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The Odd Frèchet-G Family of Probability Distributions

Muhammad Ahsan ul Haq[1, 2*] *and M. Elgarhy* [3]

[1] Quality Enhancement Cell, National College of Arts, Lahore Pakistan.
[2] College of Statistical & Actuarial Sciences, University of the Punjab, Lahore Pakistan.
[3] Graduate Studies and Scientific Research, University of Jeddah, Jeddah, KSA.

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Abstract: We propose a new generator from Frèchet random variable that is known as the *Odd Frèchet-G (OFr-G) family* of distributions. The new class of family can be more flexible since the density shapes are left skewed, symmetrical and reversed-J. Some special models derived and discussed. Several of its important properties are derived. The maximum likelihood equations are derived for OFr-G family parameters. The importance and flexibility of the derived models is assessed using two real dataset examples.

Keywords: Generalized family, Frèchet distribution, entropy, moments, maximum likelihood.

1 Introduction

Extreme value distributions (EVDs) are essential for demonstrating and measuring events which happen with little probability and have been broadly utilized as a part of hazard administration, finance, insurance, financial aspects, sports, hydrology, material sciences, broadcast communications, and numerous data sets of extreme events. Class of EVDs involves three types of EVDs: Gumbel (type I), Frèchet (type II) and Weibull (type III). In literature, these EVDs are generalized by incorporating location, scale and power parameters resulting in generalized extreme value distributions (GEVDs): generalized Gumbel, generalized Frèchet and generalized Weibull.

In the modern era, there has been an expanded attention for introducing new generators for univariate distributions by inducing at least one extra shape parameter(s) to the baseline distribution. This addition of parameter(s) has been demonstrated valuable in investigating tail properties and furthermore to increase the goodness-of-fit of the proposed family. Some commonly available generators are beta-G [1], gamma-G (type 1) [2], Kumaraswamy-G [3], gamma-G (type 2) [4], McDonald-G [5], gamma-G (type 3) [6], exponentiated generalized-G [7], Transformed-Transformer (T-X) [8], Weibull-G [9], Garhy-G [10], exponentiated Weibull-G [11], Kumaraswamy Weibull-G [12], type II half logistic-G [13] and exponentiated extended-G family [14].

The present study is unfolding as follows; Section 2 based on the derivation of new family of distributions. Further, Section 3 based on some special models of the OFr-G family. Asymptotes and shapes are given in section 4. Moreover, some suitable structural properties such as; explicit expressions for the moments, incomplete moments, and probability weighted moments, quantile measures and mean deviation are derived in Section 5. Maximum Likelihood Estimation and real data set examples are presented in Section 6 and in the end study concluded in Section 7.

2 The new family

Consider the probability density function (pdf) and cumulative distribution function (cdf) of Frèchet (Fr) distribution with real shape parameter $\theta>0$ and scale parameter $\sigma>0$ are given respectively,

*Corresponding author e-mail: ahsanshani36@gmail.com

$$G(x, \theta, \sigma) = e^{-\left(\frac{\sigma}{x}\right)^{\theta}}, \qquad x > 0, \theta, \sigma > 0 \tag{1}$$

$$g(x) = \theta\sigma^{\theta}x^{-(\theta+1)}e^{-\left(\frac{\sigma}{x}\right)^{\theta}}, \tag{2}$$

Cumulative distribution function $G(x;\xi)$ and survival function $\bar{G}(x;\xi) = 1 - G(x;\xi)$ of the baseline distribution are depending on a parameter vector ξ and let a random variable T relating a stochastic system having a baseline G distribution. The odds x that the system will not be working at time t is $G(t)/\bar{G}(t)$. We want model the random variable X of this odds using Frèchet model (with scale parameter σ = 1) given by (1). We can write

$$Pr(X \le x) = \Pi_X(x) = F_X\left[\frac{G(t)}{\bar{G}(t)}\right]$$

and then by replacing x in the Frèchet cdf by the ratio $G(x;\xi)/\bar{G}(x;\xi)$, the cdf of the new family, OFr-G, follows as

$$F(x{:}\,\theta,\xi) = \int_0^{\left[\frac{G(x;\xi)}{1-G(x;\xi)}\right]} \frac{\theta}{x^{\theta+1}} e^{-x^{-\theta}}\, dx = e^{-\left[\frac{1-G(x;\xi)}{G(x;\xi)}\right]^{\theta}} \tag{3}$$

The corresponding pdf to (3) is given by

$$f(x{:}\,\theta,\xi) = \frac{\theta g(x;\xi)[1-G(x;\xi)]^{\theta-1}}{G(x;\xi)^{\theta+1}} e^{-\left[\frac{1-G(x;\xi)}{G(x;\xi)}\right]^{\theta}} \tag{4}$$

where $g(x{:}\,\xi)$ consider a pdf of baseline distribution. Hereafter, a random variable X with density function (4) is denoted by $X \sim OFr - G(\theta, \xi)$.

