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Testing ODL_{mgf} Class of Life Distributions based on U-test

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Abstract: New test-statistic for testing exponentiality against overall decreasing life class of life distributions in moment generating function (ODL_{mgf}), based on U-statistic is studied. Selected critical values are tabulated for sample size $n = 5(5)50$. The Pitman asymptotic relative efficiencies $(PARE)$ are studied for other testes. The power of the test is estimated by simulation. The problem in case of right-censored data is also handled, and the power estimates of this test are also simulated for some commonly used distributions in reliability. Finally, real data are given to elucidate the use of the proposed test statistic for complete and incomplete data in the reliability analysis.

Keywords: life distributions, ODL_{mgf} aging class , U-statistic, asymptotic normality, efficiency, Monte Carlo method, power and censored data.

1 Introduction and Motivation

Over the past few decades, the idea of the measurement of success or failure concerning reliability has developed, leading scientists to use this advantage to establish new branches of reliability such as classes of life distributions. Many statisticians and reliability analysts have shown great interest in modeling survival data using classification of life distributions based on different aspects of aging concepts that describe how a population of units or system improved or deteriorates with age. The applications of classes of life distributions can be seen in reliability, engineering, biological science, maintenance and biometrics.

Note that the exponential distribution forms the backbone of statistical reliability theory and maintenance modeling see for example Barlow and Proschan (1981) and Zacks (1992). We mention for the common classes of life distributions that contains most of previously known classes like increasing failure rate (IFR), increasing failure rate average (IFRA), new better than used (NBU), new better than used in expectation (NBUE) and decreasing mean residual life (DMRL). For some properties and interrelationships of these criteria we refer to Bryson and Siddiqui (1969).

Abouammoh et al. (2000) introduced the NRBU, RNBU, NRBUE, HNRBUE classes of life distributions and studied the relation between them. Abouammoh and Khalique (1998) investigated a test statistic for testing exponentiality versus NRBU based on total time of test (TTT)-transform empirically. Mahmoud, et al. (2002) investigated the test statistic for NRBU based on U-Statistic. Diab et al. (2005) discussed a test statistic of RNBU by using U-test. Hendi and Abouammoh (2001) investigated the two test statistics for testing exponentiality versus NBRUE and HNBRUE classes of life distribution based on U-Statistic, and Abu-Youssef (2002) among others.

In reliability theory, ageing life is usually characterized by a nonnegative random variable $X \geq 0$ with cumulative distribution function (cdf) F and survival function (sf) $\overline{F}$ =$1-F$. For any random variable X, let

$$X_t = [X-t|X>t], \quad t \in \{x : F(x) < 1\},$$

denote a random variable whose distribution is the same as the conditional distribution of $X-t$ given that $X>t$. When X is the lifetime of a device, X_t can be regarded as the residual lifetime of the device at time t, given that the device has

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survived up to time t. Its survival function is (see, for instance, Deshpande et al (1986))

$$\overline{F_t}(x) = \frac{\overline{F}(t+x)}{\overline{F}(t)}, \quad \overline{F}(t) > 0,$$

where $\overline{F}(x)$ is the survival function of X.

X_t converges weakly to a nonnegative random variable $\widetilde{X}$ with survival function,

$$\overline{W}_F(x) = \frac{1}{\mu}\,\overline{V}(x) \text{ where } \overline{V}(x) = \int_x^\infty \overline{F}(u)du\,, \quad x \geq 0.$$

Where $\mu = \int_0^\infty \overline{F}(u)\,du$.

From the above discussion, we see that there are three random quantities related to life and these are the life itself X, the random residual life X_t, and the equilibrium life $\widetilde{X}$, Hence it would be of interest to compare the residual life X_t to its equilibrium form $\widetilde{X}$.

Formally, if X and Y are two random variables with distributions F and G (survival functions are $\overline{F}$ and $\overline{G}$), respectively, then

we say that X is smaller than Y in the moment generating function ordering (denoted by $X \leq_{mgf} Y$) if and only if

$$\int_0^\infty e^{\lambda x}\overline{F}(x)\,dx \leq \int_0^\infty e^{\lambda x}\overline{G}(x)\,dx, \qquad \text{for all } \lambda > 0.$$

Sepehrifar et al. (2012) defined overall decreasing life class ODL and investigated the probabilistic characteristics of this class of life distribution.

