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Poisson Exponentiated Erlang-Truncated Exponential Distribution

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Abstract: In this article, the Poisson exponentiated Erlang truncated exponential distribution was developed and its properties such as the quantile, moment, incomplete moment and inequality measures were derived. The parameters of the new model were estimated using maximum likelihood estimation and simulation studies were performed to examine the finite sample properties of the parameters. An application of the model was demonstrated using real data set. Finally, a bivariate extension of the model was proposed.

Keywords: Poisson, Erlang truncated exponential, bivariate, stochastic ordering, Order statistics

1 Introduction

The Erlang-truncated exponential (ETE) distribution [6] was developed by mixing the Erlang distribution with left truncated one-parameter exponential distribution. The ETE distribution, like the exponential distribution, has a constant failure rate which makes it practically impossible for the model to provide a reasonable parametric fit to data sets with decreasing failure rate, increasing failure rate and non-monotonic failure rate such as the bathtub and unimodal failure rates which are common in reliability studies and other related fields of studies. To improve the goodness-of-fit of the ETE distribution in modeling data with different failure rates, researchers in recent times have proposed new modifications of the model. Among them are: transmuted ETE distribution [11], extended ETE distribution [13], generalized ETE distribution [8] and Marshall-Olkin generalized ETE distribution [12].
Different researchers have proposed methods for modifying existing standard or classical distributions to make them more flexible for modeling lifetime data. These techniques have the ability to improve the goodness-of-fit of the modified distributions in order to provide a reasonable fit to the data set. Some of these methods include: transformed-transformer (T-X) method [1], exponentiated generalized T-X method [10], exponentiated T-X method [2], exponentiated generalized exponential-X family [9] and exponentiated generalized class [5].
In this study, another extension of the ETE distribution called the Poisson exponentiated Erlang-truncated exponential (PEETE) distribution has been proposed by compounding the Poisson distribution with the exponentiated ETE distribution. The motivation for proposing the new distribution is to provide a model for modeling data with different kinds of failure rate, varied degrees of skewness and kurtosis. The rest of the paper is organized as follows: In section 2, the probability density function (PDF), the cumulative distribution function (CDF), the survival function and hazard rate function of the PEETE distribution were defined. In section 3, statistical properties of the new model were derived. In section 4, the parameters of the model were estimated using the method of maximum likelihood. In section 5, Monte Carlo simulations were performed to examine the finite sample properties of the estimators of the parameters. In section 6, an application of the model was demonstrated using real data set. In section 7, a bivariate extension of the model was proposed. The concluding remarks were finally given in section 8.

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2 Model Definition

Let M represent the number of independent subsystems of a system functioning at a given time. Suppose that M has a zero truncated Poisson distribution with probability mass function given by

$$\mathbb{P}(M=m)=\frac{\theta^m}{m!\,(e^{\theta}-1)}, m=1,2,...,\theta>0. \tag{1}$$

Let the failure time of each subsystem follow the exponentiated Erlang-truncated exponential distribution with CDF given by

$$G(x)=\left(1-e^{-\beta(1-e^{-\lambda})x}\right)^{\alpha}, \alpha,\beta,\lambda,x>0. \tag{2}$$

If Z_j is the failure time of the j^{th} subsystem and X represents the time to failure of the first out of the M operating subsystems such that $X=\min\{Z_1,Z_2,\ldots Z_M\}$. Then the conditional CDF of X given M is

$$\begin{aligned}F(x|M=m)&=1-\mathbb{P}(X>x|M)\\&=1-\mathbb{P}(Z_1>x,...,Z_M>x)\\&=1-[\mathbb{P}(Z_1>x)]^m\\&=1-[1-\mathbb{P}(Z_1<x)]^m\\&=1-\left[1-\left(1-e^{-\beta(1-e^{-\lambda})x}\right)^{\alpha}\right]^m, x>0.\end{aligned}$$

Thus, the marginal CDF of X is given by

$$\begin{aligned}F(x)&=\frac{1}{(e^{\theta}-1)}\sum_{m=1}^{\infty}\frac{\theta^m}{m!}\left\{1-\left[1-\left(1-e^{-\beta(1-e^{-\lambda})x}\right)^{\alpha}\right]^m\right\}\\&=\frac{1-e^{-\theta\left(1-e^{-\beta(1-e^{-\lambda})x}\right)^{\alpha}}}{1-e^{-\theta}}, x>0,\end{aligned} \tag{3}$$

where $\theta,\alpha,\beta,\lambda>0$. The corresponding PDF of the PEETE distribution is obtained by differentiating equation 3 and is given by

$$f(x)=\frac{\theta\alpha\beta(1-e^{-\lambda})e^{-\beta(1-e^{-\lambda})x}\left(1-e^{-\beta(1-e^{-\lambda})x}\right)^{\alpha-1}e^{-\theta\left(1-e^{-\beta(1-e^{-\lambda})x}\right)^{\alpha}}}{1-e^{-\theta}}, x>0. \tag{4}$$

Lemma 1. The PDF of the PEETE distribution can be written in a mixture form as

$$f(x)=\frac{\alpha}{1-e^{-\theta}}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\omega_{ij}f_{ETE}(x;\beta_{j+1},\lambda), x>0, \tag{5}$$

where $f_{ETE}(x;\beta_{j+1},\lambda)=\beta_{j+1}(1-e^{-\lambda})e^{-beta_{j+1}(1-e^{-\lambda})x}$ is the PDF of the ETE distribution with parameters $\beta_{j+1}=\beta(j+1)>0$ and $\lambda>0$ and

$$\omega_{ij}=\frac{(-1)^{i+j}\theta^{i+1}\Gamma(\alpha(i+1))}{i!\,(j+1)!\,\Gamma(\alpha(i+1)-j)}.$$

