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Kumaraswamy Half-Logistic Distribution: Properties and Applications

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Abstract: This study elucidates a three-parameter probabilistic model generalized from Kumaraswamy family using half logistic distribution as a baseline model named as Kumaraswamy half logistic distribution. The properties of the observed model are also explored. Further, we explain the behavior of failure rate, cumulative failure rate, and survival rate functions. Monte Carlo simulation study is being conducted to estimate the parameters under ML estimation method. Moreover, two practical applications illustrate the flexibility and better fit of the observed model.

Keywords: Kumaraswamy-G, Half-logistic distribution, Order statistics, ML estimation

1 Introduction

Probability models are frequently used for the prediction of lifetime products in various fields of applied sciences. These models are also used to explain the failure rate and survival rate of the certain product. Therefore, many generalizations are formed by adding additional shape parameters to increase the flexibility of these probabilistic models. Many generalized families of distributions have formed from last few decades such as Macdonald-G family [1], Exponentiated Exponential-T family [2], Beta-G family [3], Marshall-Olkin-G family [4], Exponentiated Generalized family [5], Weibull-G family [6], Beta Marshall-Olkin family [7], Kumaraswamy Marshall Olkin G family [8], Transmuted Kumaraswamy family [9], Exponentiated Marshall Olkin G family [10] and many others. One of them is Kumaraswamy-G family of distribution was given by Cordeiro and Castro [11]. The cdf of the generalized form is

$$F(x) = 1 - \{1 - G^{a}(x,\xi)\}^{b}, \tag{1}$$

The density function (pdf) of the corresponding cdf is

$$f(x) = \alpha b g(x,\xi)\,[G(x,\xi)]^{a-1}\{1 - G^{a}(x,\xi)\}^{b-1} \tag{2}$$

where $\alpha > 0$ and $b > 0$ are two additional shape parameters while ξ represents the parameters of base line distribution.

Since Half-Logistic distribution is formed by [12] using the absolute transformation of the logistic distribution, therefore, having much importance in statistics, physics, hydrology and logistic regression. Moreover, this distribution is highly considered in modeling datasets of numerous areas. The pdf and cdf of half-logistic distribution is respectively

$$g(x) = \frac{2\lambda e^{-\lambda x}}{(1+e^{-\lambda x})^{2}}; \qquad x > 0, \lambda > 0, \tag{3}$$

$$G(x) = \frac{1-e^{-\lambda x}}{1+e^{-\lambda x}}, \tag{4}$$

where λ is its shape parameter.

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Many generalized models have formed for the modeling of data sets by using observed generator such as Kumaraswamy Weibull [13], Kumaraswamy Gumbel [14], Kumaraswamy Birnbaum-Saunders [15], Kumaraswamy Pareto [16], Kumaraswamy generalized Rayleigh [17], Kumaraswamy inverse Rayleigh [18], Kumaraswamy modified inverse Weibull [19], Kumaraswamy Laplace [20], Kumaraswamy exponential-Weibull [21], Kumaraswamy exponentiated inverse Rayleigh [22] and many other distributions exists in the literature. The main purpose of this study to introduce a generalized form of half-logistic distribution and describes its flexible behavior. The rest of study contain following divisions such as Section 2 explain the behavior of pdf and cdf for observed distribution, its hazard and survival rates including limiting behavior of the observed model. Section 3 explains some properties of Kw-HL distribution such as moment, generating function, and incomplete moments, random number generator and quartile function, entropies and order statistics. Section 4 contains Monte Carlo simulation study for the estimation of parameters by maximum likelihood estimates (MLE). Section 5 illustrates the real life application and flexibility of model as compared to other models. The whole study is being concluded in section 6.

2 The Kw-HL Distribution

If X belongs to Half Logistic distribution with parameter $\lambda > 0$ cumulative distribution function (cdf) of Kw-HL distribution can be obtained by inserting (4) in (1).

$$F(x;a,b,\lambda) = 1-\left\{1-\left(\frac{1-e^{-\lambda x}}{1+e^{-\lambda x}}\right)^{a}\right\}^{b} \tag{5}$$

Its corresponding pdf is

$$f(x;a,b,\lambda) = \frac{2\alpha b\lambda e^{-\lambda x}}{(1+e^{-\lambda x})^2}\left[\frac{1-e^{-\lambda x}}{1+e^{-\lambda x}}\right]^{a-1}\left\{1-\left(\frac{1-e^{-\lambda x}}{1+e^{-\lambda x}}\right)^{a}\right\}^{b-1} \tag{6}$$

where $a > 0$, $b > 0$ and $\lambda > 0$ are shape parameters.