Definition 1.1:A life distribution F on $(0,\infty)$, with $F(0^-) = 0$ is called overall decreasing life (ODL), if

$$\int_x^\infty \overline{W}(t)dt \leq \mu\overline{W}(x), \qquad x \geq 0,$$

where $\mu = \int_0^\infty \overline{F}(u)du < \infty$.

We define the following new definition.

Definition 1.2. We say that X is overall decreasing life class of life distributions in moment generating function order (ODL_{mgf}), if $E(X_t) \leq_{mgf} E\left(\widetilde{X}\right)$

This definition means that, $\left(X \in ODL_{mgf}\right)$, if, and only if,

$$\int_0^\infty e^{\lambda x}\int_x^\infty \overline{W}(t)dtdx \leq \mu \int_0^\infty e^{\lambda x}\overline{W}(x)dx.$$

The corresponding dual class of life distribution is overall increasing life class of life distributions in moment generating function order and is denoted by (OIL_{mgf}).

The rest of the article is structured as follows. In Section 2, we present a test statistic based on a U-Statistic for testing $H_0 : F$ is exponential against $H_1 : F$ is ODL_{mgf} and not exponential, the Pitman asymptotic relative efficiencies (PARE) are calculated. In Section 3 Monte Carlo null distribution critical points are simulated for sample sizes $n = 5(5)50$ and the power estimates of this test are also calculated for some common alternatives distribution followed by some numerical example. In section 4, we dealing with right-censored data and selected critical values are tabulated, the power estimates for censor data of this test are tabulated also we discuss some applications to elucidate the usefulness of the proposed test in reliability analysis for censored data.

2 Testing Against ODL_{mgf} Class

In this section, we test the null hypothesis $H_0 : F$ is exponential with mean μ against $H_1 : F$ is ODL_{mgf} and not exponential. The following lemma is needed.

Lemma 2.1. If F belongs to ODL_{mgf} class and X is a random variable with distribution function F then

$$\Delta(\lambda) = \frac{1}{\lambda^2}\mu\phi(\lambda) - \frac{1}{\lambda^3}\phi(\lambda) + \frac{1}{2\lambda}\mu_{(2)} - \frac{1}{\lambda}\mu^2 + \frac{1}{\lambda^3} \succcurlyeq 0,$$

where $\phi(\lambda) = E\left(e^{\lambda x}\right) = \int_0^\infty e^{\lambda x} dF(x)$, and $\mu_{(2)} = 2\int_0^\infty x\overline{F}(x)\,dx$.

Proof: Since F is ODL_{mgf} we have

$$\int_0^\infty e^{\lambda x} \int_x^\infty \overline{W}(t)dtdx \le \mu \int_0^\infty e^{\lambda x}\overline{W}(x)dx, \tag{1}$$

we can write,

$$\begin{aligned} R.H.S &= \mu \int_0^\infty e^{\lambda x}\overline{W}(x)dx, \\ &= \int_0^\infty e^{\lambda x} \int_x^\infty \overline{F}(u)dudx, \\ &= \frac{1}{\lambda^2}\phi(\lambda) - \frac{1}{\lambda}\mu - \frac{1}{\lambda^2}. \end{aligned} \tag{2}$$

On the other hand,

$$\begin{aligned} L.H.S &= \int_0^\infty e^{\lambda x} \int_x^\infty \overline{W}(t)dtdx, \\ &= \frac{1}{\lambda\mu}\left[\int_0^\infty e^{\lambda x} \int_x^\infty \overline{F}(u)dudx - \int_0^\infty \int_x^\infty \overline{F}(u)dudx\right] \\ &= \frac{1}{\lambda^3\mu}\phi(\lambda) - \frac{1}{2\lambda\mu}\mu_{(2)} - \frac{1}{\lambda^3\mu} - \frac{1}{\lambda^2}. \end{aligned} \tag{3}$$

Substituting from (2) and (3) into (1), we get

$$\frac{1}{\lambda^3}\phi(\lambda) - \frac{1}{2\lambda}\mu_{(2)} - \frac{1}{\lambda^3} \le \frac{1}{\lambda^2}\mu\phi(\lambda) - \frac{1}{\lambda}\mu^2$$

which completes the proof.