The hrf of the OFr-G family is

$$h(x{:}\,\theta,\xi) = \frac{\theta g(x;\xi)[1-G(x;\xi)]^{\theta-1} e^{-\left[\frac{1-G(x;\xi)}{G(x;\xi)}\right]^{\theta}}}{G(x;\xi)^{\theta+1}\left[1 - e^{-\left[\frac{1-G(x;\xi)}{G(x;\xi)}\right]^{\theta}}\right]} \tag{5}$$

The quantile function, say x_u of a random variable has pdf (4) is given by

$$x_u = G^{-1}\left\{\frac{1}{1 + [-\log(u)]^{\frac{1}{\theta}}}\right\} \tag{6}$$

where u is a uniform U(0,1) distribution. We can generate random numbers from our model by using (6)

3 Some special cases

We derived some models of the proposed OFr-G in this section. For example, odd Frèchet-Weibull, odd Frèchet-Lomax, odd Frèchet-Pareto and odd Frèchet-Gamma distributions.

3.1. Odd Frèchet-Weibull (OFr-W) distribution

We consider the Weibull distribution with scale parameter $\alpha > 0$ and shape parameter $\beta > 0$. The pdf and cdf are $g(x) = \alpha\beta x^{\beta-1}e^{-(\alpha x^{\beta})}$, $G(x) = 1 - e^{-(\alpha x^{\beta})}$, respectively. The pdf of OFr-W is

$$f_{OFr-W}(x;\theta,\alpha,\beta) = \frac{\alpha\beta\theta x^{\beta-1} e^{-\theta(\alpha x^{\beta})}}{\left\{1-e^{-(\alpha x^{\beta})}\right\}^{\theta+1}} e^{-\left\{\frac{e^{-(\alpha x^{\beta})}}{1-e^{-(\alpha x^{\beta})}}\right\}^{\theta}} \qquad x>0,\theta,\alpha,\beta>0$$

3.2 Odd Frèchet-Lomax (OFr-L) distribution

Let us consider the Lomax distribution with scale parameter $\beta>0$ and shape parameter $\alpha>0$. Then, the pdf and cdf are given by $g(x) = (\alpha/\beta)[1+(x/\beta)]^{-\alpha-1}, G(x) = 1-[1+(x/\beta)]^{-\alpha}$ respectively. Then, the OFr-L pdf is

$$f_{OFr-L}(x;\theta,\alpha,\beta) = \frac{\alpha\theta[1+(x/\beta)]^{-(\alpha\theta+1)}}{\beta[1-[1+(x/\beta)]^{-\alpha}]^{\theta+1}} e^{-\left\{\frac{[1+(x/\beta)]^{-\alpha}}{1-[1+(x/\beta)]^{-\alpha}}\right\}^{\theta}} \qquad x>0,\theta,\alpha,\beta>0$$

3.3. Odd Frèchet-Gamma (OFr-Gam) distribution

Now, we consider the parent Gamma distribution with pdf and cdf given by

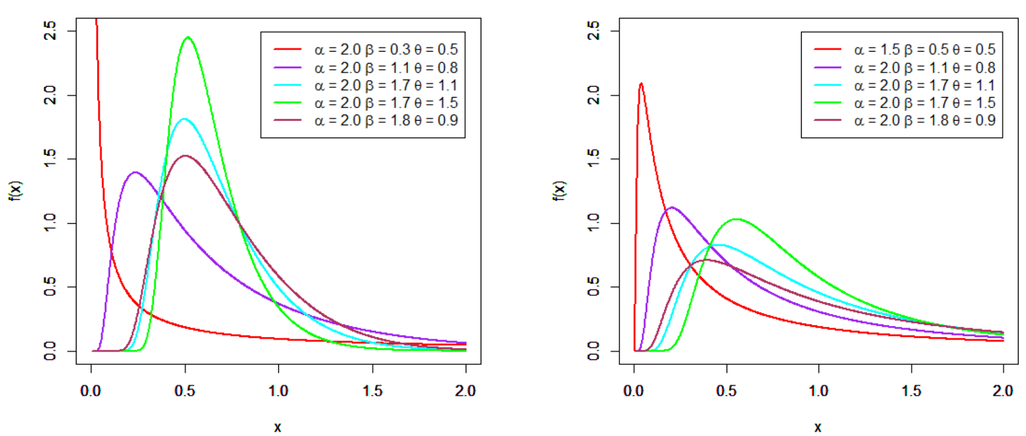
$$g(x;\ \alpha,\beta) = \frac{1}{\beta^{\alpha}\Gamma(\alpha)} x^{\alpha-1} e^{-(x/\beta)}$$

$$G(x;\ \alpha,\beta) = \frac{\Upsilon\left(\alpha,\frac{x}{\beta}\right)}{\Gamma(\alpha)}, \qquad x>0,\alpha,\beta>0$$

Where $\Upsilon\left(\alpha,\frac{x}{b}\right) = \int_0^{\frac{x}{b}} t^{\alpha-1}e^{-t}dt$ denote the incomplete gamma function. Then the OFr-Gam pdf is

$$f_{OFr-Gam}(x;\theta,\alpha,\beta) = \frac{\theta}{\beta^{\alpha}\Gamma(\alpha)} x^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)} \left\{\frac{\Upsilon\left(\alpha,\frac{x}{\beta}\right)}{\Gamma(\alpha)}\right\}^{-(\theta+1)} \left\{1-\frac{\Upsilon\left(\alpha,\frac{x}{\beta}\right)}{\Gamma(\alpha)}\right\}^{\theta-1} e^{-\left\{\frac{\Gamma(\alpha)}{\Upsilon\left(\alpha,\frac{x}{\beta}\right)}-1\right\}^{\theta}}$$

The pdf and hrf graphs of OFr-W and OFr-L distributions are presented in Figure (1 & 2). This indeed indicates that the proposed OFr-G family can be very valuable in fitting real-life data sets.



