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Stage Structure and Refuge Effects in the Dynamical Analysis of a Fractional Order Rosenzweig-MacArthur Prey-Predator Model

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Abstract: In this paper, a fractional order prey-predator model with stage structure incorporating a prey refuge is established and analyzed. The predation is modelled using a Hollings type II functional response. The existence, uniqueness, non-negativity and boundedness of the solutions of the model is established. In addition to investigating the stability of the equilibrium points, conditions for the stability and Hopf bifurcation are obtained. The impact of fractional order, prey refuge and conversion coefficient on the stability of the fractional-order system are theoretically and numerically investigated.

Keywords: fractional-order system, stability, Hopf bifurcation prey-predator model, prey refuge, numerical simulation.

1 Introduction

The incorporation of stage structure in a prey-predator model is a way to introduce life history into the model and this inclusion can provide a richer dynamics to facilitate better understanding of the interactions in the ecological system. It can take into account significant biological parameters such as different death rates for mature and immature predators. Some studies of the dynamical behaviour of prey-predator models with stage structure include [1,2,3,4,5,6,7,8]. The interaction between predator and their prey was investigated by incorporating stage-structure for the predators.

Some prey-predator models assume that a predator consumes its captured food at a constant rate immediately after killing the prey [9,10]. The Rosenzweig-MacArthur (R-M) model [9,10] assumes that the consumption process is not constant. There have been many studies on the R-M model and this includes [11,12,13]. Further, the R-M model normally assumes the attack rate of predator increases at a decreasing rate with prey density until it becomes constant due to satiation (i.e. Holling type II functional response) [14].

Rosenzweig-MacArthur model was extended to a prey-predator model with stage structure for a predator with the assumption that predators can be divided into two stages: immature and mature [5,15]. Only mature predators are assumed to attack preys and have reproductive ability. Immature predators, on the other hand, are assumed not to have the ability to attack the prey and also have no reproductive ability. They are also assumed to obtain their living resources from their parents. This type of biological scenario is commonly observed in mammals and birds as follows [5,15]:

$$\begin{aligned} \frac{dx}{dt} &= rx\left(1-\frac{x}{k}\right) - \frac{bxz}{1+ax}, \\ \frac{dy}{dt} &= \frac{cbxz}{1+ax} - (D+d_1)y, \\ \frac{dz}{dt} &= Dy - d_2 z. \end{aligned} \tag{1}$$

All the parameters are non-negative constants for all time $t \geq 0$. The state variables and parameters for system (1) are described in Table 1.

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Table 1: Parameters table for system (1).

Parameter	Description
x	Density of the prey at time t.
y	Density of the immature predator at time t.
z	Density of the mature predator at time t.
r	Intrinsic growth rate of the prey.
k	Carrying capacity of the prey.
c	Coefficient in converting prey into a new immature predator.
d_1	The death rate of the immature predator.
d_2	The death rate of the mature predator.
D	Rate of immature predator becoming mature predator.
$\frac{bx}{1+ax}$	Holling type II functional response of the mature predator.

Including a refuge (a safe area for preys) can be more realistic as it takes into account the reduced mortality. Investigations of the dynamical behaviour of prey-predator models incorporating refuge have been reported in a number of papers which include [16,7,13,17,18,19,20,21].

Fractional order differential equations have the ability to provide a reasonably accurate description of certain phenomena [6,22,23,24,25,26]. This is because systems with memory have a connection with fractional differential equations [18]. In [5,15] a global stability of a prey-predator system with stage structure for the predator was proposed. However, fractional order case and a prey refuge were not dealt with. In this paper, we study a fractional order prey-predator model with stage structure of the predator incorporating a prey refuge by extending the integer-order model (1) as follows:

$$\begin{aligned} {}^cD^{\alpha}x(t) &= rx\left(1-\frac{x}{k}\right) - \frac{b(1-\delta)xz}{1+a(1-\delta)x}, \\ {}^cD^{\alpha}y(t) &= \frac{cb(1-\delta)xz}{1+a(1-\delta)x} - (D+d_1)y, \\ {}^cD^{\alpha}z(t) &= Dy - d_2 z, \end{aligned} \tag{2}$$

with initial conditions

$$x(0) = x_0 \geq 0,\ y(0) = y_0 \geq 0,\ z(0) = z_0 \geq 0,$$

$\delta \in [0,1)$ and δx is the population density of the prey at time t which is protected due to the refuge. Here ${}^cD^{\alpha}$ is the standard Caputo differentiation and $\alpha \in (0,1]$. The parameters of fractional-order system (2) are all non-negative. The Caputo fractional derivative of order α is defined as [22,27]:

$${}^cD^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)\, ds, \quad n-1 < \alpha < n,\ n \in \mathbb{N}.$$

As far as we are aware, the dynamical analysis of a fractional order Rosenzweig-MacArthur model with stage structure and a prey refuge has not been previously investigated. Thus, in this paper, we propose and analyse a fractional order Rosenzweig-MacArthur model with stage structure which includes a prey refuge.

2 Analysis

2.1 Existence and uniqueness

The sufficient condition for existence and uniqueness of the solution of fractional-order system (2) are investigated as follows.

Theorem 1. *The sufficient condition for existence and uniqueness of the solutions of the fractional-order system (2) in the region $\Theta \times (0,T]$ with initial conditions $X(0) = X_0$ and $t \in (0,T]$ is*

$$H = \frac{T^{\alpha}}{\Gamma(\alpha+1)} \max\left\{ r\left(1+\frac{2\eta}{k}\right) + b(1-\delta)\eta(1+c);\ 2D+d_1;\ b(1-\delta)\eta(1+c)+d_2 \right\} < 1.$$

*Proof.*The existence and uniqueness are based on contraction mapping principle and this principle has been used by [25]. The fractional-order system (2) can be written as follows

$$D^{\alpha}X(t) = F(X(t)), \quad t \in (0,T], \quad X(0) = X_0,$$

where

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad X_0 = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}, \quad F(x) = \begin{bmatrix} rx\left(1-\frac{x}{k}\right) - \frac{b(1-\delta)xz}{1+a(1-\delta)x} \\ \frac{cb(1-\delta)xz}{1+a(1-\delta)x} - (D+d_1)y \\ Dy - d_2 z \end{bmatrix}.$$

Define the maximum norm as follows

$$\|N\| = \max_{t\in(0,T]} |N(t)|.$$

The norm of the matrix $M = [m_{ij}[t]]$ is defined by

$$\|M\| = \max_{j} \sum_{i=1} |m_{ij}|.$$

The existence and uniqueness of the solution are studied in the region $\Theta \times (0,T]$ where

$$\Theta = \left\{(x,y,z) \in \mathbb{R}^3_+ : \max(|x|,|y|,|z|) \leq \eta\right\}.$$

Thus, the solution of fractional-order system (2) is obtained as

$$X = X_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}F(X(s))ds = G(x).$$