The test presented here depends on a sample $X_1, X_2, ..., X_n$ from a population with distribution F. Using the previous Lemma we use $\Delta(\lambda)$ as a measure of departure from exponentiality where

$$\Delta(\lambda) = \int_0^\infty \left[\frac{1}{\lambda^3}(\lambda\mu - 1)\phi(\lambda) + \frac{1}{2\lambda}\mu_{(2)} - \frac{1}{\lambda}\mu^2 + \frac{1}{\lambda^3}\right] dF(t)$$

2.1 Empirical Test Statistic for ODL_{mgf} Alternative

To estimate $\Delta(\lambda)$ let $X_1, X_2, ..., X_n$, be a random sample from F. It is noted that , under $H_0 : \Delta(\lambda) = 0$,while under $H_1 : \Delta(\lambda) > 0$ The empirical estimate of $\Delta(\lambda)$ can be written as

$$\widehat{\Delta}_n(\lambda) = \frac{1}{n(n-1)(n-2)} \sum_{i=1}^{n} \sum_{j=1}^{n-1} \sum_{k=1}^{n-2} \left\{\frac{1}{\lambda^3}(\lambda X_j - 1)e^{\lambda X_k} + \frac{1}{2\lambda}X_j^2 - \frac{1}{\lambda}X_jX_k + \frac{1}{\lambda^3}\right\} \tag{4}$$

To make the test $\widehat{\Delta}_n(\lambda)$ scale invariant, let $\widehat{\delta}_n(\lambda) = \frac{\widehat{\Delta}_n(\lambda)}{\overline{X}^2}$ where $\overline{X} = \frac{1}{n}\sum_{i=1}^n X_i$ is the sample mean. Then

$$\widehat{\delta}_n(\lambda) = \frac{1}{n(n-1)(n-2)\overline{X}^2} \sum_{i=1}^{n} \sum_{j=1}^{n-1} \sum_{k=1}^{n-2} \left\{\frac{1}{\lambda^3}(\lambda X_j - 1)e^{\lambda X_k} + \frac{1}{2\lambda}X_j^2 - \frac{1}{\lambda}X_jX_k + \frac{1}{\lambda^3}\right\}. \tag{5}$$

The following theorem summarizes the asymptotic normality of $\widehat{\delta}_n(\lambda)$

Theorem 2.1. As n $\to \infty$, $\sqrt{n}\left[\widehat{\delta}_n(\lambda) - \widehat{\Delta}_n(\lambda)\right]$ is asymptotically normal with mean 0 and variance

$$\begin{aligned} \sigma^2 = Var\Bigg[&\frac{1}{2\lambda^3(1-\lambda)}\Big\{\left(2\lambda - 4\lambda^2 + 4\lambda^3\right)X + \lambda\left(1 - \lambda^{2\text{'}}\right)X^2 - 2(1-\lambda)^2 e^{\lambda x} \\ &+ \left(2 - 4\lambda + 2\lambda^2 - 2\lambda^3\right)\Big\}\Bigg], \end{aligned}$$

under H_0 the variance tends to

$$\sigma^2 = \frac{2\lambda - 5}{(\lambda-1)^2(2\lambda-1)}, \quad \lambda > 0, \lambda \neq 1, \frac{1}{2}$$

Proof: Using standard U-Statistic theory see Lee (1989)

$$\sigma^2 = Var\{E[\phi(X_1,X_2,X_3)|X_1] + E[\phi(X_1,X_2,X_3)|X_2] + E[\phi(X_1,X_2,X_3)|X_3]\},$$

set,

$$\phi(X_1,X_2,X_3) = \frac{1}{\lambda^3}(\lambda X_1 - 1)e^{\lambda X_2} + \frac{1}{2\lambda}X_1^2 - \frac{1}{\lambda}X_1X_2 + \frac{1}{\lambda^3},$$

then,

$$\begin{aligned}\eta_1(X) &= E[\phi(X_1,X_2,X_3)|X_1],\\ &= \frac{1}{\lambda^3}\int_0^\infty (\lambda x - 1)e^{-(1-\lambda)x}dx + \frac{1}{2\lambda}X_1^2 - \frac{1}{\lambda}X_1\int_0^\infty xe^{-x}dx + \frac{1}{\lambda^3},\\ &= \frac{1-\lambda(1-\lambda)}{\lambda^2(1-\lambda)}X_1 + \frac{1}{2\lambda}X_1^2 - \frac{1}{\lambda^2(1-\lambda)},\end{aligned}$$