(a) (b)

Fig 1: pdf curves of (a) OFr-W and (b) OFr-L

The figure 1(a) represents the behavior of OFr-W density function and explains the tractability and flexibility of model graphically with its sub-families. The pdf plot shows that for, the newly developed model has exponentially decreasing behavior, unimodal positively skewed behavior. Figure 1(b) represents the behavior of pdf of OFr-L distribution which is unimodal.

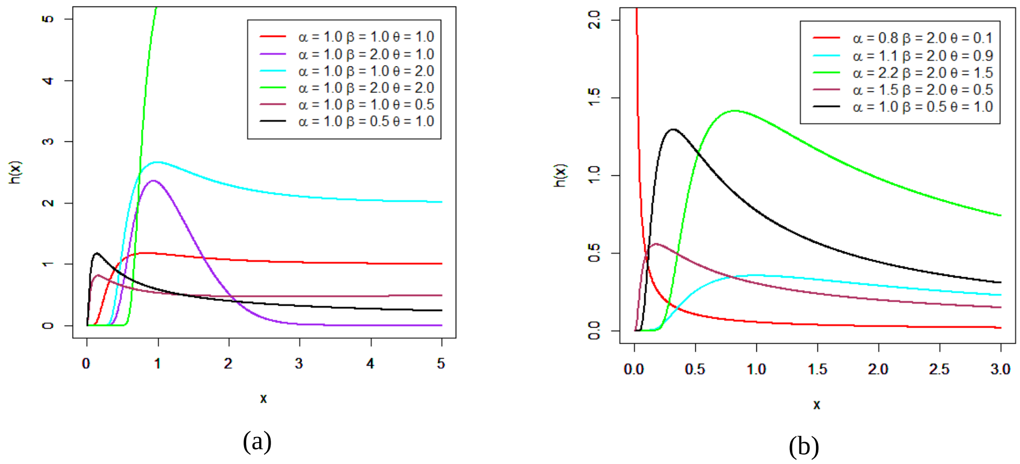


Fig 2: hrf curves of (a) OFr-W and (b) OFr-L

The figure 2(a) and figure 2(b) represents the behavior of hazard rate function and explains the tractability and flexibility of model graphically with its sub-families. The hazard increasing and decreasing and has J- shape.

Theorem 1 make available some relationships of the OFr-G family with other distributions.

Theorem 1: Let X ∼OFr−G(θ,).

(a) If $Y = G(X;\ ), then\ F_Y(y) = e^{-\left(\frac{1-y}{y}\right)^{\theta}} \qquad 0 < y < 1,$

(b) If $Y = \frac{G(X;\ )}{\bar{G}(X;\ )}, then\ Y \sim Frechet\ (\theta, 1), and$

(c) If $Y = \frac{\bar{G}(X;\ )}{G(X;\ )}, then\ Y \sim Weibull\ (\theta, 1)$

4 Asymptotes and shapes

Corollary 1: *The asymptotic of equations (3), (4) and (5) as* $x \to 0$ *are given by*

$$F(x) \sim e^{-[G(x)]^{-\theta}}\ as\ x \to 0,$$

$$f(x) \sim \frac{\theta g(x) e^{-[G(x)]^{-\theta}}}{[G(x)]^{\theta+1}}\ \ as\ \mathrm{x} \to 0,$$

$$h(x) \sim \frac{\theta g(x) e^{-[G(x)]^{-\theta}}}{[G(x)]^{\theta+1}\ 1 - e^{-[G(x)]^{-\theta}}}\ \ as\ \mathrm{x} \to 0.$$

Corollary 2. *The asymptotics of equations (3), (4) and (5) as $x \to \infty$ are given by*

$$1 - F(x) \sim [1 - G(x)]^{\theta} \qquad as\ x \to \infty,$$

$$f(x) \sim \theta\, g(x)[1 - G(x)]^{\theta - 1} \qquad as\ x \to \infty,$$

$$h(x) \sim \frac{\theta\, g(x)}{1 - G(x)} \qquad as\ x \to \infty.$$

5 Structural properties

We established some structural properties of the OFr-G family of distributions that can be more productive than processing those directly by numerical integration of its density function.

5.1. Useful expansions

In this subsection, a useful expansion of the probability density and distribution functions for OFr-G family is covered.

Firstly, we obtain an expansion for *pdf* defined in (4) as follows:

Since the exponential series is

$$e^{-\left[\frac{1-G(x;\xi)}{G(x;\xi)}\right]^{\theta}} = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \left[\frac{[1 - G(x;\xi)]}{G(x;\xi)}\right]^{\theta i}. \tag{7}$$

Then,

$$f(x{:}\, \theta, \xi) = \theta g(x;\xi) \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \frac{[1 - G(x;\xi)]^{\theta(i+1)-1}}{G(x;\xi)^{\theta(i+1)+1}},$$

We can rewrite the last equation as

$$f(x{:}\, \theta, \xi) = \theta g(x;\xi) \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \frac{[1 - G(x;\xi)]^{\theta(i+1)-1}}{\left[1 - [1 - G(x;\xi)]\right]^{\theta(i+1)+1}},$$

