (current) occasion based on the sample S_m of size m as

$$T_m = R_{2m(\alpha)} \frac{R^*_{n(\alpha)}}{R^*_{1m(\alpha)}} \tag{7}$$

where, $R^*_{1m(\alpha)} = R_{1m(\alpha)} \exp\left(\frac{\overline{Z}-\overline{z}_m}{\overline{Z}+\overline{z}_m}\right)$

Considering the convex linear combination of the estimators T_u and T_m, we have a class of estimators of $R_{2(\alpha)}$ as

$$T = \phi T_u + (1-\phi)T_m \tag{8}$$

where ϕ is an unknown constant to be determined to achieve the minimum mean square error of the class of estimators T. Considering the occurrence of non-response either on the first occasion or on the second occasion, we have derived estimators of $R_{2(\alpha)}$ on the current (second) occasion as special cases, which are discussed below:

3.1 Special Cases

3.1.1 Case I: Non-response occurs only on the First Occasion

In this case non-response occurs only on the first occasion while we have complete response in the fresh sample of size u and as per our assumption; we also have complete response in the matched sample of size m retained from the sample of size n drawn on the first occasion. Therefore, we consider $r_2 = 0$ and suggest the estimator of $R_{2(\alpha)}$ based on the fresh sample of size u drawn on current occasion as

$$T_u^{'} = R_{u(\alpha)} \exp\left(\frac{\overline{Z}-\overline{z}_u}{\overline{Z}+\overline{z}_u}\right) \tag{9}$$

Consequently the final estimators of $R_{2(\alpha)}$ based on the matched portion of sample of size m and the fresh sample of size u on the second (current) occasion are constructed as

$$T^{'} = \phi^{'} T_u^{'} + (1-\phi^{'})T_m \tag{10}$$

where $\phi^{'}$ is real constant to be determined by the minimization of the mean square error of the estimator $T^{'}$.

3.1.2 Case II: Non-response occurs only on the Second (Current) Occasion

In this situation non-response is found only on the second (current) occasion while all the units on the first occasion will respond i.e., $r_1 = n$. Accordingly, we suggest the following estimator of $R_{2(\alpha)}$ based on the matched portion of the sample S_m retained from first occasion as

$$T_m^{''} = R_{2m(\alpha)} \frac{R_{n(\alpha)}^{''}}{R_{1m(\alpha)}^{*}} \tag{11}$$

where, $R_{n(\alpha)}^{''} = R_{n(\alpha)} \exp(\frac{\overline{Z}-\overline{z}_n}{\overline{Z}+\overline{z}_n})$ and $R_{1m(\alpha)}^{*} = R_{1m(\alpha)} \exp(\frac{\overline{Z}-\overline{z}_m}{\overline{Z}+\overline{z}_m})$.
Hence, in this case the final estimator of $R_{2(\alpha)}$ based on the samples S_u and S_m on the second (current) occasion is considered as

$$T^{''} = \phi^{''} T_u + (1-\phi^{''}) T_m^{''} \tag{12}$$

where, $\phi^{''}$ is real constant to be determined by the minimization of the mean square error of the estimator $T^{''}$.

4 Properties of the Proposed Estimators T, $T^{'}$ and $T^{''}$:

Since T_u, T_m, $T_u^{'}$ and $T_m^{''}$ all are exponential type estimators, they are biased for $R_{2(\alpha)}$, therefore, the resulting estimators T, $T^{'}$ and $T^{''}$ defined in Equations (8), (10) and (12) are also biased for $R_{2(\alpha)}$. The bias B(.) and mean square errors M(.) of the proposed estimators up to the first order of approximations are derived under large sample approximations (ignoring f.p.c.) using the following assumptions:

$$\begin{gathered}
\overline{y}_{u-r_2} = \overline{Y}_2(1+e_1), \overline{x}_{u-r_2} = \overline{X}_2(1+e_2), \overline{y}_u = \overline{Y}_2(1+e_1^{'}), \overline{x}_u = \overline{X}_2(1+e_2^{'}), \overline{z}_u = \overline{Z}(1+e_3), \\
\overline{y}_{n-r_1} = \overline{Y}_1(1+e_4), \overline{x}_{n-r_1} = \overline{X}_1(1+e_5), \overline{y}_n = \overline{Y}_1(1+e_4^{'}), \overline{x}_n = \overline{X}_1(1+e_5^{'}), \overline{z}_n = \overline{Z}(1+e_6), \\
\overline{y}_{1m} = \overline{Y}_1(1+e_7), \overline{x}_{1m} = \overline{X}_1(1+e_8), \overline{z}_m = \overline{Z}(1+e_9), \overline{y}_{2m} = \overline{Y}_2(1+e_{10}), \overline{x}_{2m} = \overline{X}_2(1+e_{11}) \\
\text{such that } E(e_i) = E(e_j^{'}) = 0 \text{ and } |e_i| < 1 \text{ and } \left|e_j^{'}\right| < 1 \;\forall\; i = 1,2,...,11 \text{ and } j = 1,2,4,5.
\end{gathered}$$

Under the above transformations, the estimators T_u, T_m, $T_u^{'}$ and $T_m^{''}$ take the following forms:

$$T_u = R_{2(\alpha)}(1+e_1)(1+e_2)^{(-\alpha)} \exp(\frac{-e_3}{2+e_3}) \tag{13}$$

$$T_m = \frac{R_{2(\alpha)}(1+e_{10})(1+e_{11})^{(-\alpha)}(1+e_4)(1+e_5)^{(-\alpha)} \exp(\frac{-e_6}{2+e_6})}{(1+e_7)(1+e_8)^{(-\alpha)} \exp(\frac{-e_9}{2+e_9})} \tag{14}$$

$$T_u^{'} = R_{2(\alpha)}(1+e_1^{'})(1+e_2^{'})^{(-\alpha)} \exp(\frac{-e_3}{2+e_3}) \tag{15}$$

$$T_m^{''} = \frac{R_{2(\alpha)}(1+e_{10})(1+e_{11})^{(-\alpha)}(1+e_4^{'})(1+e_5^{'})^{(-\alpha)} \exp(\frac{-e_6}{2+e_6})}{(1+e_7)(1+e_8)^{(-\alpha)} \exp(\frac{-e_9}{2+e_9})} \tag{16}$$

Therefore, we have the following theorems.