also,

$$\begin{aligned}\eta_2(X) &= E[\phi(X_1,X_2,X_3)|X_2],\\ &= \frac{1}{\lambda^3}e^{\lambda X_2}\int_0^\infty (\lambda x - 1)e^{-x}dx + \frac{1}{2\lambda}\int_0^\infty X^2e^{-x}dx - \frac{1}{\lambda}X_2\int_0^\infty xe^{-x}dx + \frac{1}{\lambda^3},\\ &= \frac{-(1-\lambda)}{\lambda^3}e^{\lambda X_2} - \frac{1}{\lambda}X_2 + \frac{1+\lambda^2}{\lambda^3},\end{aligned}$$

and,

$$\eta_3(X) = E[\phi(X_1,X_2,X_3)|X_3] = 0$$

Set,

$$\begin{aligned}\psi(X) &= \eta_1(X) + \eta_2(X) + \eta_3(X)\\ &= \frac{1}{2\lambda^3(1-\lambda)}\Big\{\left(2\lambda - 4\lambda^2 + 4\lambda^3\right)X + \lambda\left(1-\lambda^2\right)X^2 - 2(1-\lambda)^2e^{\lambda x}\\ &\quad + \left(2 - 4\lambda + 2\lambda^2 - 2\lambda^3\right)\Big\}.\end{aligned}$$

It is easy to show that $E[\psi(X)] = 0$, under H_0 and

$$\begin{aligned}\sigma^2 &= Var[\psi(X)]\\ &= Var\left[\frac{1}{2\lambda^3(1-\lambda)}\Big\{\left(2\lambda - 4\lambda^2 + 4\lambda^3\right)X + \lambda\left(1-\lambda^2\right)X^2 - 2(1-\lambda)^2e^{\lambda x}\right.\\ &\quad + \left(2 - 4\lambda + 2\lambda^2 - 2\lambda^3\right)\Big\}.\end{aligned}$$

Then,

$$\sigma_0^2 = \frac{2\lambda - 5}{(\lambda-1)^2(2\lambda-1)}.$$

Setting $\lambda = 0.01$, we obtain $\sigma_0^2 = 5.18481$

2.2 The Pitman Asymptotic Relative Efficiency

To asses the quality of this procedures, we compute the PAE of our test , by using the concept of “Pitman’s asymptotic eficiency (PAE)”which is defined as

$$PAE(\triangle_r(\theta)) = \frac{1}{\sigma_0}\left|\frac{\partial}{\partial\theta}\triangle_r(\theta)\right|_{\theta\to\theta_0}.$$

And compared with some other tests for the following alternative distributions:

(i) Linear failure rate family (LFR) $\overline{F}_1(x) = e^{-x-\frac{\theta}{2}x^2}, x \geq 0, \theta \geq 0,$
(ii) Makeham family $\overline{F}_2(x) = e^{-x-\theta(x+e^{-x}-1)}, x \geq 0, \theta \geq 0,$
(iii) Weibull family $\overline{F}_3(x) = e^{-x^{\theta}}, x > 0, \theta \geq 0,$
(iv) Gamma family $\overline{F}_4(x) = \int_x^{\infty} e^{-u}u^{\theta-1}du/\Gamma(\theta), x > 0, \theta \geq 0.$

Note that for $\theta = 0$, $\overline{F}_1(x)$ and $\overline{F}_2(x)$ goes to the exponential distribution and for $\theta = 1$, $\overline{F}_3(x)$ and $\overline{F}_4(x)$ reduce to the exponential distribution.

Since,

$$\Delta_\theta(\lambda) = \int_0^{\infty} \left[\frac{1}{\lambda^3}(\lambda\mu_\theta - 1)E_\theta\left(e^{\lambda x}\right) + \frac{1}{2\lambda}\mu_2(\theta) - \frac{1}{\lambda}\mu_\theta^2 + \frac{1}{\lambda^3}\right] dF_\theta(t).$$

Where,

$$\mu_\theta = \int_0^{\infty} \overline{F}_\theta(x)\,dx, \quad \mu_2(\theta) = 2\int_0^{\infty} x\overline{F}_\theta(x)\,dx, \quad E_\theta\left(e^{\lambda x}\right) = \int_0^{\infty} e^{\lambda x} dF_\theta(x).$$

