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# An Analytical Method for Solving Nonlinear Sine-Gordon Equation

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**Abstract:** In this paper, an analytical method called the Natural Decomposition Method (NDM) for solving nonlinear Sine-Gordon equation is introduced. This analytical method is a combination of the Natural transform Method (NTM) and a well-known technique, the Adomian Decomposition Method (ADM). The proposed analytical method reduces the computational size, avoids round-off errors, linearization, transformation or taking some restrictive assumptions. The series solutions of the Sine-Gordon equation are successfully obtained using the analytical method, and the results are compared with the results of the existing methods.

**Keywords:** Natural Decomposition Method, Adomian polynomial, nonlinear Sine- Gordon equation

## 1 Introduction

The Sine-Gordon equation is one of the most crucial nonlinear evolution equation that plays a vital role in physical science and engineering. The equation is a nonlinear hyperbolic partial differential equation involving the sine of unknown function and the d'Alembert operator. The Sine-Gordon equation was first discovered in the nineteenth century in the course of study of various problems of differential geometry [1]. In the early 1970s it was first realized that the Sine-Gordon equation led to kink and antikink (so-called solitons) [2]. The Sine-Gordon equation appears in many physical applications in relativistic field theory, Josephson junction, mechanical transmission line and so on. The standard nonlinear Sine-Gordon equation is given by

$$v_{tt}(x,t)-\alpha^2 v_{xx}(x,t)-\beta\sin(v(x,t))=0, \qquad (1)$$

subject to the initial conditions:

$$v(x,0)=f(x), \quad v_t(x,0)=g(x), \qquad (2)$$

where $\alpha$ and $\beta$ are constant.

In recent years, many analytical method have been used to solve nonlinear partial differential equations such as Adomian Decomposition Method (ADM) [3,4,5,6], the Homotopy Analysis Method (HAM) [7], the Variational Iteration Method (VIM) [8,9,10,11,12], the Laplace Decomposition Method (LDM) [13], the Natural Decomposition Method (NDM) [14,15,16,17,18,19], the Homotopy Perturbation Method (HPM) [20,21,22], the ($\frac{G'}{G}$ )-Expansion Method [23], the Tanh Method (TM) [24], the Exp- Function Method (EFM) [25], the Natural Homotopy Perturbation Method (NHPM) [26,27,28], the Reduce Differential Transform Method (RDTM) [29,30, 31], the Generalized Kudryashov method (GKM) [32], and so on. Other related references are available in [33, 34,35,36,37,38,39,40,41].

In this paper, an analytical method for solving the nonlinear Sine-Gordon equation is introduced. The proposed analytical method is applied directly without any unnecessary linearization , discretization of variables, transformation or taking some restrictive assumptions. It avoids round-off errors and reduces the computational size. In this analytical method, the nonlinear terms are elegantly computed using Adomian polynomials. The analytical method gives a series solution which converges rapidly to an exact or approximate solution with elegant computational terms.

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## 2 Natural Transform

In 2008, Khan and Khan [37] introduced an integral transform called the N-transform and it was renamed as the Natural transform by Belgacem and Silambarasan [33, 34] in 2012. The Natural transform is an integral transform which is similar to Laplace transform [39] and the Sumudu integral transform [35,40]. It converge to Laplace transform when the variable $u = 1$ and to Sumudu transform when the variable $s = 1$.

**Definition 1**[33, 34] *The Natural transform of the function $v(t) \in A$ for $t \geq 0$ is defined by the following integral*

$$\mathbb{N}^+[v(t)] = V(s,u) = \frac{1}{u}\int_0^\infty e^{-\frac{st}{u}} v(t)dt, \;\; s>0,\; u>0, \quad (3)$$

*where s and u are the Natural transform variables and*
$A = \left\{ v(t) : \exists M, \tau_1, \tau_2 > 0, |v(t)| < Me^{\frac{|t|}{\tau_j}}, \right.$
$\left. \text{if } t \in (-1)^j \times [0,\infty) \right\}.$

Some few properties of the Natural transform method are given below. See[33, 34].