Theorem 4.1. Bias of the estimators T, $T^{'}$ and $T^{''}$ to the first order of approximations are obtained as

$$B(T) = \phi B(T_u) + (1-\phi) B(T_m) \tag{17}$$

$$B(T^{'}) = \phi^{'} B(T_u^{'}) + (1-\phi^{'}) B(T_m) \tag{18}$$

$$B(T^{''}) = \phi^{''} B(T_u) + (1-\phi^{''}) B(T_m^{''}) \tag{19}$$

where,

$$B(T_u) = R_{2(\alpha)}[-\alpha f^* \rho_{y_2x_2} C_{y_2} C_{x_2} - \frac{1}{2} f_u \rho_{y_2z} C_{y_2} C_z + \frac{\alpha}{2} f_u \rho_{x_2z} C_{x_2} C_z + \frac{1}{2}\alpha(\alpha-1) f^* C_{x_2}^2 + \frac{3}{8} f_u C_z^2] \tag{20}$$

$$B(T_m) = R_{2(\alpha)}[\frac{1}{2}\alpha(\alpha-1)(f_2 C_{x_1}^2 + f_m C_{x_2}^2) - \frac{1}{8} f_1 C_z^2 - \alpha(f_2 \rho_{y_1x_1} C_{y_1} C_{x_1} + f_m \rho_{y_2x_2} C_{y_2} C_{x_2}) + \frac{1}{2} f_1 (\rho_{y_2z} C_{y_2} C_z - \alpha \rho_{x_2z} C_{x_2} C_z)] \tag{21}$$

$$B(T_u^{'}) = R_{2(\alpha)} f_u[-\alpha \rho_{y_2x_2} C_{y_2} C_{x_2} - \frac{1}{2} \rho_{y_2z} C_{y_2} C_z + \frac{\alpha}{2} \rho_{x_2z} C_{x_2} C_z + \frac{1}{2}\alpha(\alpha-1) C_{x_2}^2 + \frac{3}{8} C_z^2] \tag{22}$$

$$B(T_m^{''}) = R_{2(\alpha)}[\frac{1}{2}\alpha(\alpha-1)(f_m C_{x_2}^2 - f_1 C_{x_1}^2) - \frac{1}{8} f_1 C_z^2 - \alpha(f_1 \rho_{y_1x_1} C_{y_1} C_{x_1} + f_m \rho_{y_2x_2} C_{y_2} C_{x_2}) + \frac{1}{2} f_1 (\rho_{y_2z} C_{y_2} C_z - \alpha \rho_{x_2z} C_{x_2} C_z)] \tag{23}$$

Proof: The bias of the estimators T, $T^{'}$ and $T^{''}$ are given by
$B(T) = E(T - R_{2(\alpha)})$
$= \phi E(T_u - R_{2(\alpha)}) + (1-\phi) E(T_m - R_{2(\alpha)})$

$$\therefore B(T) = \phi B(T_u) + (1-\phi) B(T_m) \tag{24}$$

$B(T^{'}) = E(T^{'} - R_{2(\alpha)})$
$= \phi^{'} E(T_u^{'} - R_{2(\alpha)}) + (1-\phi^{'}) E(T_m - R_{2(\alpha)})$
therefore

$$B(T^{'}) = \phi^{'} B(T_u^{'}) + (1-\phi^{'}) B(T_m) \tag{25}$$

$B(T^{''}) = E(T^{''} - R_{2(\alpha)})$
$= \phi^{''} E(T_u - R_{2(\alpha)}) + (1-\phi^{''}) E(T_m^{''} - R_{2(\alpha)})$
therefore

$$B(T^{''}) = \phi^{''} B(T_u) + (1-\phi^{''}) B(T_m^{''}) \tag{26}$$

Using the expansions of T_u, T_m, $T_u^{'}$ and $T_m^{''}$ from equations (13), (14), (15) and (16) in the equations (24), (25) and (26) and taking expectations up to first order of approximations, we have the expressions for the bias of the proposed estimators T, $T^{'}$ and $T^{''}$ as described in the equations (17), (18) and (19).

Theorem 4.2. Mean square errors of the estimators T, $T^{'}$ and $T^{''}$ to the first order of approximations are obtained as

$$M(T) = \phi^2 M(T_u) + (1-\phi)^2 M(T_m) \tag{27}$$

$$M(T^{'}) = \phi^{'2} M(T_u^{'}) + (1-\phi^{'})^2 M(T_m) \tag{28}$$

$$M(T^{''}) = \phi^{''2} M(T_u) + (1-\phi^{''})^2 M(T_m^{''}) \tag{29}$$

where,

$$M(T_u) = R_{2(\alpha)}^2[f^* C_{y_2}^2 + \alpha^2 f^* C_{x_2}^2 + \frac{1}{4} f_u C_z^2 - 2\alpha f^* \rho_{y_2x_2} C_{y_2} C_{x_2} - f_u \rho_{y_2z} C_{y_2} C_z + \alpha f_u \rho_{x_2z} C_{x_2} C_z] \tag{30}$$

$$M(T_m) = R_{2(\alpha)}^2[f_2(C_{y_1}^2 + \alpha^2 C_{x_1}^2 - 2\alpha \rho_{y_1x_1} C_{y_1} C_{x_1}) + f_1(\frac{1}{4} C_z^2 + \rho_{y_2z} C_{y_2} C_z - \alpha \rho_{x_2z} C_{x_2} C_z) + f_m(C_{y_2}^2 + \alpha^2 C_{x_2}^2 - 2\alpha \rho_{y_2x_2} C_{y_2} C_{x_2})] \tag{31}$$

$$M(T_u^{'}) = R_{2(\alpha)}^2 f_u[C_{y_2}^2 + \alpha^2 C_{x_2}^2 + \frac{1}{4} C_z^2 - 2\alpha \rho_{y_2x_2} C_{y_2} C_{x_2} - \rho_{y_2z} C_{y_2} C_z + \alpha \rho_{x_2z} C_{x_2} C_z] \tag{32}$$

$$M(T_m^{''}) = R_{2(\alpha)}^2[f_1(\frac{1}{4} C_z^2 - C_{y_1}^2 - \alpha^2 C_{x_1}^2 + 2\alpha \rho_{y_1x_1} C_{y_1} C_{x_1} + \rho_{y_2z} C_{y_2} C_z - \alpha \rho_{x_2z} C_{x_2} C_z) + f_m(C_{y_2}^2 + \alpha^2 C_{x_2}^2 - 2\alpha \rho_{y_2x_2} C_{y_2} C_{x_2})] \tag{33}$$

Proof: It is obvious that the mean square error of the proposed estimator T is given by
$M(T) = E(T - R_{2(\alpha)})^2$
$= E[\phi(T_u - R_{2(\alpha)}) + (1-\phi)(T_m - R_{2(\alpha)})]^2$

$$\therefore M(T) = \phi^2 M(T_u) + (1-\phi)^2 M(T_m) + 2\phi(1-\phi)E[(T_u - R_{2(\alpha)})(T_m - R_{2(\alpha)})] \tag{34}$$

Substituting the expressions of T_u and T_m from equations (13) and (14) in equation (34) and taking expectations up to first order of approximations, we obtain the expression for the mean square error of the estimator T as presented in equation (27).