So

$$G(X_1) - G(X_2) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(F(X_1(s)) - F(X_2(s)))ds.$$

Thus, one gets the following inequality

$$|G(X_1) - G(X_2)| \leq \frac{1}{\Gamma(\alpha)}\int_0^t \left|(t-s)^{\alpha-1}(F(X_1(s)) - F(X_2(s)))\right| ds,$$

that yields

$$\begin{aligned}\|G(X_1) - G(X_2)\| \leq & \frac{T^{\alpha}}{\Gamma(\alpha+1)} \max\left\{r\left(1+\frac{2\eta}{k}\right) + b(1-\delta)\eta[1+c];\right. \\ & \left. 2D+d_1;\; b(1-\delta)\eta[1+c]+d_2\right\}\|X_1 - X_2\| \\ \leq & H\|X_1 - X_2\|,\end{aligned}$$

where

$$H = \frac{T^{\alpha}}{\Gamma(\alpha+1)} \max\left\{r\left(1+\frac{2\eta}{k}\right) + b(1-\delta)\eta(1+c);\; 2D+d_1;\; b(1-\delta)\eta(1+c)+d_2\right\}.$$

The Lipschitz condition is thus satisfied by $G(X)$. If $H < 1$, then the mapping $X = G(X)$ is a contraction mapping. Consequently, the existence and uniqueness of fractional-order system (2) follows.

2.2 Non-negativity and boundedness

The solutions of the system (2) are the densities of the interacting populations and so must be non-negative and bounded. This is investigated in this section.

Theorem 2.*The solutions of fractional-order system (2) starting in $\mathbb{R}^3_+$ are uniformly bounded and non-negative.*

Proof. The approach used by [18] is utilized. Define the function $W(t) = x(t) + \frac{1}{c}(y(t) + z(t))$, then

$$\begin{aligned} {}^cD^\alpha W(t) &= {}^cD^\alpha x(t) + \frac{1}{c}{}^cD^\alpha y(t) + \frac{1}{c}{}^cD^\alpha z(t) \\ &= rx\left(1 - \frac{x}{k}\right) - \frac{d_1}{c}y - \frac{d_2}{c}z. \end{aligned}$$

For each $\gamma > 0$, one has

$$\begin{aligned} {}^cD^\alpha W(t) + \gamma W(t) &= rx - r\frac{x^2}{k} - \frac{d_1}{c}y - \frac{d_2}{c}z + \gamma x + \frac{\gamma}{c}y + \frac{\gamma}{c}z \\ &= -r\frac{x^2}{k} + rx + \gamma x + \frac{1}{c}(\gamma - d_1)y + \frac{1}{c}(\gamma - d_2)z. \end{aligned}$$

Let us choose, $\gamma < \min\{d_1, d_2\}$. Thus

$$\begin{aligned} {}^cD^\alpha W(t) + \gamma W(t) &\le -\frac{r}{k}\left(x - \frac{k(r+\gamma)}{2r}\right)^2 + \frac{k(r+\gamma)^2}{4r} \\ &\le \frac{k(r+\gamma)^2}{4r}. \end{aligned}$$

Now applying the standard comparison theorem for fractional order [28], one gets

$$0 \le W(t) \le W(0)E_\alpha(-\gamma(t)^\alpha) + \frac{k(r+\gamma)^2}{4r}(t)^\alpha E_{\alpha,\alpha+1}(-\gamma(t)^\alpha),$$

E_α is the Mittag-Leffler function. In accordance with Lemma 5 and Corollary 6 in [28], by taking $t \to \infty$, this gives

$$0 \le W(t) \le \frac{k(r+\gamma)^2}{4\gamma r}.$$

Hence, the solutions of fractional-order system (2) starting in $\mathbb{R}^3_+$ are uniformly bounded within the region W_1, where

$$W_1 = \left\{(x,y,z) \in R^3_+ : \ W(t) \le \frac{k(r+\gamma)^2}{4\gamma r} + \varepsilon, \ \varepsilon > 0\right\}. \tag{3}$$

We now seek to show that the solutions of the fractional-order system (2) are non-negative.
From Eq. 1 of system (2), one gets

$${}^cD^\alpha x(t) = rx\left(1 - \frac{x}{k}\right) - \frac{b(1-\delta)xz}{1 + a(1-\delta)x}. \tag{4}$$

From (3), it can be observed that

$$x + \tfrac{1}{c}(y+z) \le \frac{k(r+\gamma)^2}{4\gamma r} = \theta_1. \tag{5}$$

Based on (4) and (5), one has

$$\begin{aligned} {}^cD^\alpha x(t) &\ge rx\left(1 - \frac{\theta_1}{k}\right) - cb(1-\delta)\theta_1 x \\ &\ge \left(r - \frac{r\theta_1}{k} - cb(1-\delta)\theta_1\right)x \\ &\ge \gamma_1 x, \ \text{where} \ \ \gamma_1 = r - \frac{r\theta_1}{k} - cb(1-\delta)\theta_1. \end{aligned}$$

From the standard comparison theorem for fractional order [28], and the positivity of Mittag-Leffler function $E_{\alpha,1}(t) > 0$, for any $\alpha \in (0,1]$ [29], one gets

$$x \ge x_0 E_{\alpha,1}(\gamma_1 t^\alpha).$$

Then, we have

$$x \geq 0.$$

From Eq. 2 of system (2), one has

$$\begin{aligned} {}^cD^\alpha y(t) &= \frac{cb(1-\delta)xz}{1+a(1-\delta)x} - (D+d_1)y \\ &\geq -(D+d_1)y \\ &\geq -\gamma_2 y, \text{ where } \gamma_2 = D+d_1. \end{aligned}$$

Therefore,

$$y \geq y_0 E_{\alpha,1}(-\gamma_2 t^\alpha).$$

Then, we have

$$y \geq 0.$$

From Eq. 3 of system (2), one has

$$\begin{aligned} {}^cD^\alpha z(t) &= Dy - d_2 z \\ &\geq -d_2 z. \end{aligned}$$

Therefore,

$$z \geq z_0 E_{\alpha,1}(-d_2 t^\alpha).$$

Then, we have

$$z \geq 0.$$

Hence, the solutions of the fractional-order system (2) are non-negative.