**Property 1***If $V(s,u)$ is the Natural transform and $F(s)$ is the Laplace transform of the function $f(t) \in A$, then*

$\mathbb{N}^+[f(t)] = V(s,u) = \frac{1}{u}\int_0^\infty e^{-\frac{st}{u}} f(t)dt = \frac{1}{u}F\left(\frac{s}{u}\right).$

**Property 2***If $V(s,u)$ is the Natural transform and $G(u)$ is the Sumudu transform of the function $v(t) \in A$, then*

$\mathbb{N}^+[v(t)] = V(s,u) = \frac{1}{s}\int_0^\infty e^{-t} v\left(\frac{ut}{s}\right) dt = \frac{1}{s}G\left(\frac{u}{s}\right).$

**Property 3***If $\mathbb{N}^+[v(t)] = V(s,u)$,*

*then $\mathbb{N}^+[v(\beta t)] = \frac{1}{\beta}V(\frac{s}{\beta},u)$.*

**Property 4***If $\mathbb{N}^+[v(t)] = V(s,u)$,*

*then $\mathbb{N}^+[v'(t)] = \frac{s}{u}V(s,u) - \frac{v(0)}{u}$.*

**Property 5***If $\mathbb{N}^+[v(t)] = V(s,u)$, then*

$\mathbb{N}^+[v''(t)] = \frac{s^2}{u^2}V(s,u) - \frac{s}{u^2}v(0) - \frac{v'(0)}{u}.$

**Property 6***The Natural transform is a linear operator. That is, if $\alpha$ and $\beta$ are non–zero constants, then*

$\mathbb{N}^+[\alpha f(t) \pm \beta g(t)] = \alpha\mathbb{N}^+[f(t)] \pm \beta\mathbb{N}^+[g(t)]$

$= \alpha F^+(s,u) \pm \beta G^+(s,u).$

*Moreover, $F^+(s,u)$ and $G^+(s,u)$ are the Natural transforms of $f(t)$ and $g(t)$, respectively.*

**Table 1.** The Natural transform of some functions.

| Function | Natural transform |
|---|---|
| $1$ | $\frac{1}{s}$ |
| $t$ | $\frac{u}{s^2}$ |
| $e^{at}$ | $\frac{1}{s-au}$ |
| $\frac{t^{n-1}}{(n-1)!}, n=1,2,\ldots$ | $\frac{u^{n-1}}{s^n}$ |
| $\sin(t)$ | $\frac{u}{s^2+u^2}$ |

## 3 Analysis of the Natural Decomposition Method (NDM)

In this section, the basic idea of the Natural Decomposition Method (NDM) is clearly illustrated on the following nonlinear Sine-Gordon equation

$$v_{tt}(x,t) - \alpha^2 v_{xx}(x,t) - \beta\sin(v(x,t)) = 0, \quad (4)$$

subject to the initial conditions:

$$v(x,0) = f(x), \;\; v_t(x,0) = g(x). \quad (5)$$

Applying the Natural transform to Eq.(4) subject to the given initial conditions, we get:

$$V(x,s,u) = \frac{f(x)}{s} + \frac{ug(x)}{s^2} + \frac{u^2}{s^2}\mathbb{N}^+\left[\alpha^2 v_{xx}(x,t) + \beta\sin(v(x,t))\right]. \quad (6)$$

Taking the inverse Natural transform of Eq.(6), we get:

$$v(x,t) = G(x,t) + \mathbb{N}^{-1}\left[\frac{u^2}{s^2}\mathbb{N}^+\left[\alpha^2 v_{xx}(x,t) + \beta\sin(v(x,t))\right]\right]. \quad (7)$$

Now, we assume a series solution of the form

$$v(x,t) = \sum_{n=0}^{\infty} v_n(x,t). \quad (8)$$

The nonlinear term $\sin(v(x,t))$ is decomposed as:

$$sin(v(x,t)) = \sum_{n=0}^{\infty} A_n, \quad (9)$$

where $A_n$ is Adomian polynomials and can easily be computed with following formula

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[F\left(\sum_{i=0}^{n}\lambda^i v_i(x,t)\right)\right]_{\lambda=0}. \quad (10)$$

where $n = 0,1,2,\ldots$.