2.1 Some useful expansions

By using binomial expansion

$$(1-z)^n = \sum_{j=0}^{\infty}\binom{n}{j}(-1)^j z^j \qquad \text{for } z > 0$$

We can also express pdf as follows

$$f(x) = 2\alpha b\lambda \sum_{i,j,k=0}^{\infty}(-1)^{i+j}\binom{b-1}{i}\binom{a+ai-1}{j}\binom{-(a+ai+1)}{k}\left(e^{-\lambda x}\right)^{j+k+1}$$

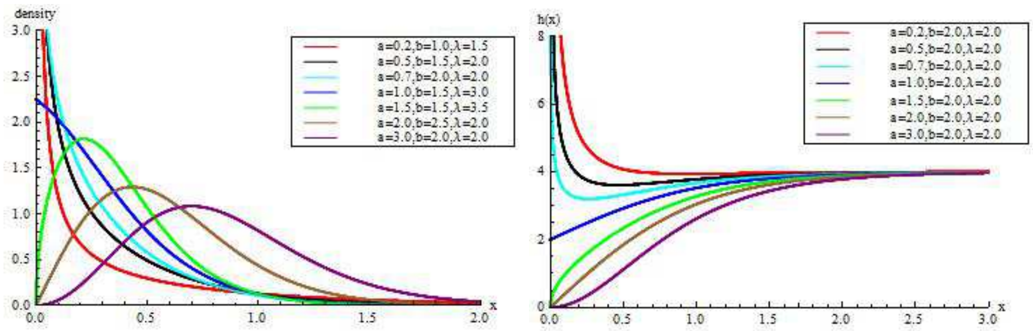
$$f(x) = 2\alpha b\lambda \sum_{i,j,k=0}^{\infty} w_{i,j,k}\left(e^{-\lambda x}\right)^{j+k+1} \tag{7}$$

where

$$w_{i,j,k} = (-1)^{i+j}\binom{b-1}{i}\binom{a+ai-1}{j}\binom{-(a+ai+1)}{k}$$

Eq. (8) represents the failure rate function for X variable of Kw-HL distribution.

$$h(x) = \frac{\frac{2\alpha b\lambda e^{-\lambda x}}{(1+e^{-\lambda x})^2}\left[\frac{1-e^{-\lambda x}}{1+e^{-\lambda x}}\right]^{a-1}}{1-\left(\frac{1-e^{-\lambda x}}{1+e^{-\lambda x}}\right)^{a}} \tag{8}$$



(a) PDF curves (b) hrf curves

Fig. 1: pdf & hrf curves at different selection of parameters

The cumulative hazard rate function and survival functions of Kw-HL model is

$$H(x) = -\log\left[1-\left(\frac{1-e^{-\lambda x}}{1+e^{-\lambda x}}\right)^{a}\right] \tag{9}$$

$$S(x) = 1-\left(\frac{1-e^{-\lambda x}}{1+e^{-\lambda x}}\right)^{a} \tag{10}$$

respectively.

Figure (1a) and (1b) represent the behavior of density function and hazard rate function of KW-HL distribution at a different combination of parameters. The observed model has bathtub failure rate.

2.2 Limiting behavior Kw-HL density and hazard rate functions

Lemma 1: For x approaches to origin, limits of Kw-HL density function is as follows

$$\lim_{x\to 0} f(x) = \begin{cases} \infty & \text{for } a<1 \\ \frac{ab\lambda a}{2^{a}} & \text{for } a=1 \\ 0 & \text{for } a>1 \end{cases} \tag{11}$$

Proof: As pdf of Kw-HL distribution is

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\left[\frac{2\alpha b\lambda e^{-\lambda x}}{(1+e^{-\lambda x})^{2}}\left[\frac{1-e^{-\lambda x}}{1+e^{-\lambda x}}\right]^{a-1}\left\{1-\left(\frac{1-e^{-\lambda x}}{1+e^{-\lambda x}}\right)^{a}\right\}^{b-1}\right]$$

The quantity

$$\lim_{x\to 0} e^{-\lambda x} \cong 1 \quad , \quad \lim_{x\to 0} 1+e^{-\lambda x} \cong 2 \quad \text{and} \quad \lim_{x\to 0}\left\{1-\left(\frac{1-e^{-\lambda x}}{1+e^{-\lambda x}}\right)^{a}\right\}^{b-1} \cong 1$$

The above expression becomes

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\left((2\alpha b\lambda/4)\left[\frac{1-e^{-\lambda x}}{2}\right]^{a-1}\right)$$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\left((2\alpha b\lambda/4)\left[\frac{1-\left(1-\lambda x+\frac{(\lambda x)^{2}}{2!}-\frac{(\lambda x)^{3}}{3!}+\ldots\right)}{2}\right]^{a-1}\right)$$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\left(\left(\frac{\alpha b\lambda}{2}\right)(\lambda x)^{a-1}\left[\frac{1-\frac{\lambda x}{2!}+\frac{(\lambda x)^2}{3!}-\ldots}{2}\right]^{a-1}\right)$$

Now results can be formed easily.