Now, using the generalized binomial series, we can write

$$\left[1 - [1 - G(x;\xi)]\right]^{-[\theta(i+1)+1]} = \sum_{j=0}^{\infty} \binom{\theta(i+1)+j}{j} [1 - G(x;\xi)]^{j}, \tag{8}$$

Then,

$$f(x{:}\, \theta, \xi) = \theta g(x;\xi) \sum_{i,j=0}^{\infty} \frac{(-1)^i}{i!} \binom{\theta(i+1)+j}{j} [1 - G(x;\xi)]^{\theta(i+1)+j-1} \quad ,$$

The binomial series is

$$(1 - z)^{\beta - 1} = \sum_{k=0}^{\infty} (-1)^k \binom{\beta - 1}{k} z^k, \tag{9}$$

for $|z| < 1$, and β is a positive real non integer. Then, by applying the binomial theorem (8) for $[1 - G(x;\xi)]^{\theta(i+1)+j-1}$ in (4), the density function of *OFr-G* becomes

$$f(x{:}\,\theta,\xi) = \theta g(x;\xi) \sum_{i,j,k=0}^{\infty} \frac{(-1)^{i+k}}{i!} \binom{\theta(i+1)+j}{j} \binom{\theta(i+1)+j-1}{k} [G(x;\xi)]^{k},$$

the *pdf* of *OFr-G* can be defined as an infinite linear combination of *pdf* of exponentiated generated i.e.

$$f(x) = \sum_{k=0}^{\infty} \eta_k g(x,\xi) G(x,\xi)^k, \tag{10}$$

then,

$$f(x) = \sum_{k=0}^{\infty} W_k h_{k+1}(x), \tag{11}$$

where,

$$W_k = \sum_{i,j=0}^{\infty} \frac{\theta(-1)^{i+k}}{i!\,(k+1)} \binom{\theta(i+1)+j}{j} \binom{\theta(i+1)+j-1}{k}$$

and,

$$W_k = \frac{\eta_k}{[k+1]}, \qquad h_{k+1}(x) = (k+1) g(x,\xi) G(x,\xi)^k.$$

Secondly, an expansion of $[F(x)]^h$ is obtained as following: Again, the binomial expansion is worked out for $[F(x)]^h$, with h is integer.

$$[F(x)]^h = \sum_{q=0}^{\infty} \frac{(-1)^q}{q!} \left[\frac{1 - G(x,\xi)}{G(x,\xi)}\right]^{q\theta}.$$

We can rewrite the last equation as

$$[F(x)]^h = \sum_{q=0}^{\infty} \frac{(-1)^q}{q!} \left[\frac{1 - G(x,\xi)}{1 - [1 - G(x,\xi)]}\right]^{q\theta}.$$

Now,

$$\left[1 - [1 - G(x;\xi)]\right]^{-q\theta} = \sum_{u=0}^{\infty} \binom{q\theta + u - 1}{u} [1 - G(x;\xi)]^u,$$

then,

$$[F(x)]^h = \sum_{q=0}^{\infty} \frac{(-1)^q}{q!} \binom{q\theta + u - 1}{u} [1 - G(x,\xi)]^{q\theta+u}.$$

Since the generalized binomial series is

$$[1-G(x,\xi)]^{q\theta+u}=\sum_{z=0}^{\infty}(-1)^z\binom{q\theta+u}{z}G(x,\xi)^z$$

Then, $[F(x)]^h$ takes the following form

$$[F(x)]^h=\sum_{q,u,z=0}^{\infty}\frac{(-1)^q}{q!}\binom{q\theta+u-1}{u}\binom{q\theta+u}{z}G(x,\xi)^z.$$

Finally,

$$[F(x)]^h=\sum_{z=0}^{\infty}s_z G(x,\zeta)^z, \tag{12}$$

where,

$$S_z=\sum_{q,u=0}^{\infty}\frac{(-1)^{q+z}}{q!}\binom{q\theta+u-1}{u}\binom{q\theta+u}{z}$$

5.2. The probability weighted moments (PWMs)

The PWMs can be obtained using the following relation

$$\tau_{r,s}=E[X^r\,\mathrm{F}(x)^s]=\int_{-\infty}^{\infty}x^r f(x)(\mathrm{F}(x))^s dx. \tag{13}$$

The PWMs of *OFr-G* is obtained by substituting (10) and (12) into (13), replacing *h* with *s*, leads to:

$$\tau_{r,s}=\int_{-\infty}^{\infty}\sum_{k,z=0}^{\infty}s_z\eta_k x^r g(x,\zeta)(\mathrm{G}(x,\zeta))^{z+k}dx.$$

Then,

$$\tau_{r,s}=\sum_{k,z=0}^{\infty}s_z\eta_k\tau_{r,z+k},$$

where, $\tau_{r,z+k}=\int_{-\infty}^{\infty}x^r g(x,\zeta)\left[\mathrm{G}(x,\zeta)\right]^{z+k}dx.$

In addition; another formula will be yielded by using quantile function as follows

$$\tau_{r,s}=\int_0^1\sum_{k,z=0}^{\infty}s_z\eta_k\left[Q_G(\mathrm{u})\right]^r u^{z+k}du.$$

5.3. Moments and moment generating function

The moments are obtained as follows

$$\mu_r' = \int_{-\infty}^{\infty} x^r f(x) dx. \tag{14}$$

Then substituting (10) into (14) yields:

$$\mu_r' = \int_{-\infty}^{\infty} \sum_{k=0}^{\infty} \eta_k x^r g(x,\zeta)(\mathrm{G}(x,\zeta))^k dx,$$

then,

$$\mu_r' = \sum_{k=0}^{\infty} \eta_k \tau_{r,k},$$

where, $\tau_{r,k}$ is the probability-weighted moments of the $G(x,\zeta)$ distribution.

