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Is It Possible to Construct a Fractional Derivative Such That the Index Law Holds?

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Abstract: The aim of this note is to make a brief consideration about the Index Law in fractional differentiation. We are not interested in any particular definition of fractional derivative, and that is why we will not introduce any. We make an exception in the section of examples, but in any case the full document can be understood without it. We show, roughly speaking, that it does not exist any linear operator which is an n-th root of the usual derivative in a very general framework.

Keywords: Fractional calculus, impossibility result, index law.

1 Introduction

From now on, we will denote by D the operator that consists in differentiating a function one time.

Probably the first reference to fractional calculus, at least in an informal way, is the famous dialogue between L'Hôpital and Leibniz. The most natural answer that anybody would have expected to the question *"What happens if the order of differentiation is $n = \frac{1}{2}$?"* would be something like *"It has to be something that, when it is applied twice, it gives the usual derivative"*.

The previous answer invites us to make the following discussion. Let's think about derivatives as operators that, when applied to a function, they give another function. We would expect to have the property that, after composing two times the $\frac{1}{2}$ derivative $D^{\frac{1}{2}}$, we get $D^{\frac{1}{2}} \circ D^{\frac{1}{2}} = D$. In general, when we compose n times the derivative of order $\frac{1}{n}$, which we will call $D^{\frac{1}{n}}$, we would like to get the usual derivative D. An interesting discussion about what should be called "fractional derivative" is performed in [1].

We know that the usual definitions for fractional derivatives (Caputo, Riemann-Liouville,...), which are presented in the classical monographs [2], [3], [4], [5] do not satisfy, in general, that property. So, in general, the following identity does not hold

$$D^{\frac{1}{n}} \circ \cdots \circ D^{\frac{1}{n}} = D,$$

where the left composition has n factors.

2 Some Examples

We will compute explicitly, for some particular definitions of fractional derivatives, the kernel of the operator consisting in applying n times the corresponding fractional derivative of order $\frac{1}{n}$. When we work with fractional derivatives which are non-local we will develop all our computations using 0 as the base point. Furthermore, we will assume that the orders of differentiation are positive real numbers, $\alpha \in \mathbb{R}^+$. As we said before, these basic definitions of fractional integrals and derivatives can be found in the monographs [2], [3], [4], [5].

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2.1 Riemann-Liouville Fractional Derivative

We know that the Riemann-Liouville fractional integral is given by, provided that it exists,

$$I^{\alpha}_{0^+} f(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds.$$

It is widely known that $I^{\alpha}_{0^+} \circ I^{\beta}_{0^+} = I^{\alpha+\beta}_{0^+}$.

Furthermore, we define the Riemann-Liouville fractional derivative as

$$D^{\alpha,RL}_{0^+} = D^{\lceil\alpha\rceil} I^{\lceil\alpha\rceil-\alpha}_{0^+}, \tag{1}$$

where $\lceil\alpha\rceil$ is the smallest integer which is greater or equal than α. It is also widely known that $D^{\alpha,RL}_{0^+} \circ I^{\alpha,RL}_{0^+} = \mathrm{Id}$, which implies that $I^{\alpha,RL}_{0^+}$ is injective and $D^{\alpha,RL}_{0^+}$ is surjective.

From (1) we know that $\dim(\ker D^{\alpha,RL}_{0^+}) \leq \lceil\alpha\rceil$. In fact, it is easy to check that the equality holds, because $f_k(t) = t^{\alpha-\lceil\alpha\rceil+k}$, with $k \in \{0,\dots,\lceil\alpha\rceil-1\}$, conform a family of $\lceil\alpha\rceil$ linearly independent elements of the kernel.

In fact, it is not very difficult to check inductively that for

$$D^{\alpha_n,RL}_{0^+} \circ \cdots \circ D^{\alpha_1,RL}_{0^+}$$

the kernel is conformed by any linear combination of the functions $f(t) = t^{\gamma}$ where

$$\gamma \in \bigcup_{j=1}^{n} \left\{ \sum_{i=1}^{j} \alpha_i - \lceil\alpha_j\rceil + k_j : k_j \in \{0,1,\dots,\lceil\alpha_j\rceil-1\} \right\}.$$

The dimension of the kernel is exactly $\lceil\alpha_1\rceil + \cdots + \lceil\alpha_n\rceil$. In particular, if all the $\alpha_i = \frac{1}{n}$, we get that $\gamma \in \{0, -\frac{1}{n}, -\frac{2}{n}, \dots, -\frac{n-1}{n}\}$.