Lemma 2: For x approaches to origin, limits of Kw-HL hazard rate is as follows

$$\lim_{x\to 0} h(x) = \begin{cases} \infty & \text{for} \quad a<1 \\ \frac{ab\lambda a}{2^a} & \text{for} \quad a=1 \\ 0 & \text{for} \quad a>1 \end{cases} \tag{12}$$

Proof: The distribution function of Kw-HL distribution is

$$\lim_{x\to 0} h(x) = \lim_{x\to 0}\frac{\frac{2\alpha b\lambda e^{-\lambda x}}{(1+e^{-\lambda x})^2}\left[\frac{1-e^{-\lambda x}}{1+e^{-\lambda x}}\right]^{a-1}}{1-\left(\frac{1-e^{-\lambda x}}{1+e^{-\lambda x}}\right)^a}$$

The results are straightforward from the equation

2.3 Some special models of Kw-HL

An approximation to other probabilistic models shows the flexibility of models for different assumed values. By considering the pdf of Kw-HL distribution from Eq. (6) here we present some special cases of observed model.

–If $a=1$ and $b=1$, Kw-HL converts into half logistic distribution $HL(\lambda)$.
–If $b=1$, observed distribution converts into exponentiated half logistic $EHL(a,\lambda)$.
–If $a=1$, observed distribution becomes generalized half logistic $GHL(a,\lambda)$.

3 Properties of Kw-HL Distribution

This section particularizes some structural quantities of Kw-HL distribution with algebraic expressions. These algebraic expressions are found more efficient to express statistical measures instead of direct integration of density function.

3.1 Moments

Theorem: Let X is a r.v belong to Kw-HL distribution with three shape parameters a, b and $\lambda>0$. The r^{th} ordinary moment of proposed model is

$$\mu_r' = 2\alpha b\lambda\sum_{i,j,k=0}^{\infty}\frac{w_{i,j,k}}{(\lambda(1+j+k))^{r+1}}\Gamma(r+1) \tag{13}$$

Proof: By definition, r^{th} moment of a distribution function is

$$\mu_r' = E(X^r) = \int_0^{\infty} x^r f(x)\ dx$$

By using Eq. (7)

$$\mu_r' = 2\alpha b\lambda\sum_{i,j,k=0}^{\infty} w_{i,j,k}\int_0^{\infty} x^r\left(e^{-\lambda x}\right)^{j+k+1}dx$$

Put $\lambda x(1+j+k)=z$ and $x=\frac{z}{\lambda(1+j+k)}$, $dx=\frac{1}{\lambda(1+j+k)}dz$

$$= 2\alpha b\lambda \sum_{i,j,k=0}^{\infty} w_{i,j,k} \int_0^{\infty} \frac{z^r}{(\lambda(1+j+k))^{r+1}} e^{-x} dz$$

$$\mu_r' = 2\alpha b\lambda \sum_{i,j,k=0}^{\infty} \frac{w_{i,j,k}}{(\lambda(1+j+k))^{r+1}} \Gamma(r+1)$$

The proof follows.

3.2 Incomplete Moment

Theorem: Let a random variable X belongs to Kw-HL distribution then its r^{th} incomplete moment is

$$\phi(x) = 2\alpha b\lambda \sum_{i,j,k=0}^{\infty} \frac{w_{i,j,k}}{(\lambda(1+j+k))^{r+1}} \Gamma(r+1, \lambda x(1+j+k)) \tag{14}$$

Proof: r^{th} incomplete moment for variable X is defined as

$$\phi(x) = \int_0^x t^r f(t)\, dt$$

$$= 2\alpha b\lambda \sum_{i,j,k=0}^{\infty} w_{i,j,k} \int_0^x t^r \left(e^{-\lambda t}\right)^{j+k+1} dt$$

Put $\lambda t(1+j+k) = z$ and $t = \frac{z}{\lambda(1+j+k)}$, $dt = \frac{1}{\lambda(1+j+k)} dz$

$$= 2\alpha b\lambda \sum_{i,j,k=0}^{\infty} w_{i,j,k} \int_0^{\lambda x(1+j+k)} \frac{z^r}{(\lambda(1+j+k))^{r+1}} e^{-z} dz$$

$$= 2\alpha b\lambda \sum_{i,j,k=0}^{\infty} \frac{w_{i,j,k}}{(\lambda(1+j+k))^{r+1}} \int_0^{\lambda x(1+j+k)} z^{r+1-1} e^{-z} dz$$

Incomplete gamma function completes the proof.