Further, another recipe can be derived, in light of the parent quantile work as follows;

$$\mu_r' = \sum_{k=0}^{\infty} \eta_k \int_0^1 (Q_G(\mathrm{u}))^r u^k du.$$

Generally, mgf is $M_X(t) = E(e^{tX})$ and by using expansion then it can be written as follows

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r'.$$

Then,

$$M_X(t) = \sum_{k,r=0}^{\infty} \frac{t^r}{r!} \eta_k \tau_{r,k}.$$

Moreover; shape will be yielded by utilizing quantile function as follows;

$$M_u(t) = \sum_{k=0}^{\infty} \eta_k \int_0^1 e^{(\mathrm{t}Q_G(\mathrm{u}))} u^k du.$$

5.4. Entropies

Entropy is a measure of variety or vulnerability of a random variable X. Three prevalent entropy measures are the Rényi, q, and Shannon. A common measure of entropy is Rényi entropy and has much importance in many fields such as statistical inference, classification, problem identification in statistics, econometrics and pattern recognition in computer sciences. The given theorem provides expression for Rényi entropy.

The Rényi entropy can be derived using the below relation

$I_R(\gamma) = \frac{1}{1-\gamma} log \int_0^\infty f^\gamma(x)\, dx \quad ,\gamma > 0$ and $\gamma \neq 1$.

By applying the binomial theory (8), (9) and exponential expansion (7) in the *pdf* (4), then the *pdf* $f(x)^\gamma$ can be expressed as follows

$$f(x)^\gamma = \sum_{i,j,k=0}^{\infty} t_{i,j,k}\, g(x,\zeta)^\gamma \, G(x,\zeta)^k ,$$

where

$$t_{i,j,k} = \frac{\theta^\gamma \gamma^k (-1)^{i+k}}{i!} \binom{\theta(i+\gamma)+\gamma+j-1}{j} \binom{\theta(i+\gamma)+j-\gamma}{k}.$$

Therefore, the Rényi entropy of OFr-G family is given by

$$I_\gamma(X) = \frac{1}{1-\gamma} \log \sum_{i,j,k=0}^{\infty} t_{i,j,k} \int_{-\infty}^{\infty} g(x,\zeta)^\gamma \, G(x,\zeta)^k \, dx .$$

The *q*- entropy is defined by the following relation

$$H_q(X) = \frac{1}{1-q} \log\left(1 - \int_{-\infty}^{\infty} f(x)^q dx\right), q > 0 \text{ and } q \neq 1.$$

Therefore, the q- entropy of OFr-G generated family of distributions is given by

$$H_q(X) = \frac{1}{1-q} \log\left(1 - \sum_{i,j,k=0}^{\infty} t_{i,j,k} \int_{-\infty}^{\infty} g(x,\zeta)^q \, G(x,\zeta)^k \, dx \right). \quad (15)$$

5.5. Order statistics

Let be random variables and its ordered values is denoted as $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$. The probability density function (pdf) of order statistics is obtained using the below function.

$$f_{i:n}(x) = Kf(x)F^{i-1}(x)\{1-F(x)\}^{n-i} = K \sum_{v=0}^{n-i} (-1)^v \binom{n-i}{v} f(x)F(x)^{j+i-1} \quad (16)$$

Where $K = n!/[(i-1)!\,(n-i)!]$.

The density of the nth ordered statistics follows the OFr-G family is derived as follows

$$f_{i:n}(x) = g_{X_{(i)}}(x,\zeta) \sum_{v=0}^{n-i} \sum_{k,z=0}^{\infty} \eta_k \, p_{z,v} \, G_{X_{(k)}}(x,\zeta)^{z+k} , \quad (17)$$

where $p_{z,v} = (-1)^v K \binom{n-k}{v} s_z$, $g(.)$ and $G(.)$ are the density and cumulative functions of the OFr-G distributions, respectively.

Moments of order statistics is defined by:

$$E(X_{(i)}{}^{r}) = \int_{-\infty}^{\infty} x^{r} f_{i:n}(x)dx \,. \tag{18}$$

by substituting (17) in (18), leads to

$$E(X_{(i)}{}^{r}) = \sum_{v=0}^{n-i}\sum_{k,z=0}^{\infty} \eta_k\, p_{z,v} \int_{-\infty}^{\infty} x^{r} g_{X_{(i)}}(x,\zeta)\, G_{X_{(i)}}(x,\zeta)^{z+k} dx \,.$$

Then,

$$E(X_{(i)}{}^{r}) = \sum_{v=0}^{n-i}\sum_{k,z=0}^{\infty} \eta_k\, p_{z,v}\, \tau_{r,z+k} \,.$$