Proof. Using the Taylor series expansion,

$$e^{-\theta\left(1-e^{-\beta(1-e^{-\lambda})x}\right)^{\alpha}}=\sum_{i=0}^{\infty}\frac{(-1)^i\theta^i\left(1-e^{-\beta(1-e^{-\lambda})x}\right)^{\alpha i}}{i!}.$$

Hence, the PDF of the PEETE distribution can be written as

$$f(x)=\frac{\theta\alpha\beta(1-e^{-\lambda})e^{-\beta(1-e^{-\lambda})x}}{1-e^{-\theta}}\sum_{i=0}^{\infty}\frac{(-1)^i\theta^i}{i!}\left(1-e^{-\beta(1-e^{-\lambda})x}\right)^{\alpha(i+1)-1}, x>0. \tag{6}$$

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For a real non-integer $\eta > 0$, the following identity holds.

$$(1-z)^{\eta-1} = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(\eta)}{j!\Gamma(\eta-j)} z^j, \ |z| < 1. \tag{7}$$

Using the identity in equation (7) and the fact that $0 < \left(1-e^{-\beta(1-e^{-\lambda})x}\right)^{\alpha(i+1)-1} < 1$, equation (6) can be expressed as

$$f(x) = \frac{\alpha}{1-e^{-\theta}} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \omega_{ij} f_{ETE}(x;\beta_{j+1},\lambda), x > 0,$$

which is the mixture representation of the PDF of the PEETE distribution. From equation (5), it can easily be seen that the PDF of the PEETE distribution is a weighted function of the ETE distribution with different parameters. The PDF of the PEETE distribution reduces to the PDF of the Poisson Erlang truncated exponential (PETE) distribution when $\alpha = 1$. Figure 1 displays the shapes of the density function of the PEETE distribution for some selected parameter values.

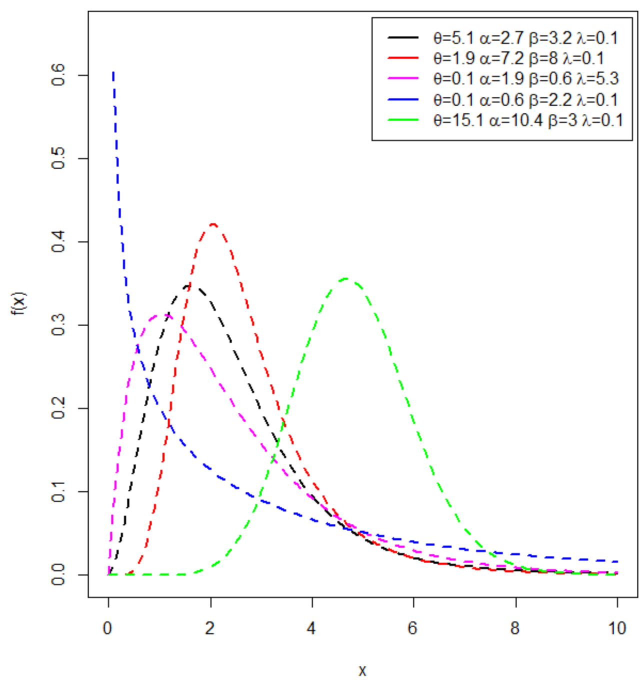

Fig. 1: PEETE distribution density function

The survival function and the hazard rate function are

$$S(x) = \frac{\left(1-e^{-\beta(1-e^{-\lambda})x}\right)^{\alpha} - e^{-\theta}}{1-e^{-\theta}}, x > 0, \tag{8}$$

and

$$\tau(x) = \frac{\theta\alpha\beta(1-e^{-\lambda})e^{-\beta(1-e^{-\lambda})x}\left(1-e^{-\beta(1-e^{-\lambda})x}\right)^{\alpha-1}}{1-e^{-\theta+\theta\left(1-e^{-\beta(1-e^{-\lambda})x}\right)^{\alpha}}}, x > 0, \tag{9}$$

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respectively. The hazard rate function of the PEETE distribution exhibits different shapes such as the upside down bathtub, increasing and decreasing failure for different combinations of the parameter values. The plot of the hazard rate function is given in Figure 2.

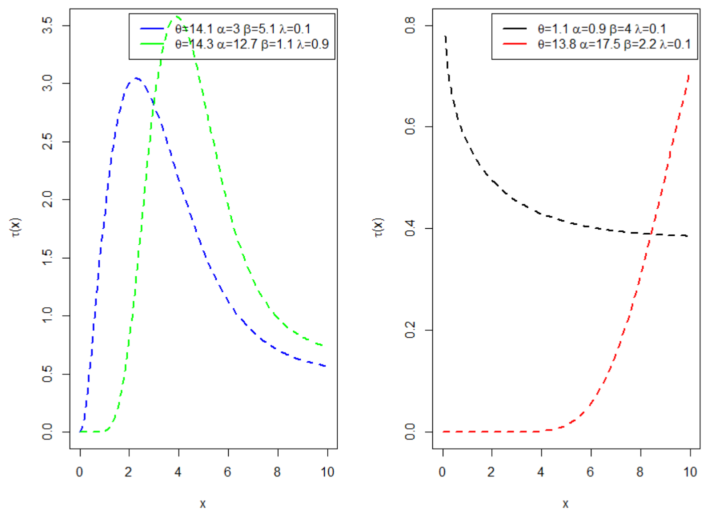

Fig. 2: PEETE hazard rate function

2.1 Statistical Properties

In this section various statistical properties of the PEETE distribution were discussed.