Some few components of $A_n$ are computed below

$$A_0 = \sin(v_0(x,t)),$$
$$A_1 = v_1(x,t)\cos(v_0(x,t)),$$
$$A_2 = v_2(x,t)\cos(v_0(x,t)) - \frac{1}{2!}v_1^2(x,t)\sin(v_0(x,t)),$$
$$A_3 = v_3(x,t)\cos(v_0(x,t)) - v_2(x,t)v_1(x,t)\sin(v_0(x,t)) - \frac{1}{3!}v_1^3(x,t)\cos(v_0(x,t)),$$
$$\vdots$$

and so on.

By substituting Eq.(8) and Eq.(9) into Eq.(7), we obtain:

$$\sum_{n=0}^{\infty} v_n(x,t) = G(x,t) + \mathbb{N}^{-1}\left[\frac{u^2}{s^2}\mathbb{N}^{+}\left[\alpha^2 \sum_{n=0}^{\infty} v_{nxx}(x,t) + \beta \sum_{n=0}^{\infty} A_n\right]\right]. \quad (11)$$

Then by comparing both sides of Eq.(11) above, we can easily generate the recursive relation as follows:

$$v_0(x,t) = G(x,t),$$
$$v_1(x,t) = \mathbb{N}^{-1}\left[\frac{u^2}{s^2}\mathbb{N}^{+}\left[\alpha^2 v_{0xx}(x,t) + \beta A_0\right]\right],$$
$$v_2(x,t) = \mathbb{N}^{-1}\left[\frac{u^2}{s^2}\mathbb{N}^{+}\left[\alpha^2 v_{1xx}(x,t) + \beta A_1\right]\right],$$
$$v_3(x,t) = \mathbb{N}^{-1}\left[\frac{u^2}{s^2}\mathbb{N}^{+}\left[\alpha^2 v_{2xx}(x,t) + \beta A_2\right]\right],$$
$$\vdots$$
$$v_{n+1}(x,t) = \mathbb{N}^{-1}\left[\frac{u^2}{s^2}\mathbb{N}^{+}\left[\alpha^2 v_{nxx}(x,t) + \beta A_n\right]\right], \quad (12)$$

$\forall n \geq 0.$

Thus, the series solutions of Eq.(4)-(5) is given by

$$v(x,t) = \sum_{n=0}^{\infty} v_n(x,t). \quad (13)$$

## 4 Applications

In this section the application of the Natural Decomposition Method to nonlinear Sine-Gordon equations are clearly illustrated to show its simplicity and high accuracy.

**Example 1***Consider the following nonlinear Sine-Gordon equation of the form:*

$$v_{tt}(x,t) - v_{xx}(x,t) - sin(v(x,t)) = 0, \quad (14)$$

*subject to the initial conditions:*

$$v(x,0) = \frac{\pi}{2}, \quad v_t(x,0) = 0. \quad (15)$$

*Applying the Natural transform of Eq.(14) subject to the given initial conditions, we get:*

$$V(x,s,u) = \frac{\pi}{2s} + \frac{u^2}{s^2}\mathbb{N}^{+}\left[v_{xx}(x,t) + sin(v(x,t))\right]. \quad (16)$$

*Taking the inverse Natural transform of Eq.(16), we get:*

$$v(x,t) = \frac{\pi}{2} - \mathbb{N}^{-1}\left[\frac{u^2}{s^2}\mathbb{N}^{+}\left[v_{xx}(x,t) + sin(v(x,t))\right]\right]. \quad (17)$$