6. Estimation

This study adopts maximum likelihood estimation method so that it is mostly used and provides maximum information about the properties of estimated parameters. Moreover, normal approximation of these estimators can frankly be managed systematically and mathematically for large sample theory. Let $x_1, x_2, \ldots. x_n$ be the observed values from the OFr-G family with parameters θ and ξ. The total log-likelihood function for φ is given by

$$l_n = log(\varphi) = n\log\theta + \sum_{i=1}^{n}\log[g(x_i;\xi)] + (\theta-1)\sum_{i=1}^{n}\log[1-G(x_i;\xi)] - (\theta+1)\sum_{i=1}^{n}\log[G(x_i;\xi)] - \sum_{i=1}^{n}\left[\frac{1-G(x_i;\xi)}{G(x_i;\xi)}\right]^{\theta} \tag{19}$$

Now we have to maximize log-likelihood function given in (19) to get the MLEs of OFr-G family of distributions. For this purpose, we take the first derivative of the above log-likelihood equation with respect to parameters and equate to zero respectively. The mechanisms of the score function $U_n(\varphi) = \left(\frac{\partial l_n}{\partial\theta}, \frac{\partial l_n}{\partial\xi}\right)^T$ are

$$\frac{\partial l_n}{\partial\theta} = \frac{n}{\theta} + \sum_{i=1}^{n}\log\frac{1-G(x_i;\xi)}{G(x_i;\xi)} - \sum_{i=1}^{n}\left[\frac{1-G(x_i;\xi)}{G(x_i;\xi)}\right]^{\theta}\log\frac{1-G(x_i;\xi)}{G(x_i;\xi)} \tag{20}$$

$$\frac{\partial l_n}{\partial\xi} = \sum_{i=1}^{n}\left[\frac{g^{(\xi)}(x_i;\xi)}{g(x_i;\xi)}\right] + (1-\theta)\sum_{i=1}^{n}\left[\frac{G^{(\xi)}(x_i;\xi)}{1-G(x_i;\xi)}\right] - (\theta+1)\sum_{i=1}^{n}\left[\frac{G^{(\xi)}(x_i;\xi)}{G(x_i;\xi)}\right] + \theta\sum_{i=1}^{n}\left[\frac{G^{(\xi)}(x_i;\xi)\left[1-G^{(\xi)}(x_i;\xi)\right]^{\theta-1}}{G(x_i;\xi)^{\theta+1}}\right] \tag{21}$$

Where $G^{(\xi)}(\cdot)$ means the derivative of the function h with respect to ξ.

The exact solution of equations (20,21) for unknown parameters is not possible. So it is well-situated to use nonlinear optimization algorithms such as a Newton-Raphson algorithm for maximizing the above likelihood function numerically. We can use R (optimal function or maxBFGS function), or MATHEMATICA (Maximize function).

6.1. Simulation study

To inspect the performance of OFr-W distribution. We conduct a simulation study by using Monte Carlos simulation method with 30,000 repetitions on the basis of bias and mean square error of estimated parameters from maximum likelihood estimation method. The simulation is done as follows:

- Generate data from $F(x) = u$, where is uniformly distributed (0, 1) from (6).

- Simulation is conducted for sample sizes n=30, 75, 100 and 300.
- The repetition of the experiment is 30,000 times for each sample size.

In each trial, the estimates of the parameters will be gotten by maximum likelihood estimation. The estimated values, biases and MSEs are be reported from these experiments. The bias and MSEs are computed by

$$Bias_{\epsilon}(n)=\frac{1}{N}\sum_{i=1}^{N}(\hat{\epsilon}_i-\epsilon) \quad \text{and} \quad MSE_{\epsilon}(n)=\frac{1}{N}\sum_{i=1}^{N}(\hat{\epsilon}_i-\epsilon)^2$$

Table 1 represents the outcomes of Monte Carlos simulation study. We evaluate the mean of estimated parameters, mean square errors, and biases. These findings based on expected first order asymptotic theory as bias and MSE's decreases toward zero with an increase in sample size.

Table 1 The parameter estimation from OFr-W distribution using MLE

θ	α	β	Sample sizes (n)	Parameters	Mean	Bias	MSE
0.50	0.50	0.75	30	θ	0.5273	0.0273	0.0194
				α	0.5149	0.0149	0.0087
				β	0.9027	0.1527	0.2433
			75	θ	0.5117	0.0117	0.0068
				α	0.5070	0.0070	0.0033
				β	0.7967	0.0467	0.0481
			100	θ	0.5108	0.0108	0.0048
				α	0.5041	0.0041	0.0023
				β	0.7873	0.0373	0.0341
			300	θ	0.5021	0.0021	0.0016
				α	0.5015	0.0015	0.0007
				β	0.7626	0.0126	0.0103
0.50	0.50	2.0	30	θ	0.5277	0.0277	0.0196
				α	0.5132	0.0132	0.0084
				β	2.4106	0.4106	1.6513
			75	θ	0.5102	0.0102	0.0069
				α	0.5057	0.0057	0.0033
				β	2.1503	0.1503	0.4008
			100	θ	0.5082	0.0082	0.0048
				α	0.5042	0.0042	0.0023
				β	2.0962	0.0962	0.2386
			300	θ	0.5032	0.0032	0.0016
				α	0.5013	0.0013	0.0008
				β	2.0319	0.0319	0.0639
1.50	0.50	0.75	30	θ	1.3273	0.0773	0.1229
				α	0.5844	0.0844	0.7105
				β	0.7980	0.0480	0.0460
			75	θ	1.2856	0.0356	0.0445
				α	0.5261	0.0261	0.0176
				β	0.7689	0.0189	0.0146
			100	θ	1.2734	0.0234	0.0298
				α	0.5162	0.0162	0.0100
				β	0.7653	0.0153	0.0118
			300	θ	1.2558	0.0058	0.0101
				α	0.5056	0.0056	0.0028
				β	0.7551	0.0051	0.0035

6.2. Applications

In this section, we give two applications to represent the significance of the OFr-W and OFr-L distributions exhibited in Section 2.1. The ML estimates, as well as goodness-of-fit measures, are computed and compared with other competing models.