The $PAE\,(\Delta_\theta(\lambda))$ can be written as,

$$\begin{aligned} PAE\,(\Delta_\theta(\lambda)) = \frac{1}{\sigma_0}\Bigg| &\int_0^{\infty} \left\{\frac{1}{\lambda^3}(\lambda\mu_\theta - 1)E_\theta\left(e^{\lambda x}\right) + \frac{1}{2\lambda}\mu_2(\theta) - \frac{1}{\lambda}\mu_\theta^2 + \frac{1}{\lambda^3}\right\} dF'_\theta(t) \\ &+ \int_0^{\infty}\left\{\frac{1}{\lambda^3}(\lambda\mu_\theta - 1)\int_0^{\infty} e^{\lambda x} dF'_\theta(x) + \frac{1}{\lambda^2}\mu'_\theta \int_0^{\infty} e^{\lambda x} dF_\theta(x)\right. \\ &\left. + \frac{1}{\lambda}\int_0^{\infty} x\overline{F}'_\theta(x)\,dx - \frac{2}{\lambda}\mu_\theta\mu'_\theta\right\} dF_\theta(t)\Bigg|. \end{aligned}$$

Using MATHEMATECA 9 program to calculate the Pitman asymptotic efficiency for alternative families we obtain

(i) Linear failure rate family:

$$PAE\,(\Delta_\theta(\lambda)) = \frac{1}{\sigma_0}\left|\frac{2-\lambda}{(\lambda-1)^2}\right|, \qquad \lambda \neq 1$$

(ii) Makeham family:

$$PAE\,(\Delta_\theta(\lambda)) = \frac{1}{\sigma_0}\left|\frac{\lambda-3}{4(2-3\lambda+\lambda^2)}\right|, \qquad \lambda \neq 1,2$$

(iii) Weibull family:

$$PAE\,(\Delta_\theta(\lambda)) = \frac{1}{\sigma_0}\left|\frac{\lambda-\lambda^2+\log[1-\lambda]}{\lambda^2(\lambda-1)}\right|, \qquad \lambda \neq 0,1$$

(iv) Gamma family:

$$PAE\,(\Delta_\theta(\lambda)) = \frac{1}{\sigma_0}|0.3425|, \quad \text{at } \lambda = 0.01$$

Table 1.gives the efficiencies of our proposed test $\triangle_\theta(\lambda)$ comparing with the tests given by Mugdadi and Ahmad (2005), $\delta_{(3)}$ and Mahmoud and Abdul Alim (2008), $\delta_{F_n}^{(2)}$ respectively. We have maximum value at $\lambda = 0.01$

Table 1: Comparison between the *PAE* of our test and some other tests:

Distribution	$\triangle_\theta(\lambda)$	$\delta_{(3)}$	$\delta_{F_n}^{(2)}$
Linear failure rate	0.8917	0.408	0.217
Makeham	0.1666	0.039	0.144
Weibull	0.6669	0.170	0.05
Gamma	0.1504	-	-

Also, the Pittman asymptotic relative efficiency $(PARE)$ of our test $\triangle_\theta(\lambda)$ ın comparing to $\delta_{(3)}, \delta_{F_n}^{(2)}$ is calculated where $PARE\,(T_1,T_2) = \frac{PAE(T_1)}{PAE(T_2)}$

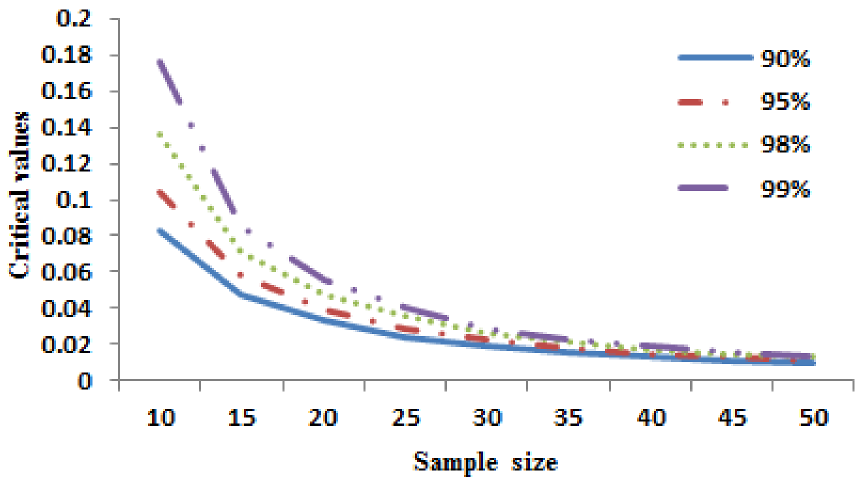

Fig. 1 Relation between critical values, sample size and confidence levels.