The proofs of the mean square errors of the estimators $T^{'}$ and $T^{''}$ defined in equations (28) and (29) can be derived in similar ways.

It is to be noted that the estimators T_u and T_m are based on two non-overlapping samples of size u and m respectively. The covariance type terms (i.e., $E[(T_u - R_{2(\alpha)})(T_m - R_{2(\alpha)})]$) are of order N^{-1}, hence for large population, they are ignored. Similarly, the other covariance type terms are also neglected.

5 Minimum Mean Square Errors of the Proposed Estimators T, $T^{'}$ and $T^{''}$

From equations (27)-(29), It is cleared that the mean square errors of the estimators T, $T^{'}$ and $T^{''}$ are functions of ϕ, $\phi^{'}$ and $\phi^{''}$. Therefore, they are to be minimized with respect to ϕ, $\phi^{'}$ and $\phi^{''}$ respectively and subsequently the optimum values of ϕ, $\phi^{'}$ and $\phi^{''}$ are obtained as:

$$\phi_{opt} = \frac{M(T_m)}{M(T_u) + M(T_m)} \tag{35}$$

$$\phi^{'}_{opt} = \frac{M(T_m)}{M(T^{'}_u) + M(T_m)} \tag{36}$$

$$\phi^{''}_{opt} = \frac{M(T^{''}_m)}{M(T_u) + M(T^{''}_m)} \tag{37}$$

Putting these optimum values of ϕ, $\phi^{'}$ and $\phi^{''}$ in the equations (27) - (29), we get the minimum mean square errors of our suggested estimators as:

$$M(T)_{opt} = \frac{M(T_u) \times M(T_m)}{M(T_u) + M(T_m)} \tag{38}$$

$$M(T^{'})_{opt} = \frac{M(T^{'}_u) \times M(T_m)}{M(T^{'}_u) + M(T_m)} \tag{39}$$

$$M(T^{''})_{opt} = \frac{M(T_u) \times M(T^{''}_m)}{M(T_u) + M(T^{''}_m)} \tag{40}$$

6 Performances of the Proposed Estimators

In practice, non-response is one of the major problems encountered by survey statisticians because non-response situations may be misleading as the estimate based on them may be biased. Thus to examine the effect of non-response on the performances of our proposed methodology, the absolute percent relative biases and percent relative losses in efficiencies of T, $T^{'}$ and $T^{''}$ with respect to
i. The estimator τ defined under the similar circumstances as the estimators T, $T^{'}$ and $T^{''}$ but in the absence of non-response and which is presented as

$$\tau = \psi T^{'}_u + (1-\psi)T^{''}_m \tag{41}$$

ii. The sample estimator R of $R_{2(\alpha)}$ under complete response which is given as

$$R = \psi^{'} R_{u(\alpha)} + (1-\psi^{'}) R_{2m(\alpha)} \tag{42}$$

are obtained, where, ψ and $\psi^{'}$ are real constants to be determined by the minimization of the mean square error of respectively.

Proceeding as above the bias and mean square error of τ and R for large N (i.e., N $\rightarrow \infty$) are derived as

$$B(\tau)_{opt} = \psi_{opt} B(T_u^{'}) + (1-\psi_{opt}) B(T_m^{''}) \tag{43}$$

$$M(\tau)_{opt} = \frac{M(T_u^{'}) \times M(T_m^{''})}{M(T_u^{'}) + M(T_m^{''})} \tag{44}$$

where,

$$\psi_{opt} = \frac{M(T_m^{''})}{M(T_u^{'}) + M(T_m^{''})} \tag{45}$$

and $B(T_u^{'})$, $B(T_m^{''})$, $M(T_u^{'})$ and $M(T_m^{''})$ are given in the equations (22), (23), (32) and (33) respectively.

Again,

$$B(R)_{opt} = \psi_{opt}^{'} B(R_{u(\alpha)}) + (1-\psi_{opt}^{'}) B(R_{2m(\alpha)}) \tag{46}$$

$$M(R)_{opt} = \frac{M(R_{u(\alpha)}) \times M(R_{2m(\alpha)})}{M(R_{u(\alpha)}) + M(R_{2m(\alpha)})} \tag{47}$$

where,

$$\psi_{opt}^{'} = \frac{M(R_{2m(\alpha)})}{M(R_{u(\alpha)}) + M(R_{2m(\alpha)})} \tag{48}$$

$$B(R_{u(\alpha)}) = \alpha f_u C_{x_2} [\frac{1}{2}(\alpha-1) C_{x_2} - \rho_{y_2 x_2} C_{y_2}] R_{2(\alpha)} \tag{49}$$

$$B(R_{2m(\alpha)}) = \alpha f_m C_{x_2} [\frac{1}{2}(\alpha-1) C_{x_2} - \rho_{y_2 x_2} C_{y_2}] R_{2(\alpha)} \tag{50}$$

$$M(R_{u(\alpha)}) = f_u [C_{y_2}^2 + \alpha^2 C_{x_2}^2 - 2\alpha \rho_{y_2 x_2} C_{y_2} C_{x_2}] R_{2(\alpha)}^2 \tag{51}$$

$$M(R_{2m(\alpha)}) = f_m [C_{y_2}^2 + \alpha^2 C_{x_2}^2 - 2\alpha \rho_{y_2 x_2} C_{y_2} C_{x_2}] R_{2(\alpha)}^2 \tag{52}$$