2.3 Equilibrium Points and Stability

The basic reproduction number R_0 is then used in this section to investigate equilibrium points and stability. For the fractional-order system (2), we have

Theorem 3. *For the fractional-order system (2), R_0 is given by*

$$R_0 = \frac{cbD(1-\delta)k}{(1+a(1-\delta)k)(D+d_1)d_2}.$$

Proof. To obtain R_0 for the fractional-order system (2), we utilize the next generation method [30]. The fractional-order system (2) can be rewritten

$$\begin{aligned} {}^cD^\alpha y(t) &= \frac{cb(1-\delta)xz}{1+a(1-\delta)x} - (D+d_1)y, \\ {}^cD^\alpha z(t) &= Dy - d_2 z, \\ {}^cD^\alpha x(t) &= rx\left(1-\frac{x}{k}\right) - \frac{b(1-\delta)xz}{1+a(1-\delta)x}. \end{aligned} \tag{6}$$

The system (6), in turn, can be written:

$$D^\alpha X(t) = f(X) - v(X),$$

where

$$f(X) = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} \frac{cb(1-\delta)xz}{1+a(1-\delta)x} \\ 0 \\ 0 \end{bmatrix}, \quad v(X) = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} (D+d_1)y \\ -(Dy-d_2z) \\ \frac{b(1-\delta)xz}{1+a(1-\delta)x} - rx\left(1-\frac{x}{k}\right) \end{bmatrix}.$$

The matrices $F(X)$ and $V(X)$ are defined as follows:

$$F(X) = \begin{bmatrix} \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} & \frac{\partial f_1}{\partial x} \\ \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} & \frac{\partial f_2}{\partial x} \\ \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} & \frac{\partial f_3}{\partial x} \end{bmatrix}, \quad V(X) = \begin{bmatrix} \frac{\partial v_1}{\partial y} & \frac{\partial v_1}{\partial z} & \frac{\partial v_1}{\partial x} \\ \frac{\partial v_2}{\partial y} & \frac{\partial v_2}{\partial z} & \frac{\partial v_2}{\partial x} \\ \frac{\partial v_3}{\partial y} & \frac{\partial v_3}{\partial z} & \frac{\partial v_3}{\partial x} \end{bmatrix}.$$

Thus, one has

$$F(X)=\begin{bmatrix}0 & \dfrac{cb(1-\delta)x}{1+a(1-\delta)x} & \dfrac{cb(1-\delta)z}{(1+a(1-\delta)x)^2}\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},$$

$$V(X)=\begin{bmatrix}D+d_1 & 0 & 0\\ -D & d_2 & 0\\ 0 & \dfrac{b(1-\delta)x}{1+a(1-\delta)x} & r\left(\dfrac{2x}{k}-1\right)-\dfrac{b(1-\delta)z}{(1+a(1-\delta)x)^2}\end{bmatrix}.$$

To obtain the eigenvalues of $F\cdot V^{-1}$, at the predator-extinction equilibrium point $E_1(k,0,0)$, the equation

$$\left|F\cdot V^{-1}-\mu I\right|=0,$$

has to be solved. μ are the eigenvalues and I is the identity matrix, $F\cdot V^{-1}$ is the next generation matrix for model (2). μ_1, μ_2 and μ_3 can be computed as $\mu_1=0$, $\mu_2=0$ and $\mu_3=\frac{cbD(1-\delta)k}{(1+a(1-\delta)k)(D+d_1)d_2}$. The spectral radius of matrix $F\cdot V^{-1}$ is $\rho\left(F\cdot V^{-1}\right)=\max(\mu_i),\ i=1,2,3.$
In accordance with Theorem 2 in [30], the basic reproduction number of the fractional-order model (2) is

$$R_0=\frac{cbD(1-\delta)k}{(1+a(1-\delta)k)(D+d_1)d_2}.$$

The basic reproduction number has a clear biological interpretation. It is the mean number of offspring by every predator.

In order to obtain the equilibrium points of the fractional-order system (2), we set

$${}^cD^{\alpha}x(t)=0,\ {}^cD^{\alpha}y(t)=0 \text{ and } {}^cD^{\alpha}z(t)=0.$$

Then, the fractional-order system (2) has three equilibrium points as follows:

1. The trivial equilibrium point $E_0(0,0,0)$ always exists.
2. The predator-extinction equilibrium point $E_1(k,0,0)$ always exists.
3. The coexistence equilibrium point $E_2(x^*,y^*,z^*)$ where

$$x^*=\frac{d_2(D+d_1)}{(1-\delta)d_3},\ \text{where}\ \ d_3=cbD-a(D+d_1)d_2.$$

$$y^*=\frac{cbr(1-\delta)d_2^2(1+a(1-\delta)k)(D+d_1)}{b(1-\delta)^2kd_3^2}(R_0-1).$$

$$z^*=\frac{D}{d_2}y^*.$$

If $R_0>1$, the coexistence equilibrium point $E_2(x^*,y^*,z^*)$ exists.

The local stability analysis for the fractional-order system (2) around equilibrium points is obtained by calculating the Jacobian matrix corresponding to equilibrium points. The Jacobian matrix of the fractional-order system (2) at any point (x,y,z) is as follows

$$J(x,y,z)=\begin{pmatrix} r-\dfrac{2rx}{k}-\dfrac{b(1-\delta)z}{(1+a(1-\delta)x)^2} & 0 & -\dfrac{b(1-\delta)x}{1+a(1-\delta)x}\\ \dfrac{cb(1-\delta)z}{(1+a(1-\delta)x)^2} & -(D+d_1) & \dfrac{cb(1-\delta)x}{1+a(1-\delta)x}\\ 0 & D & -d_2\end{pmatrix}.$$

The stability of trivial equilibrium point $E_0(0,0,0)$, predator-extinction equilibrium point $E_1(k,0,0)$ and coexistence equilibrium point $E_2(x^*,y^*,z^*)$ can be stated in the following theorems.

Theorem 4. *The fractional-order system (2) around the trivial equilibrium point $E_0(0,0,0)$ is unstable saddle point.*

*Proof.*By virtue of Matignon's condition [31,32], the trivial equilibrium point E_0 of the fractional-order system (2) is an unstable saddle point if one of the eigenvalues $\mu_i,\ i=1,2,3$, of the Jacobian $J(E_0)$ satisfy $|\arg(\mu_i)|<\frac{\alpha\pi}{2}$. The Jacobian matrix of system (2) around the trivial equilibrium point E_0 is as follows

$$J(E_0)=\begin{pmatrix} r & 0 & 0\\ 0 & -(D+d_1) & 0\\ 0 & D & -d_2\end{pmatrix}.$$

The eigenvalues of the characteristic equation of $J(E_0)$ are $\mu_1=r$, $\mu_2=-(D+d_1)$ and $\mu_3=-d_2$. It can be observed that $|arg(\mu_1)|=0<\frac{\alpha\pi}{2}$ for all $0<\alpha\le 1$.

Theorem 5.*The fractional-order system (2) around the predator-extinction equilibrium point $E_1(k,0,0)$ is locally asymptotically stable if $R_0<1$ and unstable if $R_0>1$.*

*Proof.*Matignon's condition [31,32] states that the predator-extinction equilibrium point E_1 of the fractional-order system (2) is locally asymptotically stable if all the eigenvalues $\mu_i,\ i=1,2,3$, of the Jacobian $J(E_1)$ satisfy $|\arg(\mu_i)|>\frac{\alpha\pi}{2}$. The Jacobian matrix of system (2) around the predator-extinction equilibrium point E_1 is given by

$$J(E_1)=\begin{pmatrix} -r & 0 & -\dfrac{b(1-\delta)k}{1+a(1-\delta)k}\\ 0 & -(D+d_1) & \dfrac{cb(1-\delta)k}{1+a(1-\delta)k}\\ 0 & D & -d_2\end{pmatrix}.$$

The eigenvalues of $J(E_1)$ are $\mu_1=-r$ and the other two eigenvalues $\mu_{2,3}$ are the roots of the following equation:

$$\mu^2+u\mu+v=0,$$

where

$$u=D+d_1+d_2,\quad v=d_2(D+d_1)(1-R_0).$$

The predator-extinction equilibrium point E_1 is locally asymptotically stable if $\mu_{2,3}<0$, that is, $d_2(D+d_1)(1-R_0)>0$ which gives $R_0<1$.