Table 2. show that the asymptotic relative efficiencies for our test:

Distribution	$PARE\left(\triangle_\theta(\lambda),\delta_{(3)}\right)$	$PARE\left(\triangle_\theta(\lambda),\delta_{F_n}^{(2)}\right)$
LFR	2.18554	4.1092
Makeham	4.27179	1.1569
Weibull	3.92294	13.338

3 Monte Carlo Null Distribution Critical Points

We have simulated the upper percentile points for 90%, 95%, 98% and 99%. Table 3 gives these percentile points of statistics $\widehat{\delta}_n(\lambda)$ in Equation (2.5) with $\lambda = 0.55$. These calculations are based on 5000 simulated sample size $n = 5(5)50$ from the standard exponential distribution.

Table 3. The upper percentile of $\widehat{\delta}_n(\lambda)$ with 5000 replications at $\lambda = 0.55$.

n	90%	95%	98%	99%
5	0.21549	0.29752	0.44056	0.59486
10	0.08308	0.10459	0.13597	0.17652
15	0.04813	0.05869	0.07167	0.08472
20	0.03321	0.03965	0.04778	0.05609
25	0.02457	0.02863	0.03529	0.03999
30	0.01958	0.02242	0.02627	0.02897
35	0.01601	0.01844	0.02117	0.02339
40	0.01360	0.01516	0.01737	0.01897
45	0.01162	0.01316	0.01494	0.01616
50	0.01019	0.01156	0.01293	0.01377

It is noticed from Table 3 and Fig 1 that the critical values are increasing as the confidence level increasing and is almost decreasing as the sample size increasing at $\lambda = 0.55$.

3.1 The Power Estimates

The power of the proposed test will be estimated at $(1-\alpha)\%$ confidence level, $\alpha = 0.05$ with suitable parameters values of θ at $n = 10, 20$ and 30 with respect to three alternatives linear failure rate (LFR), Weibull and Gamma distributions based on 5000 samples.

Table 4.The power estimates of $\widehat{\delta}_n(\lambda)$

Distribution	Parameter θ	Sample Size		
		$n = 10$	$n = 20$	$n = 30$
Linear Failure Rate Family	2	0.9980	1.0000	1.0000
	3	0.9994	1.0000	1.0000
	4	0.9992	1.0000	1.0000
Weibull Family	2	1.0000	1.0000	1.0000
	3	1.0000	1.0000	1.0000
	4	1.0000	1.0000	1.0000
Gamma Family	2	0.9992	0.9998	0.9998
	3	1.0000	1.0000	1.0000
	4	1.0000	1.0000	1.0000

It is clear from Table 4 that our test has good powers for all alternatives and the powers increases as the sample size increases. The power is getting as smaller as the ODL_{mgf} approaches the exponential distribution.

3.2 Applications Using Complete (Uncensored) Data

Here, we present some of a good real examples to illustrate the use of our test statistics $\widehat{\delta}_n(\lambda)$ in the case of non censored data at 95% confidence level.

Data-set #1.

Consider the data that given in Abouammoh et al (1994).These data represent set of 40 patients suffering from blood cancer (leukemia) from one of ministry of health hospitals in Saudi Arabia. In this case, we get $\widehat{\delta}_n(\lambda) = 0.0734397$ which is greater than the critical value of the Table 3. Then we accept H_1 the alternative hypotheses which show that the data set has ODL_{mgf} property but not exponential.

Data-set #2.

Consider the data set given in Grubbs (1971), This data gives the times between arrivals of 25 customers at a facility. It is easily to show that $\widehat{\delta}_n(\lambda) = 0.155362$ which is greater than the critical value of Table 3. Then we accept H_1 which shows that the data set have ODL_{mgf} property but not exponential.

Data-set #3.

Consider the data-set given in Fisher (1966) which represent the differences in heights between cross- and self-fertilized plants of the same pair grown together in one pot. In this case, we get $\widehat{\delta}_n(\lambda) = 0.421684$ and this value exceeds the tabulated critical value in Table 3.It is evident at the significant level %95,that the data set has ODL_{mgf} property.

Data-set #4.

Consider the data set in Kochar (1985) In an experiment at Florida state university to study the effect of methyl mercury poisoning on the life lengths of fish goldfish were subjected to various dosages of methyl mercury. At one dosage level the ordered times to death in day. We can see that the value of test statistic for the data set by (5) is given by $\widehat{\delta}_n(\lambda) = 7.77932$ and this value greater than the tabulated critical value in Table 3. This means that the set of data have ODL_{mgf} property and not exponential.