Data 1: Maximum Annual Flood Discharges of the North Saskachevan River

The first data set was originally reported by Montfort [15] which represents the Maximum Annual Flood Discharges of the North Saskachevan in units of 1000 cubic feet per second, of the North Saskachevan River at Edmonton, over a period of 47 years. The data are: 19.885, 20.940, 21.820, 23.700, 24.888, 25.460, 25.760, 26.720, 27.500, 28.100, 28.600, 30.200, 30.380, 31.500, 32.600, 32.680, 34.400, 35.347, 35.700, 38.100, 39.020, 39.200, 40.000, 40.400, 40.400, 42.250, 44.020, 44.730, 44.900, 46.300, 50.330, 51.442, 57.220, 58.700, 58.800, 61.200, 61.740, 65.440, 65.597, 66.000, 74.100, 75.800, 84.100, 106.600, 109.700, 121.970, 121.970, 185.560.

We exhibit the flexibility of the derived model OFr-W distribution in contrast with other models, including the Marshal Olkin-Weibull (MOW), Kumaraswamy Weibull (KwW), beta-Weibull (BW), Odd log-logistic Marshal Olkin Weibull (OLLMOW), Odd log-logistic Weibull (OLLW) and McDonald Weibull (McW) distributions. Table 2 represents the estimates that are computed using AdequacyModel. Table 3 lists the goodness of fit measures including Anderson Darling (A*), Cramer-von Mises (W*), log-likelihood function, Akaike Information Criterion (AIC), Bayesian information criterion (BIC), to compare the fitted models. Generally, we consider the best fit using the smaller values of these statistics.

Table 2: Estimated values for the first data set

Model	**Estimates**				
OFr-W	0.081501	0.597249	2.808699	-	-
MOW	0.003256	1.507926	1.700421	-	-
OLLW	0.106858	0.494316	4.878116	-	-
KwW	0.049281	1.175201	8.908800	0.305954	-
BW	0.073591	1.060735	8.887724	0.373319	-
OLLMOW	0.138308	0.423497	5.580156	0.874088	-
McW	0.134608	1.041777	5.404278	0.216423	4.820985

Table 3: Goodness of fit measures for first data set

Model	–2l	**AIC**	**BIC**	$A*$	$W*$
OFr-W	-430.2902	436.2901	441.9037	0.154704	0.021416
MOW	-453.4936	459.4935	465.1071	1.831836	0.294515
OLLW	-438.3002	444.3003	449.9139	0.690541	0.105023
KwW	-435.4428	443.4428	450.9276	0.555256	0.081141
BW	-435.1904	443.1904	450.6752	0.521344	0.075961
OLLMOW	-438.0590	446.059	453.5438	0.648353	0.098213
McW	-432.3720	442.3719	451.728	0.336956	0.047534