3.3 Moment generating function

Theorem: Suppose r.v X have pdf of Kw-HL distribution described in Eq. (7) and $M(t)$ represents its moment generating function (mgf) as

$$M(t) = 2\alpha b\lambda \sum_{i,j,k,m=0}^{\infty} \frac{w_{i,j,k} t^m}{m!(\lambda(1+j+k))^{m+1}} \Gamma(m+1) \tag{15}$$

Proof: mgf for variable X is defined as

$$M(t) = E\left(e^{tx}\right) = \int_0^{\infty} e^{tx} f(x)\, dx$$

$$M(t) = 2\alpha b\lambda \sum_{i,j,k=0}^{\infty} w_{i,j,k} \int_0^{\infty} e^{tx} e^{-\lambda x(j+k+1)} dx$$

Since
$e^{tx} = \sum_{m=0}^{\infty} \frac{t^m}{m!} x^m$

$$M(t) = 2\alpha b\lambda \sum_{i,j,k,m=0}^{\infty} \frac{w_{i,j,k} t^m}{m!} \int_0^{\infty} x^m e^{-\lambda x(j+k+1)} dx$$

Put $\lambda x(1+j+k) = z$

$$M(t) = 2\alpha b\lambda \sum_{i,j,k,m=0}^{\infty} \frac{w_{i,j,k} t^m}{m!} \int_0^{\infty} \frac{z^m}{(\lambda(1+j+k))^{m+1}} e^{-z} dz$$

3.4 Quantile Function

If q belongs to uniform distribution with interval (0,1), then random variable $X = G(q)$ has density of Eq. (6). The quantile function of X is

$$x = -\frac{1}{\lambda} \log \left[\frac{1 - \left(1 - (1-q)^{\frac{1}{b}}\right)^{1/a}}{1 + \left(1 - (1-q)^{\frac{1}{b}}\right)^{1/a}} \right] \tag{16}$$

3.5 Skewness and Kurtosis

Skewness is used to measure the asymmetry and kurtosis is used to measure the peakedness of probabilistic models. Both measures are the descriptive measures of the shape of the probability distribution. Skewness and kurtosis can be easily determined by the following expressions based on first four mean moments calculated by Eq. (13).

$$\gamma_1(sk) = \frac{\mu_3}{\mu_2^{\frac{3}{2}}} \quad \text{and} \quad \beta_2 = \frac{\mu_4}{\mu_2^2} \tag{17}$$

3.6 Mean deviation

If X belongs to Kw-HL distribution, then we can measure the scattering of r.v X by the average deviation of observations from mean and median. It particularly knows as mean deviation about mean and mean deviation about the median. It is defined as

$$M.D(\bar{X}) = \int_0^{\infty} |x-\mu| f(x) dx \quad \text{and} \quad M.D(\tilde{X}) = \int_0^{\infty} |x-M| \ f(x) dx$$

respectively, where μ represent the expected value of random variable X and could be calculated from Eq. (13) while $M = -\frac{1}{\lambda} \log \left[\left(1 + \left(1-(1-q)^{1/b}\right)^{1/a}\right) / \left(1 + \left(1-(1-q)^{1/b}\right)^{1/a}\right) \right]$ is median of X. The measures $M.D(\bar{X})$ and $M.D(\tilde{X})$ can be calculated from

$$M.D(\bar{X}) = 2\mu F(\mu) - 2J(\mu) \quad \text{and} \quad M.D(\tilde{X}) = \mu - 2J(M)$$

where $J(t) = \int_0^t t f(t) dt$

Mean deviation is practically used to explain the behavior of Bonferroni and Lorenz curves. Mostly, these curves are applied theoretically in many fields such as economics, reliability, demography, insurance, and medicine [13].

3.7 Entropies

The entropy of a random variable X is the measure of variation of the uncertainty. A common measure of entropy is Rényi entropy.