4 Testing Against ODL_{mgf} Class for Censored Data

In this section, a test statistic is proposed to test H_0 versus H_1 with randomly right-censored data. Such a censored data is usually the only information available in a life-testing model or in a clinical study where patients may be lost (censored) before the completion of a study. This experimental situation can formally be modeled as follows.

Suppose n objects are put on test, and $X_1, X_2, ..., X_n$ denote their true life time. We assume that $X_1, X_2, ..., X_n$ be independent, identically distributed (i.i.d.) according to a continuous life distribution F. Let $Y_1, Y_2, ..., Y_n$ be (i.i.d.) according to a continuous life distribution G. Also we assume that $X's$ and $Y's$ are independent.

In the randomly right-censored model, we observe the pairs (Z_j, δ_j), $j = 1, ..., n$ where $Z_j = \min(X_j, Y_j)$ and

$$\delta_j = \begin{cases} 1 \text{ if } Z_j = X_j & \left(j^{th} \textit{ observn is nucensored}\right) \\ 0 \text{ if } Z_j = Y_j & \left(j^{th} \textit{ observn is censored}\right) \end{cases}$$

Let $Z(0) = 0 < Z(1) < Z(2) < ... < Z(n)$ denote the ordered $Z's$ and $\delta_{(j)}$ is the δ_j corresponding to $Z_{(j)}$ respectively. Using the censored data (Z_j, δ_j), $j = 1, ..., n$. Kaplan and Meier (1958) proposed the product limit estimator.

$$\overline{F}_n(X) = 1 - F_n(X) = \prod_{[j:Z_{(j)} \leq X]} \{(n-j)/(n-j+1)\}^{\delta_{(j)}}, X \in [0, Z_n]$$

Now, for testing $H_0 : \Delta(\lambda) = 0$, against $H_1 : \Delta(\lambda) > 0$, using the randomly right censored data, we propose the following test statistic:

$$\widehat{\Delta}^c(\lambda) = \frac{1}{\mu^2}\int_0^\infty \left[\frac{1}{\lambda^3}(\lambda\mu - 1)\phi(\lambda) + \frac{1}{2\lambda}\mu_{(2)} - \frac{1}{\lambda}\mu^2 + \frac{1}{\lambda^3}\right] dF(t),$$

where, $\phi(\lambda) = \int_0^\infty e^{\lambda x} dF(x)$ For computational purposes, $\widehat{\Delta}^c(\lambda)$ can be rewritten as

$$\widehat{\Delta}^c(\lambda) = \frac{1}{\mu^2}\int_0^\infty \left[\frac{1}{\lambda^3}(\lambda\mu - 1)\Phi + \frac{1}{2\lambda}\mu_{(2)} - \frac{1}{\lambda}\mu^2 + \frac{1}{\lambda^3}\right] \zeta, \tag{6}$$

where,

$$\mu = \sum_{i=1}^{n}\prod_{m=1}^{i-1} c_m^{\delta_{(m)}}(Z_{(i)} - Z_{(i-1)}), \qquad \mu_{(2)} = 2\sum_{l=1}^{n} Z_{(l)}\prod_{p=1}^{l-1} c_p^{\delta_{(p)}}(Z_{(l)} - Z_{(l-1)}),$$

and,

$$\Phi = \sum_{j=1}^{n} e^{\lambda Z_{(i)}}\left(\prod_{q=1}^{j-2} c_q^{\delta_{(q)}} - \prod_{q=1}^{j-1} c_q^{\delta_{(q)}}\right), \qquad \zeta = \sum_{k=1}^{n}\left(\prod_{v=1}^{k-2} c_v^{\delta_{(v)}} - \prod_{v=1}^{k-1} c_v^{\delta_{(v)}}\right).$$

Where $C_k = \frac{n-k}{n-k+1}$.

Table 5. gives the critical values percentiles of $\widehat{\Delta}^c(\lambda)$ test for sample sizes $n = 5(5)30(10)81, 86$.based on 5000 replications.