*Now, we assume a series solution of the form*

$$v(x,t) = \sum_{n=0}^{\infty} v_n(x,t). \quad (18)$$

*The nonlinear term $sin(v(x,t))$ is decomposed as:*

$$sin(v(x,t)) = \sum_{n=0}^{\infty} A_n. \quad (19)$$

*Where $A_n$ is the Adomian polynomials.*
*Using the recursive relation of Eq.(12) where $v(x,0) = \frac{\pi}{2}, v_t(x,0) = 0,$ and $\alpha = \beta = 1$, we obtained the following approximations:*

$$\begin{aligned}
v_0(x,t) &= \frac{\pi}{2},\\
v_1(x,t) &= \mathbb{N}^{-1}\left[\frac{u^2}{s^2}\mathbb{N}^{+}\left[v_{0xx}(x,t) + A_0\right]\right]\\
&= \mathbb{N}^{-1}\left[\frac{u^2}{s^2}\mathbb{N}^{+}\left[v_{0xx}(x,t) + sin(v_0(x,t))\right]\right]\\
&= \mathbb{N}^{-1}\left[\frac{u^2}{s^2}\mathbb{N}^{+}[1]\right]\\
&= \mathbb{N}^{-1}\left[\frac{u^2}{s^3}\right]\\
&= \frac{t^2}{2},\\
v_2(x,t) &= \mathbb{N}^{-1}\left[\frac{u^2}{s^2}\mathbb{N}^{+}\left[v_{1xx}(x,t) + A_1\right]\right]\\
&= \mathbb{N}^{-1}\left[\frac{u^2}{s^2}\mathbb{N}^{+}\left[v_{1xx}(x,t) + v_1(x,t)cos(v_0(x,t))\right]\right]\\
&= 0,\\
v_3(x,t) &= \mathbb{N}^{-1}\left[\frac{u^2}{s^2}\mathbb{N}^{+}\left[v_{2xx}(x,t) + A_2\right]\right]\\
&= \mathbb{N}^{-1}\left[\frac{u^2}{s^2}\mathbb{N}^{+}\left[v_2(x,t)cos(v_0(x,t)) - \frac{1}{2!}v_1^2(x,t)sin(v_0(x,t))\right]\right]\\
&= -\frac{1}{8}\mathbb{N}^{-1}\left[\frac{u^2}{s^2}\mathbb{N}^{+}\left[t^4\right]\right]\\
&= -3\mathbb{N}^{-1}\left[\frac{u^6}{s^7}\right]\\
&= -\frac{t^6}{240},\\
&\vdots
\end{aligned}$$

*and so on.*

*Then, the series solution of Eq.(14)–(15) is given by:*

$$\begin{aligned} v(x,t) &= \sum_{n=0}^{\infty} v_n(x,t) \\ &= v_0(x,t)+v_1(x,t)+v_2(x,t)+v_3(x,t)+\cdots \\ &= \frac{\pi}{2}+\frac{t^2}{2}-\frac{t^6}{240}+\frac{t^{10}}{34560}+\cdots \end{aligned} \tag{20}$$

*Thus,*

$$v(x,t) = \frac{\pi}{2}+\frac{t^2}{2}-\frac{t^6}{240}+\frac{t^{10}}{34560}+\cdots \tag{21}$$

*The series solution is in closed agreement with the result obtained by (ADM)[2].*

**Example 2***Consider the following nonlinear Sine-Gordon equation of the form:*

$$v_{tt}(x,t)-v_{xx}(x,t) = sin(v(x,t)), \tag{22}$$

*subject to the initial conditions:*

$$v(x,0) = \frac{\pi}{2}, \quad v_t(x,0) = 1. \tag{23}$$

*Applying the Natural transform to Eq.(22) subject to the given initial conditions, we get:*

$$V(x,s,u) = \frac{\pi}{2s}+\frac{u}{s^2}+\frac{u^2}{s^2}\mathbb{N}^+\left[v_{xx}(x,t)+sin(v(x,t))\right]. \tag{24}$$

*Taking the inverse Natural transform of Eq.(24), we get:*

$$v(x,t) = \frac{\pi}{2}+t+\mathbb{N}^{-1}\left[\frac{u^2}{s^2}\mathbb{N}^+\left[v_{xx}(x,t)+sin(v(x,t))\right]\right]. \tag{25}$$