We have designated the absolute percent relative biases B^*, $B^{*'}$ and $B^{*''}$ with respect to τ and B_R^*, $B_R^{*'}$ and $B_R^{*''}$ with respect to R and percent relative losses in precision L, $L^{'}$ and $L^{''}$ with respect to τ and L_R, $L_R^{'}$ and $L_R^{''}$ with respect to R as:

$$B^* = \left| \frac{B(\tau)_{opt}}{B(T)_{opt}} \right| \times 100 \tag{53}$$

$$B^{*'} = \left| \frac{B(\tau)_{opt}}{B(T^{'})_{opt}} \right| \times 100 \tag{54}$$

$$B^{*''} = \left| \frac{B(\tau)_{opt}}{B(T^{''})_{opt}} \right| \times 100 \tag{55}$$

$$L = \frac{M(T)_{opt} - M(\tau)_{opt}}{M(T)_{opt}} \times 100 \tag{56}$$

$$L^{'} = \frac{M(T^{'})_{opt} - M(\tau)_{opt}}{M(T^{'})_{opt}} \times 100 \tag{57}$$

$$L^{''} = \frac{M(T^{''})_{opt} - M(\tau)_{opt}}{M(T^{''})_{opt}} \times 100 \tag{58}$$

Also

$$B_R^* = \left| \frac{B(R)_{opt}}{B(T)_{opt}} \right| \times 100 \tag{59}$$

$$B_R^{*'} = \left| \frac{B(R)_{opt}}{B(T^{'})_{opt}} \right| \times 100 \tag{60}$$

$$B_R^{*''} = \left| \frac{B(R)_{opt}}{B(T^{''})_{opt}} \right| \times 100 \tag{61}$$

$$L_R = \frac{M(T)_{opt} - M(R)_{opt}}{M(T)_{opt}} \times 100 \tag{62}$$

$$L_R^{'} = \frac{M(T^{'})_{opt} - M(R)_{opt}}{M(T^{'})_{opt}} \times 100 \tag{63}$$

$$L_R^{''} = \frac{M(T^{''})_{opt} - M(R)_{opt}}{M(T^{''})_{opt}} \times 100 \tag{64}$$

Where

$$B(T)_{opt} = \phi_{opt} B(T_u) + (1 - \phi_{opt}) B(T_m) \tag{65}$$

$$B(T^{'})_{opt} = \phi^{'}_{opt} B(T^{'}_u) + (1 - \phi^{'}_{opt}) B(T_m) \tag{66}$$

$$B(T^{''})_{opt} = \phi^{''}_{opt} B(T_u) + (1 - \phi^{''}_{opt}) B(T^{''}_m) \tag{67}$$

6.1 Simulation Study Using Artificially Generated Population

An important aspect of simulation is that one builds a simulation model to replicate the actual system. Simulation allows comparison of analytical techniques and helps in concluding whether a newly developed technique is better than the existing ones. Motivated by [18] and [21] who have been adopted the artificial population generation techniques, we have generated five sets of independent random numbers of size N (N = 100) namely $x^{'}_{1_k}$, $y^{'}_{1_k}$, $x^{'}_{2_k}$, $y^{'}_{2_k}$ and $z^{'}_k (k = 1,2,...,N)$ from a standard normal distribution with the help of R-software. By varying the correlation coefficients ρ_{yx} and ρ_{xz}, we have generated the following transformed variables of the population U with the values of n=70, m=50, u=20, $\sigma_y^2 = 50$, $\mu_y = 10$, $\sigma_x^2 = 100$, $\mu_x = 50$, $\sigma_z^2 = 50$ and $\mu_z = 20$ as

$y_{1_k} = \mu_y + \sigma_y\ [\rho_{yx}\ x^{'}_{1_k} + \sqrt{1-(\rho_{yx})^2}\ y^{'}_{1_k}]$

$x_{1_k} = \mu_x + \sigma_x\ x^{'}_{1_k}$

$z_k = \mu_z + \sigma_z\ [\rho_{xz}\ x^{'}_{1_k} + \sqrt{1-(\rho_{xz})^2}\ z^{'}_k]$

$y_{2_k} = y_{1_k}$

and $x_{2_k} = x_{1_k}$.

Table 1: Absolute percent relative biases of T, $T^{'}$ and $T^{''}$ with respect to τ and R