3.7.1 Rényi Entropy

Theorem: If the random variable X is defined as Eq. (5), then the Rényi entropy is given by

$$I_R(\delta) = \frac{\delta \log 2}{1-\delta} + \frac{\delta \log a}{1-\delta} + \frac{\delta \log b}{1-\delta} - \log \lambda + \frac{1}{1-\delta} \log \left[\sum_{i,j,k=0}^{\infty} \frac{y_{i,j,k}}{j+k+\delta} \right] \tag{18}$$

Proof: If r.v X belong to Kw-HL distribution then by definition, Rényi entropy is

$$I_R(\delta) = \frac{1}{1-\delta} \log [I(\delta)]$$

where $\delta > 0$ and $\delta \neq 1$.

$$I(\delta) = \int_0^\infty f^\delta(x)\, dx$$

$$I(\delta) = \int_0^\infty \frac{2^\delta \alpha^\delta b^\delta \lambda^\delta e^{-\lambda\delta x}}{(1+e^{-\lambda x})^{2\delta}} \left[\frac{1-e^{-\lambda x}}{1+e^{-\lambda x}} \right]^{\delta(a-1)} \left\{ 1 - \left(\frac{1-e^{-\lambda x}}{1+e^{-\lambda x}} \right)^a \right\}^{\delta(b-1)} dx$$

On simplification the final expression becomes

$$I(\delta) = 2^\delta \alpha^\delta b^\delta \lambda^{\delta-1} \left[\sum_{i,j,k=0}^{\infty} \frac{y_{i,j,k}}{j+k+\delta} \right]$$

$$w_{i,j,k} = (-1)^{i+j} \binom{\delta(b-1)}{i} \binom{\delta(a-1)+ai}{j} \binom{-\delta(a+1)-ai}{k}$$

So using this expression in $I_R(\delta)$, the result follows.

3.7.2 q-Entropy

The q-entropy (Hq) is defined by

$$H_q = \frac{1}{q-1} \log(1-(1-q) I_R(\delta))$$

By putting the above expression of $I_R(\delta)$ in above equation, we obtain

$$= \frac{1}{q-1} \log \left(1-(1-q) \left\{ 2^\delta \alpha^\delta b^\delta \lambda^{\delta-1} \left[\sum_{i,j,k=0}^{\infty} \frac{y_{i,j,k}}{j+k+\delta} \right] \right\} \right) \tag{19}$$

3.8 Order Statistics

Let $X_1, X_2, X_3, ...X_n$ be a random sample and its ordered values are denoted as $X_{(1)}, X_{(2)}, X_{(3)}, ...X_{(n)}$. The probability density function (pdf) of order statistics is obtained using the below function

$$f_{s:n}(x) = \frac{n!}{(s-1)!(n-s)!}[F(x)]^{s-1}[1-F(x)]^{n-s}f(x)$$

The density of the nth ordered statistics follows the Kw-HL distribution is derived as follow

$$f_{s:n}(x) = \frac{n!}{(s-1)!(n-s)!}\left[1-\left\{1-\left(\frac{1-e^{-\lambda x}}{1+e^{-\lambda x}}\right)^{a}\right\}^{b}\right]^{s-1}\left[\left\{1-\left(\frac{1-e^{-\lambda x}}{1+e^{-\lambda x}}\right)^{a}\right\}^{b}\right]^{n-s}$$
$$\frac{2\alpha b\lambda e^{-\lambda x}}{(1+e^{-\lambda x})^2}\left[\frac{1-e^{-\lambda x}}{1+e^{-\lambda x}}\right]^{a-1}\left\{1-\left(\frac{1-e^{-\lambda x}}{1+e^{-\lambda x}}\right)^{a}\right\}^{b-1}$$

3.9 Maximum Likelihood Estimates

Since maximum likelihood estimators give the maximum information about the population parameters, therefore, this section presents the maximum likelihood estimates (MLEs) of the parameters that are inherent within the Kw-HL distribution function is given by the following: Let $X_1, ..., X_n$ be random variables of the Kw-HL distribution of size n. Then sample likelihood function of Kw-HL is obtained as

$$L(x_1, x_2, ...x_n; a, b, \lambda) = \prod_{i=1}^{n} f(x) = 2^n \alpha^n b^n \lambda^n \prod_{i=1}^{n} e^{-\lambda x}\frac{\left[1-e^{-\lambda x}\right]^{\alpha-1}}{\left[1+e^{-\lambda x}\right]^{a+1}}\left\{1-\left(\frac{1-e^{-\lambda x}}{1+e^{-\lambda x}}\right)^{a}\right\}^{b-1}$$

Log-likelihood function is $\Phi = \log[L(x_1, x_2, ...x_n; a, b, \lambda)]$

$$\Phi = n\log 2 + n\log a + n\log b + n\log\lambda - \lambda\sum x + (\alpha-1)\sum\log\left[1-e^{-\lambda x}\right] - (\alpha+1)\sum\log\left[1+e^{-\lambda x}\right]$$
$$+(b-1)\sum\log\left[1-\left(\frac{1-e^{-\lambda x}}{1+e^{-\lambda x}}\right)^{a}\right]$$

Therefore, The MLE's of parameters (a, b and λ) which maximize the above log-likelihood function must satisfy the normal equations. We take the first derivative of the above log-likelihood equation with respect to parameters and equate to zero respectively.