The stability of coexistence equilibrium point $E_2(x^*,y^*,z^*)$ is now investigated. The Jacobian matrix of system (2) around the coexistence equilibrium point E_2 is

$$J(E_2)=\begin{pmatrix} r-\dfrac{2rx^*}{k}-\dfrac{b(1-\delta)z^*}{(1+a(1-\delta)x^*)^2} & 0 & -\dfrac{b(1-\delta)x^*}{1+a(1-\delta)x^*}\\ \dfrac{cb(1-\delta)z^*}{(1+a(1-\delta)x^*)^2} & -(D+d_1) & \dfrac{cb(1-\delta)x^*}{1+a(1-\delta)x^*}\\ 0 & D & -d_2\end{pmatrix}.$$

The eigenvalues of $J(E_2)$ are the roots of the following cubic equation:

$$F(\mu)=\mu^3+B_1\mu^2+B_2\mu+B_3=0, \tag{7}$$

where

$$B_1=D+d_1+d_2-r+\frac{2rx^*}{k}+\frac{bD(1-\delta)y^*}{(1+a(1-\delta)x^*)^2d_2},$$
$$B_2=\frac{rd_2(D+d_1)(D+d_1+d_2)}{bckD(1-\delta)d_3}[bcD(1-a(1-\delta)k)+a(1+a(1-\delta)k)(D+d_1)d_2],$$
$$B_3=rd_2(D+d_1)\left(1-\frac{1}{R_0}\right).$$

Also the coefficients of equation (7) can be written in the following form,

$$B_1=D_2+d_2+z^*g'(x^*)-f'(x^*),$$
$$B_2=(D_2+d_2)(z^*g'(x^*)-f'(x^*)),$$
$$B_3=D_2d_2z^*g'(x^*).$$

Where

$$f(x)=rx\left(1-\frac{x}{k}\right),\ g(x)=\frac{b(1-\delta)x}{1+a(1-\delta)x} \text{ and } D_2=D+d_1.$$

It is clear that $B_3>0$. B_1 and B_2 are positive when, $x^*>\dfrac{k}{2}$.

The discriminant $D(F)$ of the polynomial $F(\mu)$ is

$$D(F)=18B_1B_2B_3+(B_1B_2)^2-4B_3B_1^3-4B_2^3-27B_3^2.$$

According to [33,34], one obtains the following proposition.

Proposition 1. *It is assumed that the coexistence equilibrium point E_2 exists in $\mathbb{R}^3_+$.*

1. *If $D(F)>0$, $B_1>0$, $B_3>0$ and $B_1B_2>B_3$, then the coexistence equilibrium point E_2 is locally asymptotically stable for $0<\alpha\leq 1$.*
2. *If $D(F)<0,\ B_1>0,\ B_2>0,\ B_1B_2=B_3$ and $\alpha\in(0,1)$, then the coexistence equilibrium point E_2 is locally asymptotically stable.*
3. *If $D(F)<0,\ B_1<0,\ B_2<0$ and $\alpha>\frac{2}{3}$, then the coexistence equilibrium point E_2 is unstable.*
4. *The necessary condition for the coexistence equilibrium point E_2, to be locally asymptotically stable, is $B_3>0$.*

2.4 Global stability

We now study the sufficient conditions for the global asymptotic stability of the predator-extinction equilibrium point E_1 and coexistence equilibrium point E_2 of the fractional-order system (2).

Theorem 6. *The predator-extinction equilibrium point $E_1(k,0,0)$ is globally asymptotically stable if $\delta>1-\frac{(D+d_1)d_2}{kd_3}$.*

Proof. Consider the positive definite Lyapunov function as follows

$$V(x,y,z)=\frac{1}{1+a(1-\delta)k}\left(x-k-k\ln\frac{x}{k}\right)+\frac{1}{c}y+\frac{1}{c}\frac{D+d_1}{D}z.$$

The α-order derivative of $V(x,y,z)$ along the solution of system (2) is now computed. By virtue of Lemma 3.1 [35],

$$\begin{aligned}
{}^cD^\alpha V(x,y,z)\leq&\frac{1}{1+a(1-\delta)k}\left(1-\frac{k}{x}\right){}^cD^\alpha x(t)+\frac{1}{c}{}^cD^\alpha y(t)+\frac{1}{c}\frac{D+d_1}{D}{}^cD^\alpha z(t)\\
\leq&\frac{1}{1+a(1-\delta)k}(x-k)\left(r\left(1-\frac{x}{k}\right)-\frac{b(1-\delta)z}{1+a(1-\delta)x}\right)\\
&+\frac{b(1-\delta)xz}{1+a(1-\delta)x}-\frac{D+d_1}{c}y+\frac{1}{c}\frac{D+d_1}{D}(Dy-d_2z)\\
\leq&-\frac{r}{k}\frac{(x-k)^2}{(1+a(1-\delta)k)}+\frac{(D+d_1)d_2}{cD}\left(\frac{cbD(1-\delta)k}{(D+d_1)(1+a(1-\delta)k)d_2}-1\right)z\\
\leq&-\frac{r}{k}\frac{(x-k)^2}{(1+a(1-\delta)k)}+\frac{(D+d_1)d_2}{cD}(R_0-1)z.
\end{aligned}$$

Thus, ${}^cD^\alpha V(x,y,z)\leq 0$, when $R_0<1$ which is equivalent to $\delta>1-\frac{(D+d_1)d_2}{kd_3}$. In accordance with Lemma 4.6 in Huo et al. [36], the predator-extinction equilibrium point E_1 is globally asymptotically stable if $\delta>1-\frac{(D+d_1)d_2}{kd_3}$.