2.2 Quantile Function

The quantile function is a useful function for generating random numbers from the PEETE distribution. The quantile function of the PEETE distribution is given by

$$Q_X(p) = \frac{-\log\left\{1 - \left[\frac{-\log\left(1-p(1-e^{-\theta})\right)}{\theta}\right]^{\frac{1}{\alpha}}\right\}}{\beta(1-e^{-\lambda})}, p \in [0, 1]. \quad (10)$$

Substituting $p = 0.25, 0.5$ and 0.75 into equation (10) yields the first quartile, the median and the third quartile respectively.

2.3 Moment

The moments play a useful role in statistical analyses. They are used for estimating measures of central tendency, dispersion and shapes among others.
Proposition 1. The r^{th} non-central moment of the PEETE distribution is given by

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$$\mu_r^{'} = \frac{\alpha}{1-e^{-\theta}} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \omega_{ij} \frac{\Gamma(r+1)}{[\beta_{j+1}(1-e^{-\lambda})]^r}, \, r = 1, 2, ..., \tag{11}$$

where $\Gamma(a) = \int_0^\infty y^{a-1} e^{-y} dy = (a-1)!$ is the complete gamma function.
Proof. By definition

$$\begin{aligned}
\mu_r^{'} &= \int_0^\infty x^r f(x) dx \\
&= \int_0^\infty x^r \frac{\alpha}{1-e^{-\theta}} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \omega_{ij} f_{ETE}(x;\beta_{j+1},\lambda) dx \\
&= \frac{\alpha(1-e^{-\lambda})}{1-e^{-\theta}} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \omega_{ij} \beta_{j+1} \int_0^\infty x^r e^{-\beta_{j+1}(1-e^{-\lambda})x} dx.
\end{aligned}$$

After some algebraic manipulation, the moment is obtained as

$$\mu_r^{'} = \frac{\alpha}{1-e^{-\theta}} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \omega_{ij} \frac{\Gamma(r+1)}{[\beta_{j+1}(1-e^{-\lambda})]^r}, \, r = 1, 2, ...$$

2.4 Incomplete Moment

The incomplete moment has a critical role to play in statistical analyses. It is used for estimating the mean deviation, median deviation and measures of inequalities such as the Lorenz and Bonferroni curves.
Proposition 2. The r^{th} incomplete moment of the PEETE distribution is given by

$$\varphi_r(t) = \frac{\alpha}{1-e^{-\theta}} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \omega_{ij} \frac{\gamma(r+1, \beta_{j+1}(1-e^{-\lambda})t)}{[\beta_{j+1}(1-e^{-\lambda})]^r}, \, t > 0, r = 1, 2, ... \tag{12}$$

where $\gamma(a,t) = \int_0^t y^{a-1} e^{-y} dy$ is the lower incomplete gamma function.
Proof. By definition

$$\begin{aligned}
\varphi_r(t) &= \int_0^t x^r f(x) dx \\
&= \int_0^t x^r \frac{\alpha}{1-e^{-\theta}} \sum_{i=0}^{t}\sum_{j=0}^{\infty} \omega_{ij} f_{ETE}(x;\beta_{j+1},\lambda) dx \\
&= \frac{\alpha(1-e^{-\lambda})}{1-e^{-\theta}} \sum_{i=0}^{\infty}\sum_{j=0}^{t} \omega_{ij} \beta_{j+1} \int_0^t x^r e^{-\beta_{j+1}(1-e^{-\lambda})x} dx \\
&= \frac{\alpha}{1-e^{-\theta}} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \omega_{ij} \frac{\gamma(r+1, \beta_{j+1}(1-e^{-\lambda})t)}{[\beta_{j+1}(1-e^{-\lambda})]^r}.
\end{aligned}$$

The mean deviation, $\delta_1(x)$ and median deviation, $\delta_2(x)$, can easily be computed using the relationships $\delta_1(x) = 2\mu F(\mu) - 2\varphi_1(\mu)$ and $\delta_2(x) = \mu - 2\varphi_1(M)$. Where $\mu = E(X)$ and M is the median of the PEETE random variable. $\varphi_1(\mu)$ and $\varphi_1(M)$ are computed using the first incomplete moment.

2.5 Inequality Measures

In this subsection, the Lorenz, $L_F(x)$, and Bonferroni, $B_F(x)$ curves were derived. They are the most widely used measures of income inequality of a given population.
Proposition 3. The Lorenz curve for the PEETE distribution is given by

$$L_F(x) = \frac{\alpha}{\mu(1-e^{-\theta})} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \omega_{ij} \frac{\gamma(2, \beta_{j+1}(1-e^{-\lambda})x)}{[\beta_{j+1}(1-e^{-\lambda})]}. \tag{13}$$

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Proof. By definition

$$\begin{aligned} L_F(x) &= \frac{1}{\mu}\int_0^x z f(z) dz \\ &= \frac{\alpha}{\mu(1-e^{-\theta})}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\omega_{ij}\frac{\gamma(2,\beta_{j+1}(1-e^{-\lambda})x)}{[\beta_{j+1}(1-e^{-\lambda})]}. \end{aligned}$$

Proposition 4. The Bonferroni curve for the PEETE distribution is given by

$$B_F(x) = \frac{\alpha}{\mu\left(1-e^{-\theta\left(1-e^{-\beta(1-e^{-\lambda})x}\right)^{\alpha}}\right)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\omega_{ij}\frac{\gamma(2,\beta_{j+1}(1-e^{-\lambda})x)}{[\beta_{j+1}(1-e^{-\lambda})]}. \tag{14}$$