$$J_a = \frac{\partial L}{\partial a} = \frac{n}{a} + \sum\log\left[1-e^{-\lambda x}\right] - \sum\log\left[1+e^{-\lambda x}\right] + (b-1)\sum\frac{-\left(\frac{1-e^{-\lambda x}}{1+e^{-\lambda x}}\right)^{a}\log\left[\frac{1-e^{-\lambda x}}{1+e^{-\lambda x}}\right]}{\left[1-\left(\frac{1-e^{-\lambda x}}{1+e^{-\lambda x}}\right)^{a}\right]} = 0 \quad (20)$$

$$J_b = \frac{\partial L}{\partial b} = \frac{n}{b} + \sum\log\left[1-\left(\frac{1-e^{-\lambda x}}{1+e^{-\lambda x}}\right)^{a}\right] = 0 \quad (21)$$

$$J_\lambda = \frac{\partial L}{\partial \lambda} = \frac{n}{\lambda} - \sum x + (\alpha-1)\sum\frac{\lambda e^{-\lambda x}}{1-e^{-\lambda x}} + (\alpha+1)\sum\frac{\lambda e^{-\lambda x}}{1+e^{-\lambda x}} - (b-1)\sum\frac{a\left(\frac{1-e^{-\lambda x}}{1+e^{-\lambda x}}\right)^{a-1}2\lambda e^{-\lambda x}}{\left[1-\left(\frac{1-e^{-\lambda x}}{1+e^{-\lambda x}}\right)^{a}\right]\left[1+e^{-\lambda x}\right]^2} = 0 \quad (22)$$

Since the above derived equations are in the complex form, therefore the exact solution of ML estimator for unknown parameters is not possible. So it is convenient to use nonlinear Newton Raphson algorithm for exact numerically solution to maximize the above likelihood function.

Table 1: Mean estimates, bias and MSE of Estimated parameters

a	b	λ	Sample size	Parameter	Mean	Bias	MSE
1.5	1.5	2	50	a	1.548	0.048	0.077
				b	1.524	0.024	0.049
				λ	2.083	0.083	0.239
			100	a	1.514	0.014	0.023
				b	1.509	0.009	0.016
				λ	2.029	0.029	0.066
			150	a	1.507	0.007	0.011
				b	1.505	0.005	0.007
				λ	2.015	0.015	0.030
2.5	2	2	50	a	2.611	0.111	0.313
				b	2.039	0.039	0.087
				λ	2.196	0.196	0.929
			100	a	2.534	0.034	0.083
				b	2.01	0.01	0.027
				λ	2.049	0.049	0.124
			150	a	2.516	0.016	0.04
				b	2.006	0.006	0.014
				λ	2.027	0.027	0.058
3.5	2.5	2	50	a	3.673	0.173	0.754
				b	2.548	0.048	0.138
				λ	2.433	0.433	136.1
			100	a	3.548	0.048	0.199
				b	2.518	0.018	0.043
				λ	2.09	0.09	0.221
			150	a	3.525	0.025	0.092
				b	2.508	0.008	0.021
				λ	2.041	0.041	0.091

4 Data Analysis

4.1 Simulation study

This section compares the parameters for different sample sizes at different combination of parameters on the basis of bias and MSE of Kw-HL distribution. We generate 10,000 samples by using Monte Carlos simulation. All the algorithms are coded in R language. We calculate ML estimates for a,b and λ based on generated samples. Mean of these estimates with bias and MSE are represented in the table below.

The values in Table 1 indicate that the MSE of ML estimators of a,b and λ decreases and their biases reduce towards 0 as sample size increases. While the increase in shape parameters, bias and MSE of estimated parameters increases.

4.2 Applications

We applied two data sets to illustrate the usefulness of the proposed model. The first data set was reported by [23]. The data represents the survival time of 72 infected with virulent tubercle bacilli. The data are as follows: 0.1, 0.33, 0.44, 0.56, 0.59, 0.72, 0.74, 0.77, 0.92, 0.93, 0.96, 1.00, 1.00, 1.02, 1.05, 1.07, 0.7, 0.08, 1.08, 1.08, 1.09, 1.12, 1.13, 1.15, 1.16, 1.20, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34, 1.36, 1.39, 1.44, 1.46, 1.53, 1.59, 1.60, 1.63, 1.63, 1.68, 1.71, 1.72, 1.76, 1.83, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16, 2.22, 2.30, 2.31, 2.40, 2.45, 2.51, 2.53, 2.54, 2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32, 4.58, 5.55.