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			$\rho_{yx}\rightarrow$	0.6						0.9					
	p_1	p_2	$\rho_{xz}\downarrow$	B^*	$B^{*'}$	$B^{*''}$	B_R^*	$B_R^{*'}$	$B_R^{*''}$	B^*	$B^{*'}$	$B^{*''}$	B_R^*	$B_R^{*'}$	$B_R^{*''}$
	0.05	0.05		141.1	144.3	98.2	151.2	154.7	105.2	141.7	144.9	98.3	151.1	154.5	104.7
		0.1		137.9	144.3	96.4	147.8	154.7	103.3	138.6	144.9	96.5	147.7	154.5	102.9
		0.15		134.7	144.3	94.6	144.4	154.7	101.4	135.4	144.9	94.8	144.4	154.5	101.1
	0.1	0.05		135.6	138.8	98.2	145.4	148.8	105.2	136.3	139.5	98.3	145.3	148.7	104.7
		0.1	0.2	132.5	138.8	96.4	142	148.8	103.3	133.2	139.5	96.5	142	148.7	102.9
		0.15		129.4	138.8	94.6	138.8	148.8	101.4	130.2	139.5	94.8	138.8	148.7	101.1
	0.15	0.05		130.2	133.3	98.2	139.6	142.9	105.2	130.9	133.9	98.3	139.5	142.8	104.7
$\alpha=1$		0.1		127.1	133.3	96.4	136.3	142.9	103.3	127.9	133.9	96.5	136.3	142.8	102.9
		0.15		124.2	133.3	94.6	133.1	142.9	101.4	124.9	133.9	94.8	133.1	142.8	101.1
	0.05	0.05		163.3	172.1	96.7	224.6	236.8	133.1	144.2	147.4	98.4	154.2	157.5	105.2
		0.1		155.1	172.1	93.6	213.3	236.8	128.8	141.1	147.4	96.7	150.8	157.5	103.4
		0.15		147.4	172.1	90.6	202.8	236.8	124.6	138	147.4	95.1	147.5	157.5	101.7
	0.1	0.05		157.8	166.5	96.7	217.1	229	133.1	138.7	141.8	98.4	148.2	151.6	105.2
		0.1	0.8	149.8	166.5	93.6	206.1	229	128.8	135.6	141.8	96.7	145	151.6	103.4
		0.15		142.3	166.5	90.6	195.8	229	124.6	132.6	141.8	95.1	141.7	151.6	101.7
	0.15	0.05		152.2	160.7	96.7	209.5	221.1	133.1	133.1	136.1	98.4	142.3	145.5	105.2
		0.1		144.4	160.7	93.6	198.7	221.1	128.8	130.1	136.1	96.7	139.1	145.5	103.4
		0.15		137.1	160.7	90.6	188.7	221.1	124.6	127.1	136.1	95.1	135.9	145.5	101.7
			$\rho_{yx}\rightarrow$	0.6						0.9					
	p_1	p_2	$\rho_{xz}\downarrow$	B^*	$B^{*'}$	$B^{*''}$	B_R^*	$B_R^{*'}$	$B_R^{*''}$	B^*	$B^{*'}$	$B^{*''}$	B_R^*	$B_R^{*'}$	$B_R^{*''}$
	0.05	0.05		98.8	101.5	97.5	118.6	121.8	117	102.4	104.6	98.1	117.8	120.3	112.9
		0.1		98	101.5	96.7	117.6	121.8	116	101.6	104.6	97.3	116.9	120.3	112
		0.15		97.1	101.5	95.9	116.6	121.8	115.1	100.7	104.6	96.5	115.9	120.3	111.1
	0.1	0.05		94.8	97.4	97.5	113.8	116.9	117	98.3	100.4	98.1	113.1	115.5	112.9
		0.1	0.2	94	97.4	96.7	112.8	116.9	116	97.4	100.4	97.3	112.1	115.5	112
		0.15		93.1	97.4	95.9	111.8	116.9	115.1	96.6	100.4	96.5	111.1	115.5	111.1
	0.15	0.05		90.8	93.3	97.5	109	112	117	94.1	96.1	98.1	108.3	110.6	112.9
$\alpha=-1$		0.1		90	93.3	96.7	108	112	116	93.3	96.1	97.3	107.3	110.6	112
		0.15		89.1	93.3	95.9	107	112	115.1	92.4	96.1	96.5	106.4	110.6	111.1
	0.05	0.05		105.1	103.3	101.5	115.4	113.4	111.5	105.5	104.2	101	120.6	119.1	115.6
		0.1		104.2	103.3	100.7	114.4	113.4	110.6	104.4	104.2	100.1	119.4	119.1	114.4
		0.15		103.3	103.3	99.9	113.5	113.4	109.7	103.3	104.2	99.1	118.2	119.1	113.3
	0.1	0.05		101	99.3	101.5	110.9	109.1	111.5	1011.3	100.2	101	115.8	114.5	115.6
		0.1	0.8	100.1	99.3	100.7	109.9	109.1	110.6	100.2	100.2	100.1	114.6	114.5	114.4
		0.15		99.2	99.3	99.9	108.9	109.1	109.7	99.1	100.2	99.1	113.4	114.5	113.3
	0.15	0.05		96.8	95.4	101.5	106.4	104.8	111.5	97.1	96.2	101	111.1	110	115.6
		0.1		95.9	95.4	100.7	105.4	104.8	110.6	96.1	96.2	100.1	109.9	110	114.4
		0.15		95	95.4	99.9	104.4	104.8	109.7	95	96.2	99.1	108.7	110	113.3

Table 2: Percent relative losses in precision of T, $T^{'}$ and $T^{''}$ with respect to τ and RPREs of different estimators

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			$\rho_{yx} \rightarrow$				0.6						0.9		
	p_1	p_2	$\rho_{xz} \downarrow$	L	$L^{'}$	$L^{''}$	L_R	$L_R^{'}$	$L_R^{''}$	L	$L^{'}$	$L^{''}$	L_R	$L_R^{'}$	$L_R^{''}$
	0.05	0.05		4.6	1	1	-17.3	-18.5	-21.8	4.4	0.9	0.9	-12.4	-13.4	-16.6
		0.1		5.6	2	2	-16.1	-18.5	-20.6	5.3	1.7	1.7	-11.4	-13.4	-15.6
		0.15		6.5	3	3	-14.9	-18.5	-19.3	6.1	2.6	2.6	-10.4	-13.4	-14.6
	0.1	0.05		8.2	1	1	-12.9	-14.1	-21.8	8	0.9	0.9	-8.2	-9.2	-16.6
		0.1	0.2	9.2	2	2	-11.7	-14.1	-20.6	8.8	1.7	1.7	-7.2	-9.2	-15.6
		0.15		10.2	3	3	-10.5	-14.1	-19.3	9.7	2.6	2.6	-6.2	-9.2	-14.6
	0.15	0.05		11.8	1	1	-8.4	-9.6	-21.8	11.6	0.9	0.9	-4	-5	-16.6
$\alpha = 1$		0.1		12.8	2	2	-7.2	-9.6	-20.6	12.4	1.7	1.7	-3	-5	-15.6
		0.15		13.8	3	3	-6	-9.6	-19.3	13.3	2.6	2.6	-2	-5	-14.6
	0.05	0.05		4.5	0.8	0.8	-15.2	-16.2	-19.7	4.5	0.7	0.7	-16.4	-17.3	-21.2
		0.1		5.3	1.6	1.6	-14.2	-16.2	-18.7	5.3	1.4	1.4	-15.6	-17.3	-20.3
		0.15		6.2	2.4	2.4	-13.2	-16.2	-17.7	6	2.1	2.1	-14.7	-17.3	-19.4
	0.1	0.05		8.2	0.8	0.8	-10.7	-11.7	-19.7	8.5	0.7	0.7	-11.7	-12.5	-21.2
		0.1	0.8	9.1	1.6	1.6	-9.7	-11.7	-18.7	9.2	1.4	1.4	-10.8	-12.5	-20.3
		0.15		9.9	2.4	2.4	-8.7	-11.7	-17.7	9.9	2.1	2.1	-9.9	-12.5	-19.4
	0.15	0.05		12	0.8	0.8	-6.2	-7.2	-19.7	12.4	0.7	0.7	-6.9	-7.8	-21.2
		0.1		12.8	1.6	1.6	-5.2	-7.2	-18.7	13.1	1.4	1.4	-6.1	-7.8	-20.3
		0.15		13.6	2.4	2.4	-4.2	-7.2	-17.7	13.8	2.1	2.1	-5.1	-7.8	-19.4
			$\rho_{yx} \rightarrow$				0.6						0.9		
	p_1	p_2	$\rho_{xz} \downarrow$	L	$L^{'}$	$L^{''}$	L_R	$L_R^{'}$	$L_R^{''}$	L	$L^{'}$	$L^{''}$	L_R	$L_R^{'}$	$L_R^{''}$
	0.05	0.05		3.5	-0.4	-0.4	-22	-21.5	-26.9	3.8	0	0	-20.7	-20.7	-25.5
		0.1		4.4	0.6	0.6	-20.8	-21.5	-25.6	4.8	1	1	-19.5	-20.7	-24.3
		0.15		5.4	1.6	1.6	-19.5	-21.5	-24.4	5.7	1.9	1.9	-18.3	-20.7	-23.1
	0.1	0.05		7.3	-0.4	-0.4	-17.2	-16.7	-26.9	7.7	0	0	-15.9	-15.9	-25.5
		0.1	0.2	8.3	0.6	0.6	-15.9	-16.7	-25.6	8.6	1	1	-14.7	-15.9	-24.3
		0.15		9.3	1.6	1.6	-14.7	-16.7	-24.4	9.6	1.9	1.9	-13.5	-15.9	-23.1
	0.15	0.05		11.2	-0.4	-0.4	-12.3	-11.8	-26.9	11.5	0	0	-11.1	-11.1	-25.5
$\alpha = -1$		0.1		12.1	0.6	0.6	-11.1	-11.8	-25.6	12.4	1	1	-9.9	-11.1	-24.3
		0.15		13.1	1.6	1.6	-9.8	-11.8	-24.4	13.4	1.9	1.9	-8.7	-11.1	-23.1
	0.05	0.05		6.3	2.8	2.8	-15.1	-18.5	-19.4	6.3	2.6	2.6	-17.3	-20.6	-21.9
		0.1		7.2	3.7	3.7	-14	-18.5	-18.4	7.2	3.5	3.5	-16.2	-20.6	-20.7
		0.15		8.1	4.6	4.6	-12.9	-18.5	-17.3	8.1	4.4	4.4	-15.1	-20.6	-19.6
	0.1	0.05		9.8	2.8	2.8	-10.8	-14.2	-19.4	9.9	2.6	2.6	-12.8	-16.1	-21.9
		0.1	0.8	10.7	3.7	3.7	-9.7	-14.2	-18.4	10.8	3.5	3.5	-11.7	-16.1	-20.7
		0.15		11.6	4.6	4.6	-8.6	-14.2	-17.3	11.7	4.4	4.4	-10.5	-16.1	-19.6
	0.15	0.05		13.4	2.8	2.8	-6.4	-9.8	-19.4	13.5	2.6	2.6	-8.2	-11.5	-21.9
		0.1		14.3	3.7	3.7	-5.3	-9.8	-18.4	14.4	3.5	3.5	-7.1	-11.5	-20.7
		0.15		15.2	4.6	4.6	-4.2	-9.8	-17.3	15.3	4.4	4.4	-6	-11.5	-19.6