Proof. The proof follows directly from the definition

$$B_F(x) = \frac{1}{\mu F(x)}\int_0^x z f(z) dz.$$

2.6 Entropy

Entropies have been extensively used in information theory. They are good measures of randomness or variation of a random variable. In this subsection, the Rényi entropy [14] of a random variable having the PEETE distribution is given.
Proposition 5. If the random variable X has a PEETE distribution, then the Rényi entropy of X is given by

$$I_R(\delta) = \frac{1}{1-\delta}\log\left[A\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{i+j+k}j^k\delta^{i-k-1}\theta^i\Gamma(\delta(\alpha-1)+\alpha i+1)\Gamma(k+1)}{i!\,j!\,k![\beta(1-e^{-\lambda})]\Gamma(\delta(\alpha-1)+\alpha i-j+1)}\right], \tag{15}$$

where $\delta \neq 1$, $\delta > 0$ and $A = \left(\frac{\theta\alpha\beta(1-e^{-\lambda})}{1-e^{-\theta}}\right)^{\delta}$.
Proof. Using similar concepts for expanding density,

$$f^{\delta}(x) = A\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{i+j+k}(\delta\theta)^i[\beta_j]^k\Gamma(\delta(\alpha-1)+\alpha i+1)}{i!J!K!\Gamma(\delta(\alpha-1)+\alpha i-j+1)}x^k e^{-\beta\delta(1-e^{-\lambda})x}. \tag{16}$$

By definition,

$$I_R(\delta) = \frac{1}{1-\delta}\log\left[\int_0^{\infty} f^{\delta}(x)dx\right], \delta \neq 1, \delta > 0.$$

Thus,

$$\begin{aligned} I_R(\delta) &= \frac{1}{1-\delta}\log\left[\int_0^{\infty} A\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{i+j+k}(\delta\theta)^i[\beta_j]^k\Gamma(\delta(\alpha-1)+\alpha i+1)}{i!J!K!\Gamma(\delta(\alpha-1)+\alpha i-j+1)}x^k e^{-\beta\delta(1-e^{-\lambda})x}dx\right] \\ &= \frac{1}{1-\delta}\log\left[A\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{i+j+k}(\delta\theta)^i[\beta_j]^k\Gamma(\delta(\alpha-1)+\alpha i+1)}{i!J!K!\Gamma(\delta(\alpha-1)+\alpha i-j+1)}\int_0^{\infty} x^k e^{-\beta\delta(1-e^{-\lambda})x}dx\right] \\ &= \frac{1}{1-\delta}\log\left[A\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{i+j+k}j^k\delta^{i-k-1}\theta^i\Gamma(\delta(\alpha-1)+\alpha i+1)\Gamma(k+1)}{i!\,j!\,k![\beta(1-e^{-\lambda})]\Gamma(\delta(\alpha-1)+\alpha i-j+1)}\right]. \end{aligned}$$

The Rényi entropy tends to the Shannon entropy as $\delta \longrightarrow 1$.

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2.7 Stochastic Ordering

Stochastic ordering is the commonest way to show ordering mechanism in lifetime distributions. Suppose $X_1 \sim \text{PEETE}(\theta, \alpha_1, \beta, \lambda)$ and $X_2 \sim \text{PEETE}(\theta, \alpha_2, \beta, \lambda)$, then X_1 is said to be stochastically smaller than X_2 in the

1. stochastic order $(X_1 \leq_{st} X_2)$ if the associated CDFs satisfy: $F_{X_1} \geq F_{X_2}$ for all x.
2. hazard rate order $(X_1 \leq_{hr} X_2)$ if the associated hazard rate functions satisfy: $h_{X_1} \geq h_{X_2}$ for all x.
3. likelihood ratio order $(X_1 \leq_{lr} X_2)$ if the ratio of the associated PDFs given by $\frac{f_{X_1}(x)}{f_{X_2}(x)}$ decreases in x.

When X_1 and X_2 have a common finite left end-point support, the following implications hold

$$X_1 \leq_{lr} X_2 \Longrightarrow X_1 \leq_{hr} X_2 \Longrightarrow X_1 \leq_{st} X_2.$$

Suppose that the densities of X_1 and X_2 are

$$f_{X_1}(x) = \frac{\theta\alpha_1\beta(1-e^{-\lambda})e^{-\beta(1-e^{-\lambda})x}\left(1-e^{-\beta(1-e^{-\lambda})x}\right)^{\alpha_1-1} e^{-\theta\left(1-e^{-\beta(1-e^{-\lambda})x}\right)^{\alpha_1}}}{1-e^{-\theta}}, x>0,$$

and

$$f_{X_2}(x) = \frac{\theta\alpha_2\beta(1-e^{-\lambda})e^{-\beta(1-e^{-\lambda})x}\left(1-e^{-\beta(1-e^{-\lambda})x}\right)^{\alpha_2-1} e^{-\theta\left(1-e^{-\beta(1-e^{-\lambda})x}\right)^{\alpha_2}}}{1-e^{-\theta}}, x>0,$$

respectively. Then the ratio of the two densities is

$$\frac{f_{X_1}(x)}{f_{X_2}(x)} = \frac{\alpha_1}{\alpha_2}\left(1-e^{-\beta(1-e^{-\lambda})x}\right)^{\alpha_1-\alpha_2} e^{\theta\left[\left(1-e^{-\beta(1-e^{-\lambda})x}\right)^{\alpha_2}-\left(1-e^{-\beta(1-e^{-\lambda})x}\right)^{\alpha_1}\right]}, x>0.$$