The second data set was originally used by [24]. Data consists of 30 observations of March precipitation (in inches) in Minneapolis/St Paul. The observations are: 0.77, 1.74, 0.81, 1.20, 1.95, 1.20, 0.47, 1.43, 3.37, 2.20, 3.00, 3.09, 1.51, 2.10, 0.52, 1.62, 1.31, 0.32, 0.59, 0.81, 2.81, 1.87, 1.18, 1.35, 4.75, 2.48, 0.96, 1.89, 0.90, 2.05. The summary statistics of the data sets is given in Table 2.

The comparison of the Kumaraswamy Half Logistic distribution is being made with Kumaraswamy Logistic (Kw-L), Beta Exponential (BE), Beta Weibull (BW), Type II Half Logistic Weibull (TIIHLW) and Exponentiated Half Logistic (EHL) distribution. The numerous accuracy measures including Akeike Info Criterion (AIC), Bayes Info Criterion (BIC),

Table 2: Descriptive Statistics for data sets

Data	Min.	Q1	Median	Q3	Mean	Max.
Set 1	0.080	1.080	1.560	2.302	1.837	7.000
Set 2	0.320	0.915	1.470	2.088	1.675	4.750

likelihood (L), Anderson-Darling test (A^*) and Cramer-von Mises test (W^*) are being calculated. The density functions of other existing distribution are given as follows:

Beta Exponential Distribution

$$f(x;a,b,\lambda) = \frac{\lambda}{\text{Beta}[a,b]} e^{-b\lambda x}\left(1-e^{-\lambda x}\right)^{a-1}$$

Beta Weibull Distribution

$$f(x;a,b,\gamma,\lambda) = \frac{\lambda^{\gamma}\gamma}{\text{Beta}[a,b]} e^{-(\lambda x)^{\gamma}}\left(1-e^{-(\lambda x)^{\gamma}}\right)^{a-1}\left(1-\left(1-e^{-(\lambda x)^{\gamma}}\right)\right)^{b-1} x^{\gamma-1}$$

Type II Half Logistic Weibull Distribution

$$f(x;\lambda,\delta,\gamma) = 2\lambda\delta\gamma \frac{x^{\gamma-1}e^{-\delta x^{\gamma}}\left(1-e^{-\delta x^{\gamma}}\right)^{\lambda-1}}{\left(1+(1-e^{-\delta x^{\gamma}})^{\lambda}\right)^{2}}$$

Exponentiated Half Logistic Distribution

$$f(x;\lambda,\theta) = \frac{2\theta\lambda\, e^{-\lambda x}}{(1+e^{-\lambda x})^{2}}\left(\frac{1-e^{-\lambda x}}{1+e^{-\lambda x}}\right)^{\theta-1}$$

Table 3: ML estimates for survival time of infected guinea pigs.

$EHL(\lambda,\gamma)$	1.09538	1.99161	-	-
$KwHL(a,b,\lambda)$	1.61665	$3.82948{*}10^{7}$	0.0000198	-
$BE(a,b,\lambda)$	2.55961	1.77383	0.58859	-
$TIIHLW(\lambda,\gamma,\delta)$	1.66869	1.41133	0.379459	-
$BW(a,b,\lambda,\gamma)$	1.94921	0.726736	0.906026	1.20367

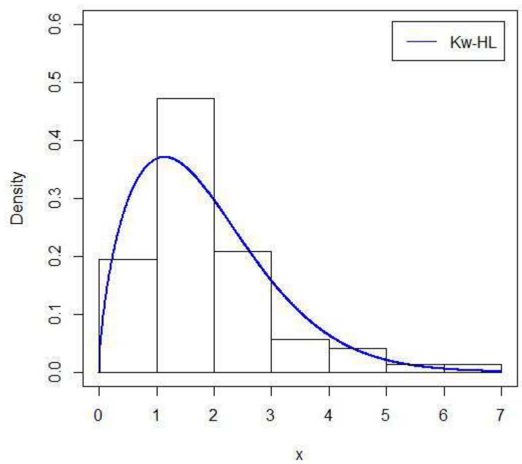
Table 4: ML estimates for March precipitation data

$EHL(\lambda,\gamma)$	1.28559	2.35058	-
$BE(a,b,\lambda)$	3.214041	1.58825	0.803263
$TIIHLW(\lambda,\gamma,\delta)$	1.82277	1.4654	0.450975
$KwHL(a,b,\lambda)$	1.80925	$2.46749{*}10^{8}$	0.0000243