Theorem 7. *The coexistence equilibrium point $E_2(x^*,y^*,z^*)$ is globally asymptotically stable if $x_{\inf}>\frac{k}{2}$.*

Proof. The approach used in [5] is adopted. To prove global stability of E_2, we define the positive definite Lyapunov function as follows

$$V(x,y,z)=\frac{1}{1+ax^*}\left(x-x^*-x^*\ln\frac{x}{x^*}\right)+\frac{1}{c}\left(y-y^*-y^*\ln\frac{y}{y^*}\right)+\frac{D+d_1}{cD}\left(z-z^*-z^*\ln\frac{z}{z^*}\right).$$

We now calculate the α-order derivative of $V(x,y,z)$ along the solution of system (2). According to Lemma 3.1 [35]. One obtains that

$$\begin{aligned}
{}^cD^{\alpha}V(x,y,z) \leq & \frac{g(x)-g(x^*)}{g(x)}{}^cD^{\alpha}x(t)+\frac{1}{c}\left(1-\frac{y^*}{y}\right){}^cD^{\alpha}y(t)+\frac{D+d_1}{cD}\left(1-\frac{z^*}{z}\right){}^cD^{\alpha}z(t)\\
\leq & \frac{g(x)-g(x^*)}{g(x)}(f(x)-g(x)z)+\frac{1}{c}\left(1-\frac{y^*}{y}\right)(cg(x)z-(D+d_1)y)\\
&+\frac{D+d_1}{cD}\left(1-\frac{z^*}{z}\right)(Dy-d_2z)\\
\leq & f(x)\frac{g(x)-g(x^*)}{g(x)}+g(x^*)z-\frac{D+d_1}{c}y^*\left(\frac{g(x)}{g(x^*)}\frac{z}{z^*}\frac{y^*}{y}+\frac{z^*}{z}\frac{y}{y^*}+\frac{g(x^*)}{g(x)}-3\right)\\
&+\frac{D+d_1}{c}y^*\frac{g(x^*)}{g(x)}-\frac{D+d_1}{c}y^*-\frac{D+d_1}{cD}d_2z.
\end{aligned}$$

Since $g(x^*)=\frac{(D+d_1)d_2}{cD}$, this yields

$$\begin{aligned}
{}^cD^{\alpha}V(x,y,z) \leq & f(x)\frac{g(x)-g(x^*)}{g(x)}-\frac{D+d_1}{c}y^*\left(\frac{g(x)}{g(x^*)}\frac{z}{z^*}\frac{y^*}{y}+\frac{z^*}{z}\frac{y}{y^*}+\frac{g(x^*)}{g(x)}-3\right)\\
&+\frac{D+d_1}{c}y^*\left(\frac{g(x^*)}{g(x)}-1\right)\\
\leq & \frac{1}{g(x)}(f(x)-f(x^*))(g(x)-g(x^*))\\
&-\frac{D+d_1}{c}y^*\left(\frac{g(x)}{g(x^*)}\frac{z}{z^*}\frac{y^*}{y}+\frac{z^*}{z}\frac{y}{y^*}+\frac{g(x^*)}{g(x)}-3\right).
\end{aligned}$$

From the inequality of arithmetic and geometric means, it is clear that

$$\frac{g(x)}{g(x^*)}\frac{z}{z^*}\frac{y^*}{y}+\frac{z^*}{z}\frac{y}{y^*}+\frac{g(x^*)}{g(x)}\geq 3,$$

with equality if and only if

$$\frac{g(x)}{g(x^*)}\frac{z}{z^*}\frac{y^*}{y}=\frac{z^*}{z}\frac{y}{y^*}=\frac{g(x^*)}{g(x)}=1,$$

that is, $x=x^*$ and $\dfrac{y}{y^*}=\dfrac{z}{z^*}$.

If $x>\frac{k}{2}$ for $t\geq t_0$. Then since $f(x)$ is strictly decreasing on $[\frac{k}{2},\infty)$ and $g(x)$, in turn, is strictly increasing on $[0,\infty)$, it follows that

$$\frac{1}{g(x)}(f(x)-f(x^*))(g(x)-g(x^*))\leq 0.$$

Thus, ${}^cD^{\alpha}V(x,y,z)\leq 0$, when $x>\frac{k}{2}$ which satisfies $x_{\inf}>\frac{k}{2}$, where $x_{\inf}\leq \liminf_{t\to\infty}x(t)$. In accordance with Lemma 4.6 in Huo et al. [36], the coexistence equilibrium point E_2 is globally asymptotically stable if $x_{\inf}>\frac{k}{2}$.

2.5 Hopf Bifurcation

Consider the fractional order commensurate system:

$$ {}^cD^{\alpha}x=f(m,x), \tag{8}$$

where $\alpha\in(0,1]$, $x\in\mathbb{R}^3$ and further suppose that E is an equilibrium point of system (8). In [34], a fractional order Hopf bifurcation is proposed. It states that system (8) undergoes a Hopf bifurcation through the equilibrium E at the value m_{cr} of m if:

– The characteristic equation of the Jacobian matrix has one real eigenvalue μ_1 and two complex-conjugate eigenvalues $\mu_{2,3}$,

$-\theta_{1,2,3}(\alpha, m_{cr}) = 0$, where $\theta_i(\alpha, m) = \frac{\alpha\pi}{2} - |\arg(\mu_i(m))|, \quad i = 1,2,3.$

$-\frac{\partial\theta_{1,2,3}}{\partial m}|_{m=m_{cr}} \neq 0.$

3 Numerical simulations

For the numerical simulation of the fractional-order system (2) the generalized Adams-Bashforth-Moulton type predictor-corrector scheme is applied. We carry out numerical simulations so as to demonstrate the qualitative behavior of the fractional-order system (2). We choose the following set of parameter values:

$$r = 3,\ a = 2,\ c = 1,\ b = 1,\ D = 1,\ d_1 = 0.1,\ d_2 = 0.2 \text{ and } k = 1.5,$$

as they were used for the integer-order system [5].

For the above set of parameter values, one gets the predator-extinction equilibrium point $E_1(1.5,0,0)$ and the coexistence equilibrium point $E_2(0.4910, 0.9008, 4.5041)$ where $\delta = 0.2$.

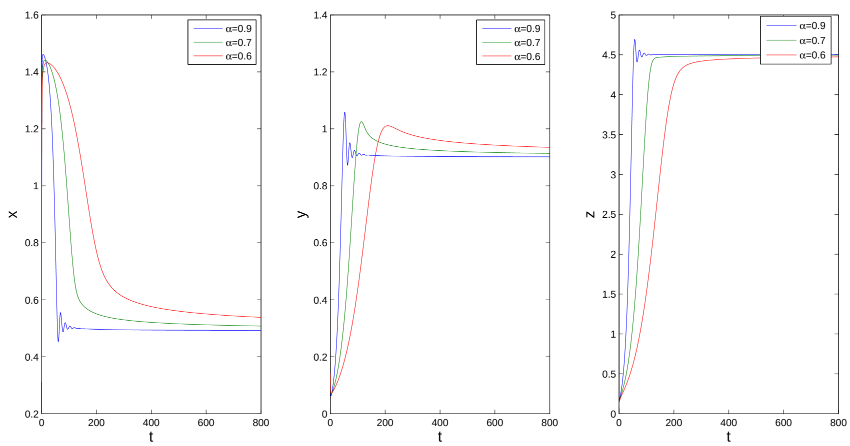

Fig. 1: State trajectories of the fractional-order system (2) with different fractional orders α and $\delta = 0.2$.