Table 5. Critical values for percentiles of $\widehat{\Delta}^c(\lambda)$ test at $\lambda = 0.55$

n	90%	95%	98%	99%
5	1.16580	1.26058	1.35252	1.41201
10	1.18099	1.26657	1.34111	1.37197
15	1.13501	1.23818	1.31547	1.35936
20	1.07032	1.18395	1.28310	1.31544
25	0.98278	1.11435	1.23061	1.27834
30	0.92051	1.06680	1.18682	1.24833
40	0.79518	0.95872	1.10180	1.17032
50	0.66343	0.83783	1.00606	1.11748
60	0.58566	0.77149	0.95579	1.04114
70	0.49344	0.68441	0.88579	0.98454
81	0.42276	0.58798	0.81735	0.95729
86	0.37183	0.54065	0.77197	0.88220

In view of Table 5, and Fig. 2 it is noticed that the critical values are increasing as the confidence level increasing and is almost decreasing as the sample size increasing.

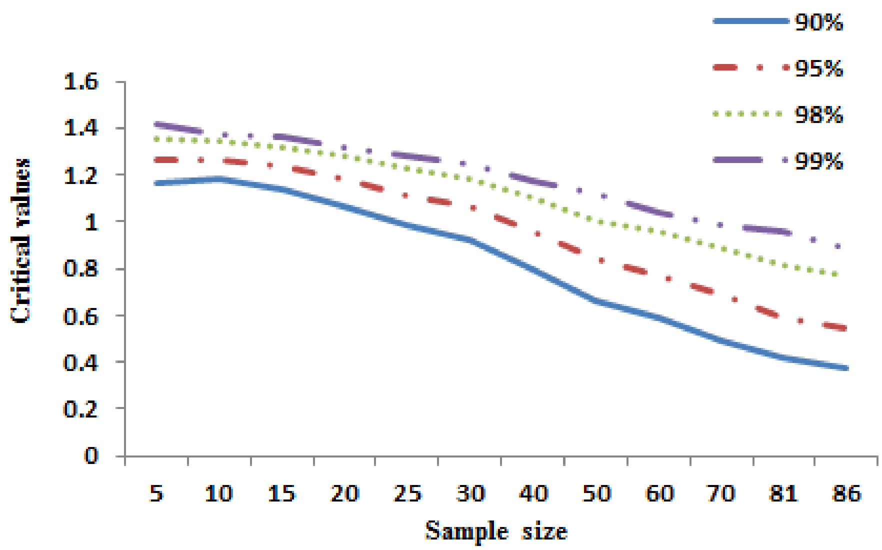

Fig. 2 Relation between critical values, sample size and confidence levels

4.1 The power estimates for $\overset{\curlywedge}{\triangle}^{c}(\lambda)$

Table 6 shows the power estimate of the test statistic $\widehat{\Delta}^{c}(\lambda)$ at $(1-\alpha)\%$ confidence level $\alpha = 0.05$ using LFR, Weibull and Gamma family. The estimates are based on 5000 simulated samples for sizes $n = 10, 20$ and 30.

Table 6. Power estimates of $\overset{\curlywedge}{\triangle}^{c}(\lambda)$ at $\lambda = 0.55$.

Distribution	Parametar θ	Sample size		
		$n = 10$	$n = 20$	$n = 30$
LFR family	2	0.9986	1.0000	1.0000
	3	0.9986	1.0000	1.0000
	4	0.9976	1.0000	1.0000
Weibull family	2	0.9982	1.0000	1.0000
	3	0.9988	1.0000	1.0000
	4	0.9994	1.0000	1.0000
Gamma family	2	0.9274	0.9008	0.916
	3	0.9258	0.9996	0.9998
	4	0.8882	1.0000	1.0000

We notice from Table 5. that our test has a good power, and the power increases as the sample size increases.

4.2 Applications for Censored data

We present two good real examples to illustrate the use of our test statistics $\widehat{\Delta}^{c}(\lambda)$ in the case of censored data at 95% confidence level.

Data-set #5.

Consider the data from Susarla and Vanryzin (1978),which represent 81 survival times (in months) of patients melanoma. Out of these 46 represents non-censored data. Now, taking into account the whole set of survival data (both censored and uncensored). It was found that the value of test statistic for the data set using formula (6) is given by $\widehat{\Delta}^{c}(\lambda) = -1.19327$ and this value is less than the tabulated critical value in Table 5. This means that the data set have the exponential property.

Data-set #6.

On the basis of right censored data for lung cancer patients from Pena (2002).These data consists of 86 survival times (in month) with 22 right censored. Now account the whole set of survival data (both censored and uncensored), and computing the test statistic given by formula (6). It was found that $\widehat{\Delta}^c(\lambda) = 2.37579 * 10^{10}$ which is greater than the tabulated value in Table 5.It is evident at the significant level %95,that the data set has ODL_{mgf} property.

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