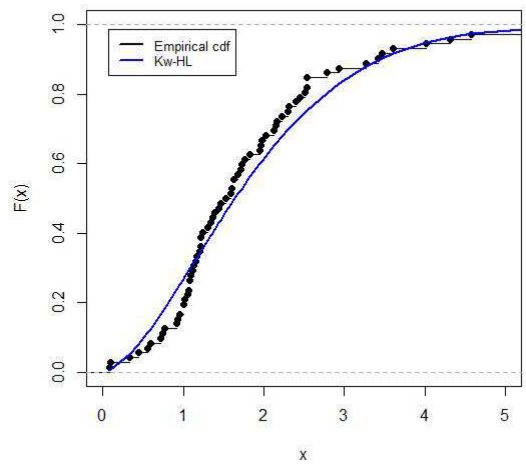
Model Selection The model selection is carried by using goodness of fit measures including maximized log-likelihood ($\hat{l}$), Akaike information criterion (AIC), Bayesian information criterion (BIC), Anderson Darling test (A_0^*) and Cramer Von Mises (W_0^*). Using these goodness of fit criteria findings of Table 5 and 6 shows that proposed model give superior fit than other models.

5 Conclusion

The study introduces a new generalization of half logistic distribution named as Kumaraswamy half logistic distribution and elaborates explicit expression for its fundamental properties. The study also explains the behavior of estimated parameters by using Monte Carlos simulation approach. Two real life applications have also presented for explaining the better fit of the observed model as compared to some existing models.

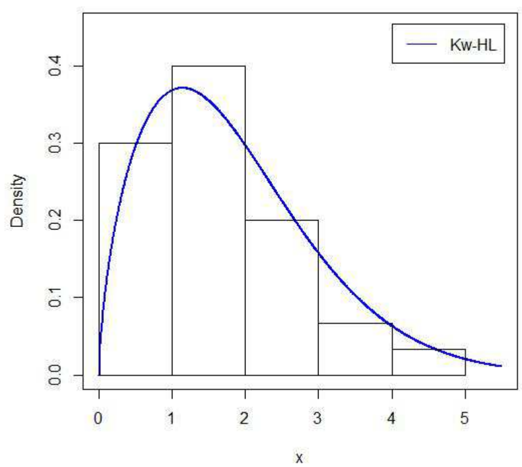


(a) PDF curves

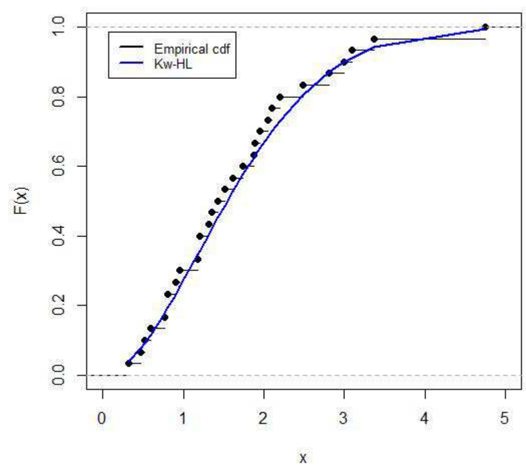


(b) CDF curves

Fig. 2: The fitted pdf and cdf of the Kw-HL distribution for first dataset



(a) PDF curves



(b) CDF curves

Fig. 3: The fitted pdf and cdf of the Kw-HL distribution for second dataset

Table 5: Some statistics for models fitted to survival time of infected guinea pigs.

Model	AIC	BIC	L	A^*	W^*
$\mathrm{EHL}(\lambda,\gamma)$	208.387	212.94	-102.194	0.59843	0.08869
$\mathrm{K}w\mathrm{HL}(a,b,\lambda)$	114.223	121.053	-54.1115	0.25968	0.06352
$\mathrm{BE}(a,b,\lambda)$	211.834	218.664	-102.917	0.70161	0.10251
$\mathrm{TIIHL}W(\lambda,\gamma,\delta)$	211.156	217.986	-102.578	0.64029	0.09285
$\mathrm{B}W(a,b,\lambda,\gamma)$	213.590	222.697	-102.795	0.69309	0.10284

Table 6: Some statistics for models fitted to March precipitation data

Model	AIC	BIC	L	A^*	W^*
$\mathrm{EHL}(\lambda,\gamma)$	80.4299	83.2323	-38.2149	0.116806	0.0153267
$\mathrm{BE}(a,b,\lambda)$	82.1707	86.3743	-38.0854	0.109457	0.0147817
$\mathrm{TIIHLW}(\lambda,\gamma,\delta)$	82.2453	86.4489	-38.1226	0.111972	0.0152154
$\mathrm{K}w\mathrm{HL}(a,b,\lambda)$	41.6990	45.9026	-17.8495	0.167878	0.0111148

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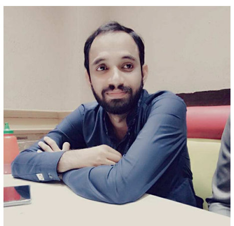

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