2.2 Caputo Fractional Derivative

We define the Caputo fractional derivative as

$$D^{\alpha,C}_{0^+} = I^{\lceil\alpha\rceil-\alpha}_{0^+} D^{\lceil\alpha\rceil}. \tag{2}$$

From (2) we know that $\dim(\ker D^{\alpha,C}_{0^+}) = \lceil\alpha\rceil$. Indeed, we can find explicitly the kernel because $f_k(t) = t^k$, with $k \in \{0,\dots,\lceil\alpha\rceil-1\}$, conform a family of $\lceil\alpha\rceil$ linearly independent elements in $\ker D^{\alpha,C}_{0^+}$.

In fact, it is not very difficult to check that for

$$D^{\alpha_n,RL}_{0^+} \circ \cdots \circ D^{\alpha_1,RL}_{0^+}$$

the kernel is conformed by any linear combination of the functions $f(t) = t^{\gamma}$ where

$$\gamma \in \bigcup_{j=1}^{n} \left\{ \sum_{i=1}^{j-1} \alpha_i + k_j : k_j \in \{0,1,\dots,\lceil\alpha_j\rceil-1\} \right\}.$$

The dimension of the kernel is exactly $\lceil\alpha_1\rceil + \cdots + \lceil\alpha_n\rceil$. In particular, if all the $\alpha_i = \frac{1}{n}$, we get that $\gamma \in \{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}\}$.

3 Main Result

We have checked in the previous section the well-known fact that the identity $D^{\frac{1}{n}} \circ \cdots \circ D^{\frac{1}{n}} = D$, where the left composition has n factors, does not hold for two of the most relevant fractional derivatives. In fact, in both cases, the left side has a kernel of dimension n which strictly contains the kernel of D, with dimension 1.

It is a very reasonable question if the previous identity can not hold for any fractional derivative. We will show that the identity is impossible over any functional space S that contains the constant functions, where the existence of primitives is ensured and where the operators involved are well-defined. For instance, one can think that S is some $C^k(\mathbb{R})$, with $k \in \mathbb{Z}^+ \cup \{\infty\} \cup \{\omega\}$, where $\mathscr{C}^{\omega}(\mathbb{R})$ denotes the analytical functions over $\mathbb{R}$. The only assumption is that the fractional derivative is linear. Now, we state and prove the main result.

Theorem 1. *Given $n \in \mathbb{Z}^+$, there is no linear map $F : S \longrightarrow S$ such that $F^n = D$ holds, where F^n denotes the composition of F with itself n times and where D represents the classical derivative.*

Proof. It is well-known that the functions that lie in the kernel of D are the constant functions. So, we are saying that $\ker D = \{f \in S : f \text{ is constant}\}$.

We will assume that there is some F, which is a linear map over S, such that $F^n = D$ holds and we will achieve a contradiction.

Note that $\ker F \neq 0$, because if $\ker F = 0$ then D would be injective because of $F^n = D$ and we would get $\ker D = 0$, which is not true. Furthermore, it is evident that $\ker F \subset \ker D$, but $\ker D$ has dimension 1 so necessarily $\ker F = \ker D$.

On the other hand, F has to be surjective because of the combined fact that D is surjective (any function in S has as preimage any of its primitives) together with $F^n = D$. Consequently, the operator F^{n-1}, which consists in $n-1$ compositions of F, is surjective.

This means that, if 1 indicates the constant function with value 1, there is an $f \in S$ with $F^{n-1}(f) = 1$ and, evidently, $f \notin \ker F = \ker D$. However, remembering that $1 \in \ker F = \ker D$ we have that

$$0 \neq D(f) = F(F^{n-1}(f)) = F(1) = 0$$

which is clearly a contradiction.

This means that such F can not exist, i.e., $D^{\frac{1}{n}}$ can not exist with the mentioned property.

4 Conclusion

The main conclusion of this note is that the perfect fractional derivative does not exist. The Index Law can not hold for fractional derivatives, although it might be true if we restrict our space to eliminate the functions that cause troubles. This restriction is usually imposed in Fractional Calculus requiring that the function considered and some of its derivatives vanish at some point.

In short, it is not a whim that in the common definitions of fractional derivatives (Riemann-Liouville, Caputo, Grünwald-Letnikov, Caputo-Fabrizio [6], [7], Losada-Nieto [8]...) there is no additivity on the orders. It is compulsory if we just demand fractional derivatives to be linear!

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