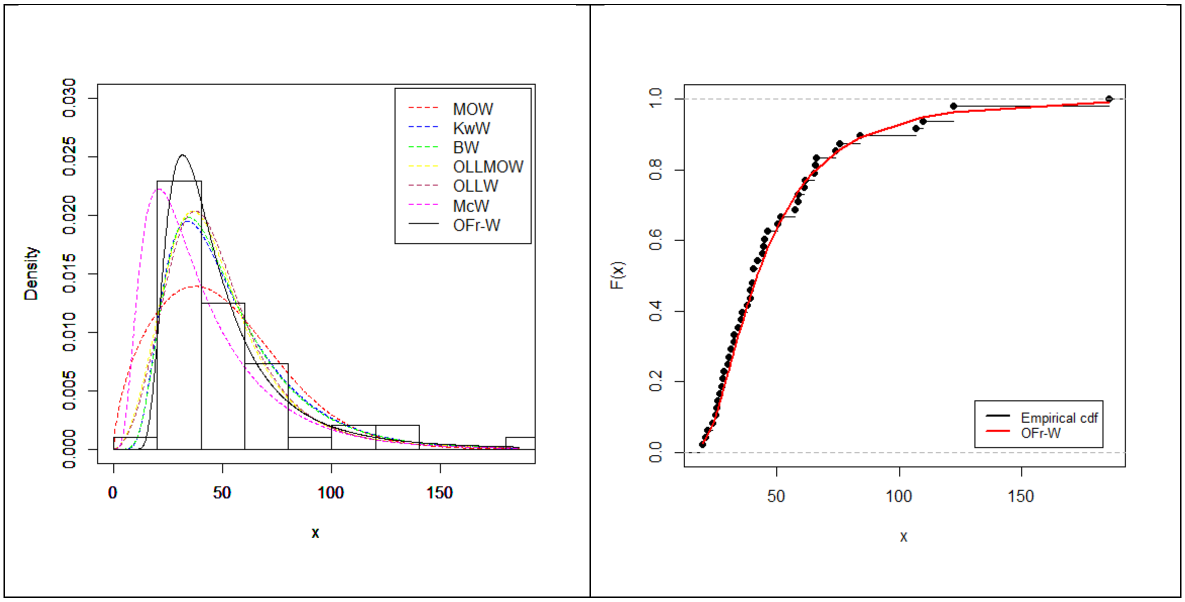


Figure 3: Fitted PDF and CDF for the First Data Set

Data 2: Actual Taxes data

The second data set consist of the monthly actual taxes revenue in Egypt from January 2006 to November 2010 [16]. It consists of the observations listed as: 5.9, 20.4, 14.9, 16.2, 17.2, 7.8, 6.1, 9.2, 10.2, 9.6, 13.3, 8.5, 21.6, 18.5, 5.1, 6.7, 17.0, 8.6, 9.7, 39.2, 35.7, 15.7, 9.7, 10.0, 4.1, 36.0, 8.5, 8.0, 9.2, 26.2, 21.9, 16.7, 21.3, 35.4, 14.3, 8.5, 10.6, 19.1, 20.5, 7.1, 7.7, 18.1, 16.5, 11.9, 7.0, 8.6, 12.5, 10.3, 11.2, 6.1, 8.4, 11.0, 11.6, 11.9, 5.2, 6.8, 8.9, 7.1, 10.8.

We fitted the OFr-L distribution in contrast with the exponential Lomax (EL), Marshal Olkin-Lomax (MOL), Kumaraswamy Lomax (KwL), Odd log-logistic Marshal Olkin Lomax (OLLMOL), Odd log-logistic Lomax (OLLL) and McDonald Lomax (McL) distributions. Based on the goodness of fit measures Table 4, we note that the OFr-L distribution provides the best fit. The fitted PDF, CDF plots are displayed in Figure 4. From these plots, we can also conclude that the OFr-L distribution is very suitable for these data.

Table 4: Estimated values for the second data

Model	Estimates			
OFrL	1.917880	21.323965	1.910503	-
EL	4.537816	11.232723	15.99319	-
MOL	6.035800	14.661110	30.31272	-
OLLL	0.428485	2.7669142	4.9719023	-
KwL	2.494748	5.479055	16.25659	1.618642
OLLMOL	0.367402	1.2489887	5.890851	5.386643

Table 5: Goodness of fit measures for first data set

Model	–2l	AIC	BIC	$A*$	$W*$
OFrL	**-376.818**	**382.8182**	**389.0508**	**0.232776**	**0.034398**
EL	-377.351	383.3510	389.5836	0.354634	0.062102
MOL	-391.268	397.2680	403.5007	1.252071	0.213027
OLLL	-380.045	386.0446	392.2772	0.513687	0.091913
KwL	-377.711	385.7111	394.0212	0.373264	0.065615
OLLMOL	-379.950	387.9502	396.2604	0.513275	0.091878

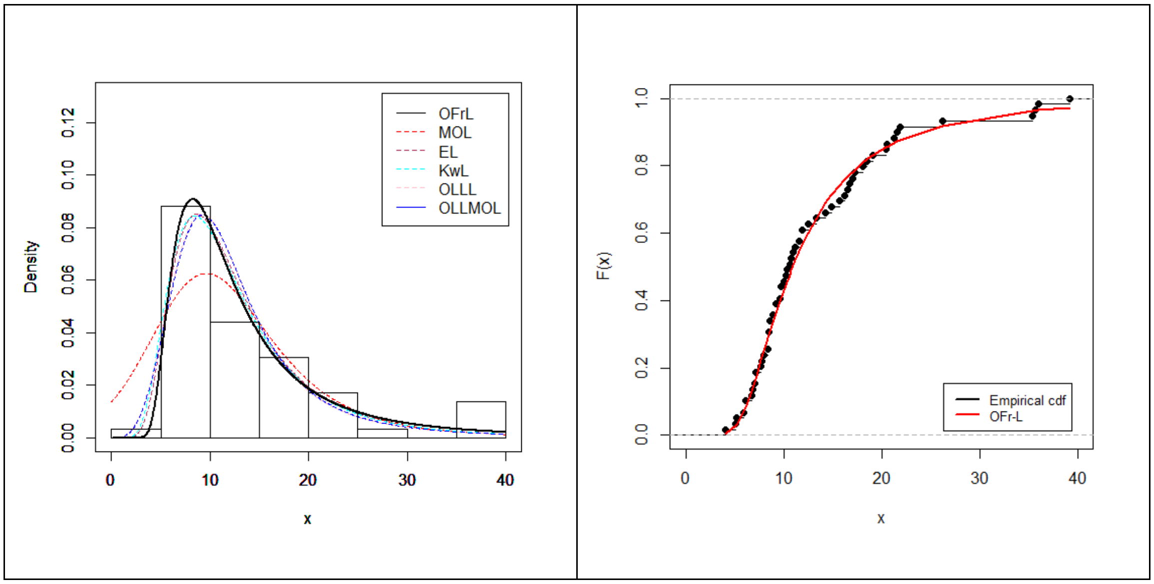


Figure 4: Fitted PDF and CDF for the Second Data Set

7. Conclusion

In this paper, we propose the Odd Generalized Frèchet-G family. We study some mathematical properties, including expansion of the density function and order statistics. Parameters of the OFr-W distribution are estimated using maximum likelihood method. Further, we fit two special models of the proposed family to real data sets to demonstrate the usefulness of the new family. These special models provide consistently better fit than other competing models. It is clear that from tables and figures the derived family provides a better fit than existing models.

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