Differentiating the ratio of the densities yields

$$\begin{aligned}\frac{d}{dx}\frac{f_{X_1}(x)}{f_{X_2}(x)} &= B(\alpha_1-\alpha_2)\left(1-e^{-\beta(1-e^{-\lambda})x}\right)^{\alpha_1-\alpha_2-1} + \\ &B\theta\left(1-e^{-\beta(1-e^{-\lambda})x}\right)^{\alpha_1-\alpha_2}\left[\left(1-e^{-\beta(1-e^{-\lambda})x}\right)^{\alpha_2-1}-\left(1-e^{-\beta(1-e^{-\lambda})x}\right)^{\alpha_1-1}\right],\end{aligned}$$

where

$$B = \frac{\alpha_1}{\alpha_2}\beta(1-e^{-\lambda})e^{-\beta(1-e^{-\lambda})x+\theta\left[\left(1-e^{-\beta(1-e^{-\lambda})x}\right)^{\alpha_2}-\left(1-e^{-\beta(1-e^{-\lambda})x}\right)^{\alpha_1}\right]}.$$

If $\alpha_2 > \alpha_1$, $\frac{d}{dx}\frac{f_{X_1}(x)}{f_{X_2}(x)} < 0$, which implies $(X_1 \leq_{lr} X_2)$.

2.8 Order Statistics

In this subsection, the order statistics of PEETE distribution were derived. Suppose $X_1, X_2, \ldots, X_n$ is random sample from PEETE and $X_{1:n} < X_{2:n} < \ldots < X_{n:n}$ are the corresponding order statistics. The PDF, $f_{r:n}(x)$, of the r^{th} order statistic $X_{r:n}$ is

$$f_{r:n}(x) = \frac{1}{B(r, n-r+1)}[F(x)]^{r-1}[1-F(x)]^{n-r} f(x),$$

where $F(x)$ and $f(x)$ are the CDF and PDF of the PEETE distribution respectively, and $B(\cdot,\cdot)$ is the beta function. Since $0 < F(x) < 1$ for $x > 0$, using the binomial series expansion of $[1-F(x)]^{n-r}$, which is given by

$$[1-F(x)]^{n-r} = \sum_{i=0}^{n-r}(-1)^i\binom{n-r}{i}[F(x)]^i,$$

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we have

$$f_{r:n}(x) = \frac{1}{B(r, n-r+1)} \sum_{i=0}^{n-r} (-1)^{l} \binom{n-r}{i} [F(x)]^{r+i-1} f(x). \tag{17}$$

Substituting the CDF and PDF of the PEETE distribution into equation (17) and using similar concept for expanding the density gives

$$f_{r:n}(x) = \theta\alpha \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \sum_{i=0}^{n-r} \sum_{j=0}^{r+i-1} \zeta_{ijkm} \frac{f_{ETE}(x; \beta_{m+1}, \lambda)}{(1-e^{-\theta})^{r+i}}, \tag{18}$$

where

$$\zeta_{ijkm} = \frac{(-1)^{i+j+k+m}[\theta(j+1)]^{k}\Gamma(n+1)\Gamma(r+i)\Gamma(\alpha(k+1))}{i!\, j!\, k!\,(m+1)!\,(r-1)!\,\Gamma(n-r-i+1)\Gamma(r+i-j)\Gamma(\alpha(k+1)-m)}.$$

The density of the r^{th} order statistic is a weighted function of the density of ETE distribution with parameters $\beta_{m+1} = \beta(m+1)$ and λ.

3 Parameter Estimation

In this section, the maximum likelihood estimators of the unknown parameters of the PEETE distribution were derived. Suppose $X_1, X_2, \ldots, X_n$ form a random sample of size n from the PEETE distribution, then the log-likelihood function is given by

$$\ell = n\log(\theta\alpha\beta(1-e^{-\lambda})) - n\log(1-e^{-\theta}) - (1-e^{-\lambda})\sum_{i=1}^{n} x_i - \theta\sum_{i=1}^{n}(1-e^{-\beta(1-e^{-\lambda})x_i})^{\alpha} + (\alpha-1)\sum_{i=1}^{n}\log(1-e^{-\beta(1-e^{-\lambda})x_i}). \tag{19}$$

Finding the partial derivatives of the log-likelihood function with respect to the parameters θ, α, β and λ, the score functions are obtained as

$$\frac{\partial\ell}{\partial\theta} = \frac{n}{\theta} - \frac{ne^{-\theta}}{1-e^{-\theta}} - \sum_{i=1}^{n}(1-e^{-\beta(1-e^{-\lambda})x_i})^{\alpha}, \tag{20}$$

$$\frac{\partial\ell}{\partial\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\log(1-e^{-\beta(1-e^{-\lambda})x_i}) - \theta\sum_{i=1}^{n}(1-e^{-\beta(1-e^{-\lambda})x_i})^{\alpha}\log(1-e^{-\beta(1-e^{-\lambda})x_i}), \tag{21}$$

$$\frac{\partial\ell}{\partial\beta} = \frac{n}{\beta} - (1-e^{-\lambda})\sum_{i=1}^{n} x_i + (\alpha-1)\sum_{i=1}^{n}\frac{x_i(1-e^{-\lambda})e^{-\beta(1-e^{-\lambda})x_i}}{1-e^{-\beta(1-e^{-\lambda})x_i}} - \theta\alpha(1-e^{-\lambda})\sum_{i=1}^{n} x_i e^{-\beta(1-e^{-\lambda})x_i}(1-e^{-\beta(1-e^{-\lambda})x_i})^{\alpha-1}, \tag{22}$$

$$\frac{\partial\ell}{\partial\lambda} = \frac{ne^{-\lambda}}{1-e^{-\lambda}} - \beta e^{-\lambda}\sum_{i=1}^{n} x_i + (\alpha-1)\beta e^{-\lambda}\sum_{i=1}^{n}\frac{x_i e^{-\beta(1-e^{-\lambda})x_i}}{1-e^{-\beta(1-e^{-\lambda})x_i}} - \theta\alpha\beta e^{-\lambda}\sum_{i=1}^{n} x_i e^{-\beta(1-e^{-\lambda})x_i}(1-e^{-\beta(1-e^{-\lambda})x_i})^{\alpha-1}. \tag{23}$$