From Fig. 1, it is observed that the fractional order affects the convergence speed of the solutions of the fractional-order system (2). It also can be observed that the convergence speed to coexistence equilibrium point $E_2(0.4910, 0.9008, 4.5041)$ increases with increasing fractional order α, $(0 < \alpha < 1)$.

In Fig. 2, all trajectories with different initial conditions converge to the predator-extinction equilibrium point $E_1(1.5,0,0)$. This shows that the predator-extinction equilibrium point E_1 is globally asymptotically stable. In this case $R_0 = 0.672 < 1$ and this coincides with Theorem 5 and $\delta = 0.8 > 1 - \frac{(D+d_1)d_2}{kd_3}$ and this coincides with Theorem 6.

The coexistence equilibrium point $E_2(0.5238, 0.9297, 4.6485)$ is globally asymptotically stable for value $c = 1$, $\delta = 0.25$ and $\alpha = 0.98$ with different initial values as shown in Fig. 3.

A bifurcation diagram is drawn around the coexistence equilibrium point $E_2(0.4910, 0.9008, 4.5041)$ with respect to the fractional order α so as to understand the role of α. The fractional-order system (2) undergoes Hopf bifurcation at the supercritical Hopf bifurcation value $\alpha^* = 0.972627$ as shown in Fig. 4. The coexistence equilibrium point E_2 is stable for $\alpha < \alpha^*$ as shown in Fig. 4; for example, when $\alpha = 0.95$, the coexistence equilibrium point E_2 is locally asymptotically stable as shown in Fig. 5 (c). For $\alpha > \alpha^*$, the system shows limit cycle behaviour as shown in Fig. 4; for example, when $\alpha = 0.98$, the coexistence equilibrium point E_2 loses its stability and stable limit cycle occurs around equilibrium point E_2 as shown in Fig. 5 (b).

When $\alpha < \alpha^*$ all trajectories of the fractional-order system (2) converges to a coexistence equilibrium point $E_2(0.4910, 0.9008, 4.5041)$ as shown in Fig. 4, 5 (c); while with α being increased to pass α^*, the coexistence equilibrium point E_2 loses its stability and stable limit cycle occurs around coexistence equilibrium point E_2 as shown in Fig. 4, 5 (b). In integer-order case when $\alpha = 1$, a stable limit cycle emerges to which all trajectories are attracted. The

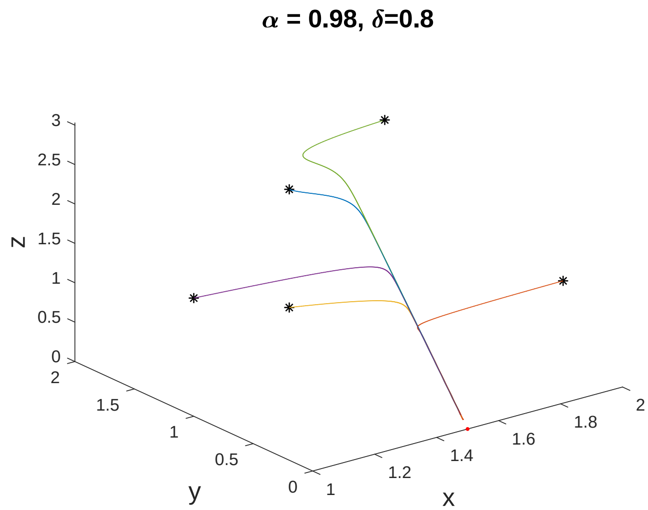

Fig. 2: Globally asymptotically stable of the predator-extinction equilibrium point $E_1(1.5,0,0)$ with different initial values.

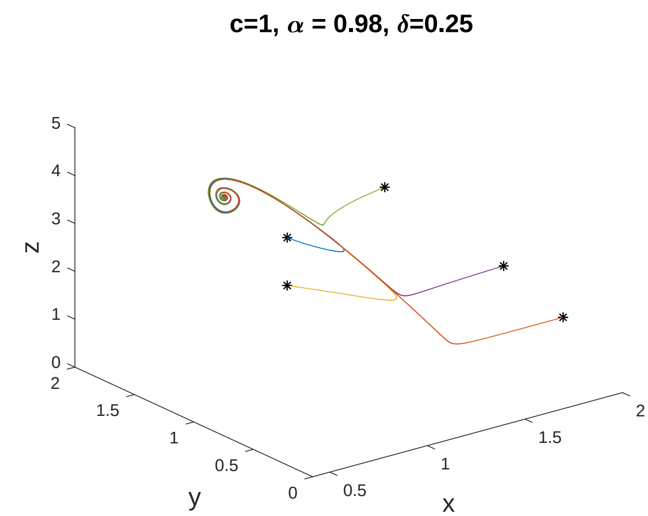

Fig. 3: Globally asymptotically stable of the coexistence equilibrium point $E_2(0.5238,0.9297,4.6485)$ with different initial values.

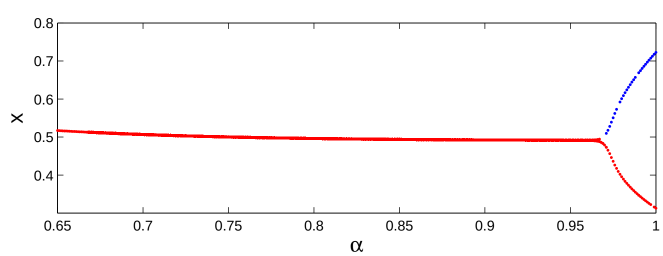

Fig. 4: Bifurcation diagram of the fractional-order system (2) with respect to α when $\delta = 0.2$.

amplitude of the stable limit cycle is, however, now bigger as indicated in Fig. 4, 5 (a). The integer-order system (1)

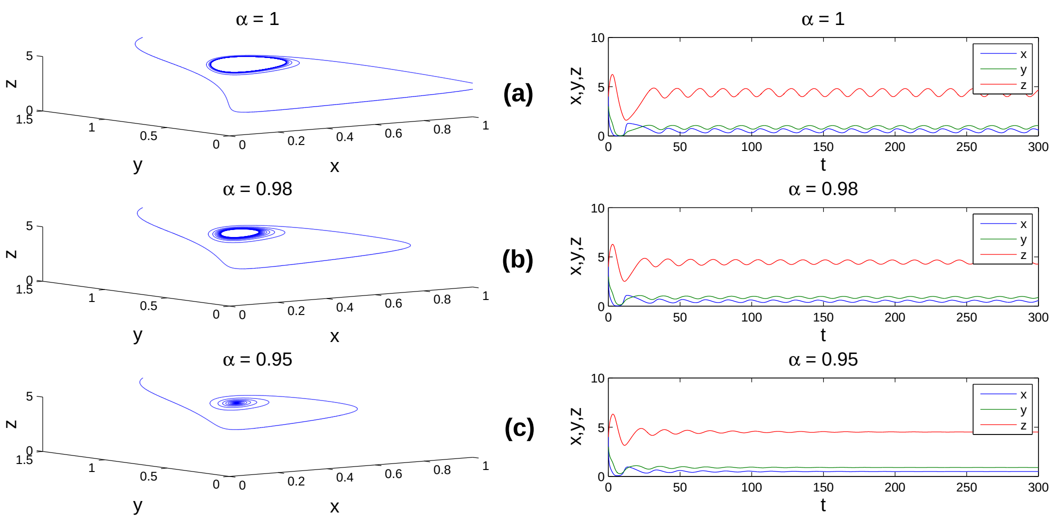

Fig. 5: Time series and phase portrait of x, y and z with different values of α when $\delta = 0.2$.

around the coexistence equilibrium point E_2 which is unstable as stated in [5,15] becomes asymptotically stable in the fractional-order system (2) which we consider in this paper.