6.2 Numerical Illustration Using Real Population

We have considered the following real populations to demonstrate the efficacy of the proposed estimation strategies in Tables 3 - 4.

i). The Educational Attainment by the United States (Table No- 233)

Y_1: Percent of persons 25 years and over who have completed a Bachelor's Degree in 2008 in a state in United States.

X_1: Percent of persons 25 years and over who have completed High School in 2008 in a state in United States.
Y_2: Percent of persons 25 years and over who have completed a Bachelor's Degree in 2009 in a state in United States.
X_2: Percent of persons 25 years and over who have completed High School in 2009 in a state in United States.
Z: Percent of persons 25 years and over who have completed an Advanced Degree in the year 2006 in a state in United States.
N = 50, $\overline{Y}_1$= 27.36, $\overline{X}_1$= 86.58, $\overline{Y}_2$= 27.59, $\overline{X}_2$= 86.88, $\overline{Z}$= 9.76, C_{y_1}= 2.08, C_{x_1}= 0.95, C_{y_2}= 2.10, C_{x_2}= 0.98, C_z= 3.33, $\rho_{y_1x_1}$= -0.09, $\rho_{y_2x_2}$= -0.11, ρ_{y_2z}= 0.95, ρ_{x_2z}= -0.16.

ii). Community Hospitals of the United States (Table No- 174)
Y_1: Number of hospitals of a state in United States in 2008.
X_1: Total number of patients admitted in all hospitals of a state in United States in 2008.
Y_2: Number of hospitals of a state in United States in 2009.
X_2: Total number of patients admitted in all hospitals of a state in United States in 2009.
Z: Total number of beds in all hospitals of a state in United States in 2008.
N= 50, $\overline{Y}_1$= 98.24, $\overline{X}_1$= 701.16, $\overline{Y}_2$= 98.20, $\overline{X}_2$= 696.61, $\overline{Z}$= 15.80, C_{y_1}= 2.65, C_{x_1}= 2.32, C_{y_2}= 2.68, C_{x_2}= 2.32, C_z= 2.22, $\rho_{y_1x_1}$= 0.76, $\rho_{y_2x_2}$= 0.74, ρ_{y_2z}= 0.79, ρ_{x_2z}= 0.98.
Survey data are collected from the Statistical Abstract of the United States 2012 published by the United States Census Bureau.

Table 3: Percent relative losses in precision from Education Survey (Table No-233)

α	p_1	p_2	L	L'	L''	L_R	L'_R	L''_R
1	0.05	0.05	12.0	11.0	11.0	-56.9	-76.6	-58.6
		0.1	20.5	19.6	19.6	-41.6	-76.6	-43.3
		0.15	27.4	26.5	26.5	-29.4	-76.6	-31.0
	0.1	0.05	12.9	11.0	11.0	-55.2	-74.9	-58.6
		0.1	21.5	19.6	19.6	-39.9	-74.9	-43.3
		0.15	28.4	26.5	26.5	-27.7	-74.9	-31.0
	0.15	0.05	13.9	11.0	11.0	-53.4	-73.1	-58.6
		0.1	22.5	19.6	19.6	-38.1	-73.1	-43.3
		0.15	29.4	26.5	26.5	-25.9	-73.1	-31.0
-1	0.05	0.05	-65.9	-67.0	-67.0	-147.4	-47.5	-149.0
		0.1	-32.9	-34.0	-34.0	-98.2	-47.5	-99.8
		0.15	-11.8	-12.9	-12.9	-66.8	-47.5	-68.4
	0.1	0.05	-64.7	-67.0	-67.0	-145.7	-45.8	-149.0
		0.1	-31.7	-34.0	-34.0	-96.5	-45.8	-99.8
		0.15	-10.7	-12.9	-12.9	-65.1	-45.8	-68.4
	0.15	0.05	-63.5	-67.0	-67.0	-143.9	-44.0	-149.0
		0.1	-30.5	-34.0	-34.0	-94.7	-44.0	-99.8
		0.15	-9.5	-12.9	-12.9	-63.3	-44.0	-68.4