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The maximum likelihood estimates of the parameters are obtained by equation the score functions to zero and the system of non-linear equations solved numerically. To construct confidence intervals for the parameters of the PEETE distribution, the observed information matrix $J(\boldsymbol{\Theta})$ was used. The observed information matrix for the parameters is given by

$$J(\boldsymbol{\Theta}) = -\begin{bmatrix} \frac{\partial^2 \ell}{\partial \theta^2} & \frac{\partial^2 \ell}{\partial \theta \partial \alpha} & \frac{\partial^2 \ell}{\partial \theta \partial \beta} & \frac{\partial^2 \ell}{\partial \theta \partial \lambda} \\ & \frac{\partial^2 \ell}{\partial \alpha^2} & \frac{\partial^2 \ell}{\partial \alpha \partial \beta} & \frac{\partial^2 \ell}{\partial \alpha \partial \lambda} \\ & & \frac{\partial^2 \ell}{\partial \beta^2} & \frac{\partial^2 \ell}{\partial \beta \partial \lambda} \\ & & & \frac{\partial^2 \ell}{\partial \lambda^2} \end{bmatrix},$$

where $\boldsymbol{\Theta} = (\theta, \alpha, \beta, \lambda)^{'}$. The elements of the observed information matrix are given in the appendix. When the regularity condition holds and the parameters are within the interior of the parameter space, but not on the boundary, the distribution of $\sqrt{n}(\hat{\boldsymbol{\Theta}} - \boldsymbol{\Theta})$ converges to the multivariate normal distribution $N_4(\mathbf{0}, I^{-1}(\boldsymbol{\Theta}))$, where $I(\boldsymbol{\Theta})$ is the expected information matrix. The asymptotic behavior is still valid when $I(\boldsymbol{\Theta})$ is replaced by the observed information matrix estimated at $J(\hat{\boldsymbol{\Theta}})$.

4 Simulation Study

In this section, a Monte Carlo simulation was performed to investigate the finite sample properties of the maximum likelihood estimators for the parameters of the PEETE distribution. The results of the simulation were obtained from 2,000 Monte Carlo replications. In each repetition, a random sample of size $n = 25$, 50, 75 and 100 were generated from the PEETE distribution. Table 1 displays the mean estimates, average bias, root mean square error (RMSE), coverage probability (CP) and average width (AW) of the confidence intervals for the parameters of the PEETE distribution. The results revealed that the average bias, the RMSE and AW decrease as the sample size increases. In addition, the CPs of the confidence intervals are quite close to the nominal 95%. The mean estimates of the parameters are generally close to the actual values as the sample size increases. Hence, the results revealed that the estimates of the parameters are stable and their asymptotic properties can be used for constructing confidence intervals even for reasonably small sample size.

5 Application

This section presents the application of the PEETE distribution using real data set. The goodness-of-fit of the model was compared with that of its sub-model (PETE distribution), Inverse Weibull Poisson (IWP) distribution [3] and Exponentiated Kumaraswamy Dagum (EKD) distribution [7] using Cramér-von (W^*) Misses distance values and Anderson-Darling (AD) test statistics as well as Akaike information criterion (AIC), corrected Akaike information criterion (AICc) and Bayesian information criterion (BIC). The data were reported by [4] and consist of the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli. Table 2 displays the data set.
The density functions of the IWP and EKD distributions are;

$$f(x) = \frac{\theta \alpha \beta x^{-\beta-1} e^{-\alpha x^{-\beta}} e^{\theta e^{-\alpha x^{-\beta}}}}{e^{\theta} - 1}, \; \theta, \alpha, \beta, x > 0, \tag{24}$$

and

$$\begin{aligned} f(x) = {} & \alpha \beta \theta a b x^{-\theta-1}(1+\beta x^{-\theta})^{-\alpha-1}\left(1-(1+\beta x^{-\theta})^{-\alpha}\right)^{a-1}\left[\left(1-(1+\beta x^{-\theta})^{-\alpha}\right)^{a}\right]^{b-1} \\ & \alpha, \beta, \theta, a, b, x > 0, \end{aligned} \tag{25}$$

respectively. The maximum likelihood estimates of the parameters of the fitted distributions and their corresponding standard errors in brackets are shown in Table 3. The estimated parameters for the PEETE distribution and its sub-model were all significant at the 5% significance level.
The goodness-of-fit statistics displayed in Table 4 revealed that the PEETE distribution provides a better fit to the data compared to the PETE, IWP and EKD distributions. The conclusion is based on the fact that it had the lowest value for the statistics. The likelihood ratio test statistic for the test of hypothesis $H_0 : \alpha = 1$ vs $H_1 : H_0$ is false gave a value 38.824. The corresponding P-value $= 4.6380 \times 10^{-10} < 0.05$. Thus, the null hypothesis was rejected in favor of the PEETE

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Table 1: Simulation results: Mean estimates, Bias, RMSE, CP and AW