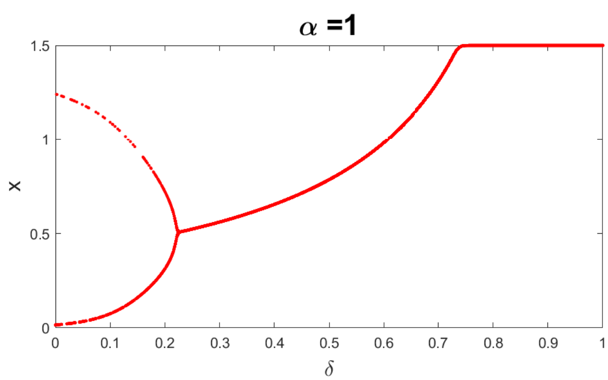

Fig. 6: Bifurcation diagram of the fractional-order system (2) with respect to δ when $\alpha = 1$.

To better understand the effect of refuge size δ around the coexistence equilibrium point $E_2(0.4910, 0.9008, 4.5041)$, a bifurcation diagram with respect to δ and $\alpha = 1$ is drawn as shown in Fig. 6. The system (2) undergoes Hopf bifurcation at the supercritical Hopf bifurcation value $\delta^* = 0.220689$ and a transcritical bifurcation value $\delta_{tr} = 0.738095$ can be seen in Fig. 6. When $\delta < \delta^*$ the system shows limit-cycle behaviour as shown in Fig. 6; for example, when $\delta = 0.19$, the coexistence equilibrium point E_2 loses its stability and stable limit cycle occurs around equilibrium point E_2 as shown in Fig. 7 (c). For $\delta > \delta^*$ the coexistence equilibrium point E_2 is locally asymptotically stable as shown in Fig. 6; for example, when $\delta = 0.25$, E_2 is locally asymptotically stable as shown in Fig. 7 (b). But for $\delta > \delta_{tc}$ the predator population goes extinct from the system and the prey population then attains the carrying capacity as indicated in Fig. 6; for example, when $\delta = 0.8$, the population follows the same trajectory as can be seen in Fig. 7 (a).

It is interesting to note that the prey refuge has stabilization effects. The coexistence equilibrium point E_2 is unstable without prey refuge as shown in 7 (d) and it becomes asymptotically stable by incorporating a prey refuge as shown in Fig. 7 (b), where $\delta < \delta_{tc}$.

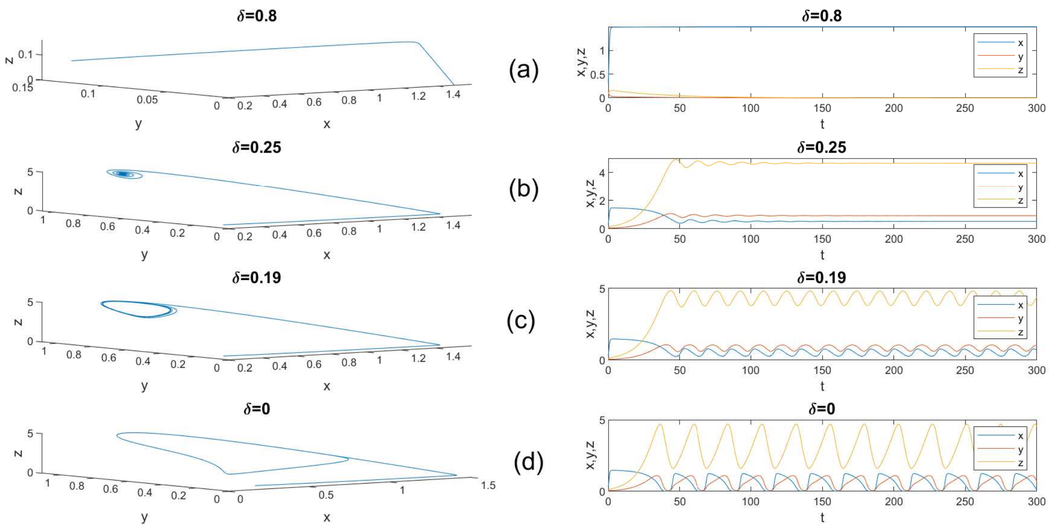

Fig. 7: Time series and phase portrait of x, y and z with different values of δ when $\alpha = 1$ and $c = 1$.

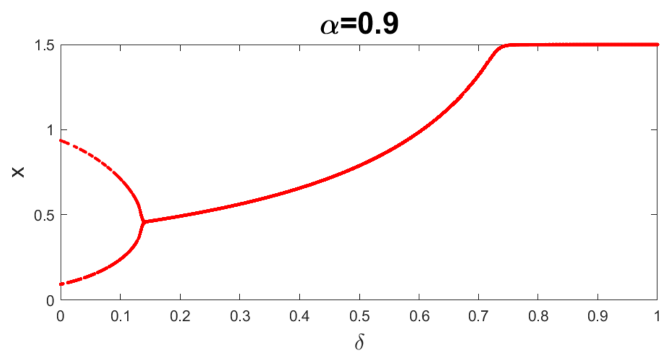

Fig. 8: Bifurcation diagram of the fractional-order system (2) with respect to δ when $\alpha = 0.9$.

Now, we draw the bifurcation diagram with respect to δ and $\alpha = 0.9$ as shown in Fig. 8. The fractional-order system (2) undergoes Hopf bifurcation at the supercritical Hopf bifurcation value $\delta^* = 0.1383$ and a transcritical bifurcation value $\delta_{tr} = 0.738095$ as shown in Fig. 8. It is seen from Fig. 8 that when $\delta < \delta^*$ the system shows limit cycle behaviour, for $\delta > \delta^*$ the equilibrium point E_2 is locally asymptotically stable and for $\delta > \delta_{tc}$ the predator population goes extinct from the system and the prey population attains the environment's carrying capacity which coincide with Fig. 9.