Table 4: Percent relative losses in precision from Hospital Survey (Table No-174)

α	p_1	p_2	L	$L^{'}$	$L^{''}$	L_R	$L_R^{'}$	$L_R^{''}$
1	0.05	0.05	1.5	0.6	0.6	-2.3	-2.9	-3.3
		0.1	2.0	1.1	1.1	-1.7	-2.9	-2.7
		0.15	2.6	1.7	1.7	-1.1	-2.9	-2.0
	0.1	0.05	2.4	0.6	0.6	-1.3	-1.9	-3.3
		0.1	3.0	1.1	1.1	-0.7	-1.9	-2.7
		0.15	3.6	1.7	1.7	-0.1	-1.9	-2.0
	0.15	0.05	3.5	0.6	0.6	-0.2	-0.8	-3.3
		0.1	4.0	1.1	1.1	0.4	-0.8	-2.7
		0.15	4.6	1.7	1.7	1.0	-0.8	-2.0
-1	0.05	0.05	21.1	15.1	15.1	-17.4	-39.8	-26.4
		0.1	21.6	15.5	15.5	-16.7	-39.8	-25.8
		0.15	22.0	15.9	15.9	-16.1	-39.8	-25.1
	0.1	0.05	26.8	15.1	15.1	-9.0	-31.4	-26.4
		0.1	27.2	15.5	15.5	-8.4	-31.4	-25.8
		0.15	27.6	15.9	15.9	-7.7	-31.4	-25.1
	0.15	0.05	32.0	15.1	15.1	-1.2	-23.6	-26.4
		0.1	32.4	15.5	15.5	-0.5	-23.6	-25.8
		0.15	32.9	15.9	15.9	-0.1	-23.6	-25.1

6.3 Efficiency Comparison

To elucidate the efficacy of our proposed estimators, we have compared our estimators with the sample estimators of $R_{2(\alpha)}$ under the similar circumstances as the estimators T, $T^{'}$ and $T^{''}$ using the real populations discussed in section 6.2. We have made comparison of
i). The estimators T and $T^{'}$ with R^{*} which is the sample estimator of $R_{2(\alpha)}$ when non-response occurs on both the first occasion as well as on the second (current) occasion and is defined as

$$R^{*} = \lambda R_{(u-r_2)(\alpha)} + (1-\lambda)R_{2m(\alpha)} \tag{68}$$

where, λ is real constant to be determined by the minimization of the mean square error of R^{*}.
ii). The estimator $T^{'}$ with R which is the sample estimator of $R_{2(\alpha)}$ when non-response occurs only on the first occasion which is defined in the equation (42).
The bias and mean square error of R^{*} for large N (i.e., N $\rightarrow \infty$) are derived as

$$B(R^{*})_{opt} = \lambda_{opt} B(R_{(u-r_2)(\alpha)}) + (1-\lambda_{opt})B(R_{2m(\alpha)}) \tag{69}$$

$$M(R^{*})_{opt} = \frac{M(R_{(u-r_2)(\alpha)}) \times M(R_{2m(\alpha)})}{M(R_{(u-r_2)(\alpha)}) + M(R_{2m(\alpha)})} \tag{70}$$

and

$$\lambda_{opt} = \frac{M(R_{2m(\alpha)})}{M(R_{(u-r_2)(\alpha)}) + M(R_{2m(\alpha)})} \tag{71}$$

where,

$$B(R_{(u-r_2)(\alpha)}) = \alpha f^{*} C_{x_2}[\frac{1}{2}(\alpha-1)C_{x_2} - \rho_{y_2x_2}C_{y_2}]R_{2(\alpha)} \tag{72}$$

$$M(R_{(u-r_2)(\alpha)}) = R^2_{2(\alpha)} f^{*}[C^2_{y_2} + \alpha^2 C^2_{x_2} - 2\alpha\rho_{y_2x_2}C_{y_2}C_{x_2}] \tag{73}$$

and $B(R_{2m(\alpha)})$ and $M(R_{2m(\alpha)})$ are given in the equations (50) and (52).
Therefore, $PRE[T(orT^{''})]=\frac{M(R^{*})_{opt}}{M[T(orT^{''})]_{opt}} \times 100$
$PRE[T^{'}]=\frac{M(R)_{opt}}{M[T^{'}]_{opt}} \times 100$

Table 5: PREs of different estimators

	Educational attainment by United States (Table No - 233)					Community hospitals of United States (Table No - 174)				
	p_1	p_2	$PRE(T)$	$PRE(T^{'})$	$PRE(T^{''})$	p_1	p_2	$PRE(T)$	$PRE(T^{'})$	$PRE(T^{''})$
$\alpha=1$	0.05	0.05	158.95	176.57	160.62	0.05	0.05	103.67	102.92	104.61
		0.1	145.36	176.57	147.05		0.1	104.42	102.92	105.37
		0.15	134.59	176.57	136.3		0.15	105.17	102.92	106.14
	0.1	0.05	157.21	174.85	160.62	0.1	0.05	102.65	101.91	104.61
		0.1	143.6	174.85	147.05		0.1	103.39	101.91	105.37
		0.15	132.8	174.85	136.3		0.15	104.13	101.91	106.14
	0.15	0.05	155.39	173.05	160.62	0.15	0.05	101.55	100.82	104.61
		0.1	141.75	173.05	147.05		0.1	102.27	100.82	105.37
		0.15	130.93	173.05	136.3		0.15	103	100.82	106.14
	p_1	p_2	$PRE(T)$	$PRE(T^{'})$	$PRE(T^{''})$	p_1	p_2	$PRE(T)$	$PRE(T^{'})$	$PRE(T^{''})$
$\alpha=-1$	0.05	0.05	250.62	147.52	252.29	0.05	0.05	118.9	139.8	128.06
		0.1	203.43	147.52	205.12		0.1	119.83	139.8	129.11
		0.15	173.47	147.52	175.19		0.15	120.76	139.8	130.16
	0.1	0.05	248.88	145.8	252.29	0.1	0.05	110.4	131.4	128.06
		0.1	201.67	145.8	205.12		0.1	111.21	131.4	129.11
		0.15	171.68	145.8	175.19		0.15	112.02	131.4	130.16
	0.15	0.05	247.07	144.01	252.29	0.15	0.05	102.49	123.6	128.06
		0.1	199.82	144.01	205.12		0.1	103.2	123.6	129.11
		0.15	169.82	144.01	175.19		0.15	103.91	123.6	130.16