θ	α	β	λ	n	Parameters	Estimates	Bias	RMSE	CP	AW
4.5	0.2	2.1	0.1	25	θ	3.2749	-1.2251	1.9328	0.8660	10.9282
					α	0.2198	0.0198	0.0424	0.9745	0.1840
					β	4.6493	2.5493	5.0133	0.9845	1661.2690
					λ	0.8563	0.7563	1.5057	0.9255	841.2298
				50	θ	3.7299	-0.7701	1.4662	0.8680	7.8690
					α	0.2129	0.0129	0.0297	0.9610	0.1210
					β	3.4281	1.3281	3.0450	0.9645	548.0638
					λ	0.4509	0.3509	0.8627	0.8085	163.6414
				75	θ	4.0460	-0.4540	1.0496	0.9160	5.7757
					α	0.2094	0.0094	0.0235	0.9560	0.0925
					β	2.7099	0.6099	1.7290	0.9680	132.8608
					λ	0.2932	0.1932	0.6123	0.7110	30.4660
				100	θ	4.2003	-0.2997	0.8034	0.9435	4.7515
					α	0.2060	0.0060	0.0191	0.9635	0.0773
					β	2.4033	0.3033	1.1106	0.9645	41.5491
					λ	0.2061	0.1061	0.3351	0.6315	6.7270
0.64	0.3	0.1	3.2	25	θ	3.1843	-2.3157	2.8077	0.7395	10.8286
					α	0.2945	-0.0055	0.0527	0.9835	0.2499
					β	0.8152	0.7152	1.1687	0.9955	300.9816
					λ	3.6602	0.4602	1.1582	0.9990	858.1100
				50	θ	3.6257	-1.8743	2.4368	0.7245	8.8408
					α	0.2985	-0.0015	0.0374	0.9745	0.1721
					β	0.5895	0.4895	0.7949	0.9890	123.2901
					λ	3.5637	0.3637	0.9282	0.9959	432.8220
				75	θ	3.8937	-1.6063	2.2051	0.7245	8.0128
					α	0.3008	0.0008	0.0310	0.9760	0.1376
					β	0.4697	0.3697	0.5991	0.9880	53.0148
					λ	3.6334	0.4334	1.3407	0.9985	174.035
				100	θ	4.1381	-1.3618	1.9598	0.7475	7.6524
					α	0.3023	0.0023	0.0270	0.9700	0.1164
					β	0.3782	0.2782	0.4563	0.9800	35.5386
					λ	3.5550	0.3550	1.0751	0.9648	165.2170
2.5	0.3	0.1	0.5	25	θ	2.2951	-0.2049	1.3294	0.9499	8.6878
					α	0.3186	0.0186	0.0635	0.9835	0.3037
					β	0.1799	0.0799	0.1633	0.9490	592.4642
					λ	0.5303	0.0303	0.1398	0.9950	2055.8740
				50	θ	2.4004	-0.0996	1.1981	0.9800	6.5996
					α	0.3132	0.0132	0.0470	0.9725	0.2116
					β	0.1439	0.0439	0.1011	0.9590	189.9837
					λ	0.5174	0.0174	0.1402	0.9758	832.5282
				75	θ	2.4144	-0.0856	1.1194	0.9570	5.4190
					α	0.3121	0.0121	0.0396	0.9605	0.1723
					β	0.1294	0.0294	0.0726	0.9935	107.3689
					λ	0.5165	0.0165	0.1262	0.9695	534.3913
				100	θ	2.4589	-0.0411	1.0211	0.9590	4.7641
					α	0.3110	0.0110	0.0354	0.9600	0.0.1476
					β	0.1241	0.0241	0.0637	0.9290	78.5896
					λ	0.5060	0.0060	0.1186	0.9790	398.3197

Table 2: Survival times of guinea pigs

0.1	0.33	0.44	0.56	0.59	0.72	0.74	0.77	0.92	0.93	0.96	1.0	1.0	1.02	1.05
1.07	1.07	1.08	1.08	1.08	1.09	1.12	1.13	1.15	1.16	1.2	1.21	1.22	1.22	1.24
1.3	1.34	1.36	1.39	1.44	1.46	1.53	1.59	1.6	1.63	1.63	1.68	1.71	1.72	1.76
1.83	1.95	1.96	1.97	2.02	2.13	2.15	2.16	2.22	2.3	2.31	2.4	2.45	2.51	2.53
2.54	2.54	2.78	2.93	3.27	3.42	3.47	3.61	4.02	4.32	4.58	5.55			

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Table 3: Maximum likelihood estimates of parameters and standard errors

Model	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{a}$	$\hat{b}$
PEETE	6.3462	2.8121	154.5800	0.0025		
	(0.0534)	(0.4075)	(5.6981×10^{-6})	(3.9763×10^{-4})		
PETE	100.6900		150.8300	3.7323×10^{-5}		
	(1.6448×10^{-12})		(1.0945×10^{-12})	(4.4278×10^{-6})		
IWP	8.3210	0.1650	1.645			
	(2.0720)	(0.0460)	(0.120)			
EKD	2.9310	1.8640	5.9970		1.3820	0.4520
	(2.0110)	(7.3390)	(7.5200)		(1.7450)	(1.8700)