It is to be noted that the fractional-order system (2) is more stable than its integer counterpart system (1) because the larger domain of stability. The coexistence equilibrium point E_2 is unstable for integer-order case when $\alpha = 1$ and $\delta = 0.19$ as shown in Fig. 7 (c) becomes asymptotically stable for the fractional order case when $\alpha = 0.9$ and $\delta = 0.19$ as shown in Fig. 9 (b).

In order to show the effect of conversion coefficient c around the coexistence equilibrium point $E_2(0.4910, 0.9008, 4.5041)$, one can draw the bifurcation diagram with respect to c and $\alpha = 1$ as shown in Fig. 10. The fractional-order system (2) undergoes Hopf bifurcation at the transcritical bifurcation value $c_{tc} = 0.623333$ and the supercritical Hopf bifurcation value $c^* = 0.979802$. It is observed from Fig. 10 that when $c < c_{tc}$ the predator population goes extinct from the system and the prey population attains the carrying capacity, for $c > c_{tc}$ the equilibrium point E_2 is locally asymptotically stable and for $c > c^*$ the system undergoes limit cycle behaviour which coincide with Fig. 11.

Now, we draw the bifurcation diagram with respect to c and $\alpha = 0.9$ as shown in Fig. 12. The fractional-order system (2) undergoes Hopf bifurcation at the transcritical bifurcation value $c_{tc} = 0.623333$ and the supercritical Hopf bifurcation

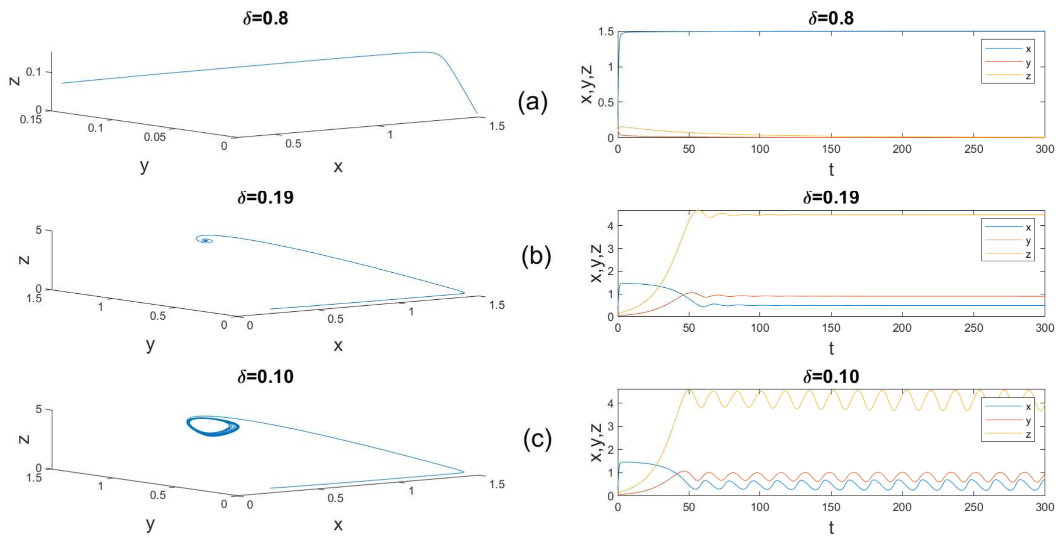

Fig. 9: Time series and phase portrait of x,y and z with different values of δ when $\alpha = 0.9$ and $c = 1$.

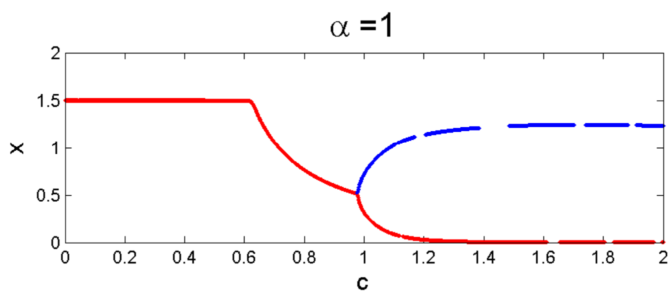

Fig. 10: Bifurcation diagram of the fractional-order system (2) with respect to c when $\alpha = 1$ and $\delta = 0.2$.

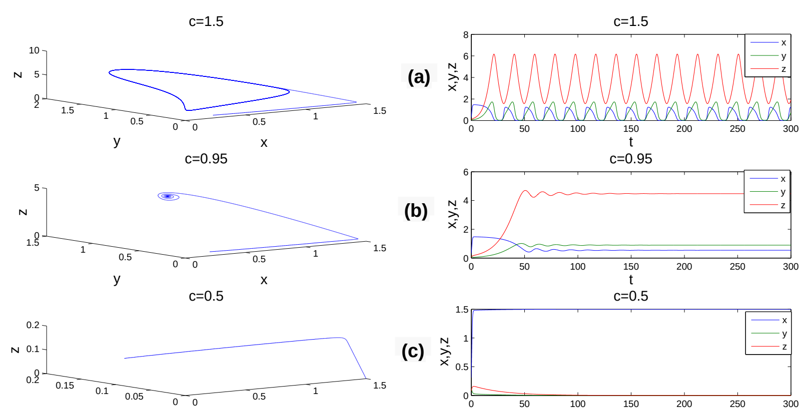

Fig. 11: Time series and phase portrait of x,y and z with different values of c when $\alpha = 1$ and $\delta = 0.2$.

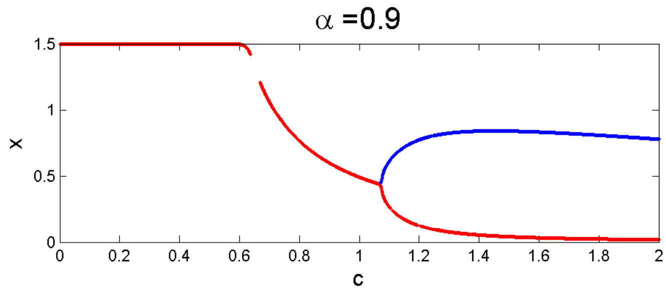

Fig. 12: Bifurcation diagram of the fractional-order system (2) with respect to c when $\alpha = 0.9$ and $\delta = 0.2$.

value $c^* = 1.0715$ as shown in Fig. 12. When $c < c_{tc}$ the predator population goes extinct from the system and the prey population attains the carrying capacity , for $c > c_{tc}$ the coexistence equilibrium point E_2 is locally asymptotically stable and the system undergoes stable limit cycle behaviour for $c > c^*$ as indicated in Fig. 12.

4 Conclusion

In the present paper, a fractional-order prey-predator model with stage structure incorporating a prey refuge is proposed and analyzed. The dynamical behaviours of the fractional order system (2) have been investigated. The stability conditions of the predator-extinction equilibrium point and the coexistence equilibrium point have been established. The global asymptotic stability of the equilibrium points of the fractional order system (2) has been investigated. Numerical studies have been conducted to verify the theoretical results. The rich dynamical behaviour indicated by the simulations are in agreement with the theoretical studies.

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