*Now, we assume a series solution of the form*

$$v(x,t) = \sum_{n=0}^{\infty} v_n(x,t). \tag{26}$$

*The nonlinear term $sin(v(x,t))$ is decomposed as:*

$$sin(v(x,t)) = \sum_{n=0}^{\infty} A_n. \tag{27}$$

*Where $A_n$ is the Adomian polynomials.*
*Then using the recursive relation of Eq.(12) where $v(x,0) = \frac{\pi}{2}, v_t(x,0) = 1,$ and $\alpha = \beta = 1$, we obtained the following approximations:*

$$\begin{aligned} v_0(x,t) &= \frac{\pi}{2}+t, \\ v_1(x,t) &= \mathbb{N}^{-1}\left[\frac{u^2}{s^2}\mathbb{N}^+\left[v_{0xx}(x,t)+A_0\right]\right] \\ &= \mathbb{N}^{-1}\left[\frac{u^2}{s^2}\mathbb{N}^+\left[v_{0xx}(x,t)+sin(v_0(x,t))\right]\right] \\ &= \mathbb{N}^{-1}\left[\frac{u^2}{s^2}\mathbb{N}^+\left[cos(t)\right]\right] \\ &= \mathbb{N}^{-1}\left[\frac{1}{s}\right]-\mathbb{N}^{-1}\left[\frac{s}{s^2+u^2}\right] \\ &= 1-cos(t), \\ v_2(x,t) &= \mathbb{N}^{-1}\left[\frac{u^2}{s^2}\mathbb{N}^+\left[v_{1xx}(x,t)+A_1\right]\right] \\ &= \mathbb{N}^{-1}\left[\frac{u^2}{s^2}\mathbb{N}^+\left[v_{1xx}(x,t)+v_1(x,t)cos(v_0(x,t))\right]\right] \\ &= \mathbb{N}^{-1}\left[\frac{u^2}{s^2}\mathbb{N}^+\left[sin(t)cos(t)-sin(t)\right]\right] \\ &= \mathbb{N}^{-1}\left[\frac{u}{s^2+u^2}\right]-\frac{3}{4}\mathbb{N}^{-1}\left[\frac{u}{s^2}\right]-\frac{1}{8}\mathbb{N}^{-1}\left[\frac{2u}{s^2+4u^2}\right] \\ &= sin(t)-\frac{3t}{4}-\frac{sin(2t)}{8} \\ &\vdots \end{aligned}$$

*and so on.*
*Then, the series solution of Eq.(22)–(23) is given by:*

$$\begin{aligned} v(x,t) &= \sum_{n=0}^{\infty} v_n(x,t) \\ &= v_0(x,t)+v_1(x,t)+v_2(x,t)+v_3(x,t)+\cdots \\ &= \frac{\pi}{2}+t+1-cos(t)+sin(t)-\frac{3t}{4}-\frac{sin(2t)}{8}+\cdots \\ &= \frac{\pi}{2}+t+\frac{t^2}{2!}-\frac{t^4}{4!}+\cdots \end{aligned} \tag{28}$$

*Thus,*

$$v(x,t) = \frac{\pi}{2}+t+\frac{t^2}{2!}-\frac{t^4}{4!}+\cdots \tag{29}$$

*The series solution is in closed agreement with the result obtained by (ADM)[2].*

## 5 Conclusion

In this paper, the Natural Decomposition Method (NDM) is successfully applied to nonlinear Sine-Gordon equation. The NDM gives a series solution which converges rapidly to an exact or approximate solution. Furthermore, the NDM reduces the computational size and avoids round-off errors. Series solutions of the nonlinear Sine-Gordon equation are successfully obtained using the analytical method, and the results are compared with the results of the existing techniques. Thus, the proposed analytical method can easily be used to solve many nonlinear partial differential equations due to its efficiency and high accuracy.

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