7 Perspective

The following conclusions can be drawn from the above study:
I. From simulation study:
a) Table-1 interprets that for fixed values of ρ_{yx} and ρ_{xz}, the absolute percent relative biases of T, $T^{'}$ and $T^{''}$ (i.e., B^{*}, $B^{*'}$ and $B^{*''}$ with respect to τ and B_R^{*}, $B_R^{*'}$ and $B_R^{*''}$ with respect to R) are getting reduced with different choices of the non-response rate p_1 and p_2 on the first and second (current) occasion respectively. This behavior is highly desirable as it conclude that for different choices of non-response rate on the first occasion or on the second occasion or on both the occasions, our proposed methodology provides estimates along with reduced bias. This phenomenon establishes that the suggested methodology is competent enough in reducing the negative effect of non-response to great extent.
b) Table-1 also exhibits that for fixed non-response rates p_1 and p_2 and for fixed value of ρ_{xz}, our estimators T, $T^{'}$ and $T^{''}$ provide estimates with less bias for increasing values of ρ_{yx}. The same behavior is noticed if we increase ρ_{xz} with the fixed value of ρ_{yx} kipping p_1 and p_2 as fixed. These behaviors states that the estimators T, $T^{'}$ and $T^{''}$ will give estimates with reduced absolute bias if the study variables y and x are highly correlated with the auxiliary character z.
c) Table-2 explains that if we fix the correlation coefficients between x and z and y and z, we observe that the loss in the precision of our proposed estimators are not too high, even the percent relative losses of T, $T^{'}$ and $T^{''}$ with respect to the sample estimator R, are becoming negative (i.e. gains), for different values of non-response rates p_1 and p_2. The same behavior is noticed if we fix the non-response rates p_1 and p_2 while the values of ρ_{yx} and ρ_{xz} are varying.
II. Using the real populations:
a) From Table-3 and Table-4 we can observe the same behaviors in the percent relative losses of T, $T^{'}$ and $T^{''}$ as in Table-2. This behavior is highly desirable, as it pays in terms of enhance precision of estimates as well as reduces the cost of survey.
b) In Table-5, it can be observed that for various choices of p_1 and p_2 (non-response rates on first and second occasions) our proposed estimators T, $T^{'}$ and $T^{''}$ are more efficient than the sample estimators of $R_{2(\alpha)}$ under the similar circumstances.

Thus, the above analysis indicates the effectiveness of our proposed methods of imputation and vindicates the competent of suggested estimators in reducing the negative effect of non-response situations more precisely. It is to be noted that the absolute percent relative biases are less and losses in precision are not appreciable and even in many cases negative losses (gains) are noticeable which establishes the practicability of our proposed estimation procedures. So, they may be recommended to the survey statisticians and practitioners for their use in real life problems.

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References

[1] H. Spencer, First Principles, MacMillan, London, 1862.
[2] E. Domar, Econometrica **14**, 137-147 (1946).
[3] R.M Solow, Quarterly Journal of Economics **70**, 65-94 (1956).
[4] C.J Dalgaard and H. Strulik, Rediscovering the Solow model: An energy network approach, Discussion Papers, Department of Economics, University of Copenhagen.
[5] J.R Banavar, A. Maritan, A. Rinaldo, Nature **399**, 130-132 (1999).
[6] C.L. Dalgaard and H. Strulik, Resource and Energy Economics **33**, 782-797 (2011).
[7] J.R Banavar, J. Damuth, A. Maritan, A. Rinaldo, Proceedings of the National Academy of Sciences of the United States of America **99**, 10506-10509 (2002).
[8] G.B West, J.H Brown, B.J Enquist, Science **276**, 122-126 (1997).
[9] G.B West, J.H Brown, B.J Enquist, Science **284**, 1677-1679 (1999).
[10] G.B West, J.H Brown, and B.J Enquist, Nature **413**, 628-631 (2001).
[11] M. Kleiber, Hilgardia **6**, 315-332 (1932).
[12] L.M.A Bettencourt, J. Lobo, D. Helbing, C. Kühnert, G.B West, Proceedings of the National Academy of Sciences of the United States of America **104**, 7301-7306 (2007).
[13] T.W. Swan, Economic Record **32**, 334-361 (1956).
[14] C.T.H Baker, G.A Bocharov, C.A.H Paul, Journal of Theoretical Medicine **2**, 117-128 (1997).
[15] C. Bianca, Math. Models Methods Appl. Sci. **21**, 389-419 (2011).
[16] C. Bianca, M. Pennisi, Nonlinear Analysis: Real World Applications **13**, 1913-1940 (2012).
[17] C. Bianca and L. Fermo, Computers & Mathematics with Applications **61**, 277-288 (2011).
[18] H. Smith, An introduction to delay differential equations with applications to the life sciences, Texts in Applied Mathematics, Volume 57, Springer, Berlin, 2011.
[19] A. Casal and M. Freedman, IEEE Transactions on Automatic Control **25**, 967-973 (1980).
[20] S. Ruan and J. Wei, Dynamics of Continuous, Discrete & Impulsive Systems. Series A. Mathematical Analysis **10**, 863-874 (2003).
[21] R. Rand and A. Verdugo, Communications in Nonlinear Science and Numerical Simulation **12**, 859-864 (2007).
[22] O. Galor, From stagnation to growth: unified growth theory, In: Aghion, P., Durlauf, S. (Eds.), Handbook of Economic Growth, vol. 1A. Amsterdam, North-Holland, 2005.
[23] C. Bianca, Nonlinear Analysis: Real World Applications **13**, 2593-2608 (2012).
[24] C. Bianca, Appl. Math. Inf. Sci. **6**, 495-499 (2012).
[25] C. Bianca, Physics of Life Reviews (2012), http://dx.doi.org/10.1016/j.plrev.2012.09.002.

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