Table 4: Goodness-of-fit statistics

Model	AIC	AICc	BIC	AD	W*
PEETE	**195.7286**	**196.6379**	**204.8353**	**0.3645**	**0.0547**
PETE	232.5525	233.1495	239.3825	0.5908	0.0954
IWP	215.7000	216.0000	222.5000	2.0980	0.3100
EKD	197.0000	197.9000	208.4000	0.4150	0.0628

distribution. This implies that the PEETE distribution is significantly better than the PETE distribution. The estimated asymptotic variance-covariance matrix for the parameters of the PEETE distribution is

$$J^{-1} = -\begin{bmatrix} 2.8470\times10^{-3} & 2.1744\times10^{-2} & -3.0404\times10^{-7} & 1.9133\times10^{-5} \\ 2.1744\times10^{-2} & 1.6607\times10^{-1} & -2.3221\times10^{-6} & 1.4613\times10^{-4} \\ -3.0404\times10^{-7} & -2.3221\times10^{-6} & 3.2468\times10^{-11} & -2.0428\times10^{-9} \\ 1.9133\times10^{-5} & 1.4613\times10^{-4} & -2.0428\times10^{-9} & 1.5811\times10^{-7} \end{bmatrix}.$$

Hence, the approximate 95% confidence interval for the parameters θ, α, β and λ are $[6.2416, 6.4508]$, $[2.0134, 3.6108]$, $[154.5800, 154.5800]$ and $[0.0017, 0.0033]$ respectively. Figure 3 displays the empirical density and the fitted densities of the guinea pig data. From the figure, it can be seen that the PEETE distribution provides a better fit to the data compared to the other fitted models.

6 Bivariate Extension

Suppose the pair (X,Y) are bivariate random variables of the PEETE distribution with marginal distribution functions $F_X(x)$ and $F_Y(y)$ and Copula C. Given that the copula related to (X,Y) is a member of Ali-Mikhail-Haq family of copula defined by

$$C(u,v) = \frac{uv}{1-\phi(1-u)(1-v)}, |\phi| < 1.$$

If the marginal distribution functions are

$$F_X(x) = \frac{1-e^{-\theta_1\left(1-e^{-\beta_1(1-e^{-\lambda_1})x}\right)^{\alpha_1}}}{1-e^{-\theta_1}}, \theta_1, \alpha_1, \beta_1, \lambda_1, x > 0,$$

and

$$F_Y(y) = \frac{1-e^{-\theta_2\left(1-e^{-\beta_2(1-e^{-\lambda_2})y}\right)^{\alpha_2}}}{1-e^{-\theta_2}}, \theta_2, \alpha_2, \beta_2, \lambda_2, y > 0.$$

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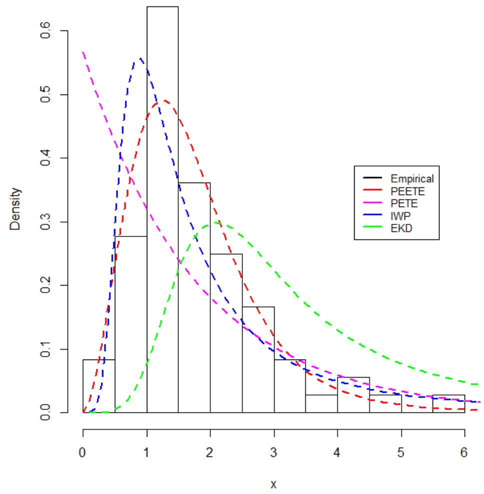

Fig. 3: Empirical and fitted densities of guinea data

Then joint CDF of the bivariate PEETE distribution is given by

$$F_{XY}(x,y) = \frac{(1-e^{-\theta_1})^{-1}(1-e^{-\theta_2})^{-1}\left(1-e^{-\theta_1\left(1-e^{-\beta_1(1-e^{-\lambda_1})x}\right)^{\alpha_1}}\right)\left(1-e^{-\theta_2\left(1-e^{-\beta_2(1-e^{-\lambda_2})y}\right)^{\alpha_2}}\right)}{1-\phi\left[1-(1-e^{-\theta_1})^{-1}\left(1-e^{-\theta_1\left(1-e^{-\beta_1(1-e^{-\lambda_1})x}\right)^{\alpha_1}}\right)\right]\left[1-(1-e^{-\theta_2})^{-1}\left(1-e^{-\theta_2\left(1-e^{-\beta_2(1-e^{-\lambda_2})y}\right)^{\alpha_2}}\right)\right]},$$
$$\theta_1, \alpha_1, \beta_1, \lambda_1, \theta_2, \alpha_2, \beta_2, \lambda_2, x, y > 0, |\phi| < 1.$$

The conditional distribution functions are given by

$$F_{X|Y}(x|y) = \frac{(1-e^{-\theta_1})^{-1}\left(1-e^{-\theta_1\left(1-e^{-\beta_1(1-e^{-\lambda_1})x}\right)^{\alpha_1}}\right)}{1-\phi\left[1-(1-e^{-\theta_1})^{-1}\left(1-e^{-\theta_1\left(1-e^{-\beta_1(1-e^{-\lambda_1})x}\right)^{\alpha_1}}\right)\right]\left[1-(1-e^{-\theta_2})^{-1}\left(1-e^{-\theta_2\left(1-e^{-\beta_2(1-e^{-\lambda_2})y}\right)^{\alpha_2}}\right)\right]},$$
$$\theta_1, \alpha_1, \beta_1, \lambda_1, \theta_2, \alpha_2, \beta_2, \lambda_2, x, y > 0, |\phi| < 1,$$

and

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$$F_{Y|X}(y|x) = \frac{(1-e^{-\theta_2})^{-1}\left(1-e^{-\theta_2\left(1-e^{-\beta_2(1-e^{-\lambda_2})y}\right)^{\alpha_2}}\right)}{1-\phi\left[1-(1-e^{-\theta_1})^{-1}\left(1-e^{-\theta_1\left(1-e^{-\beta_1(1-e^{-\lambda_1})x}\right)^{\alpha_1}}\right)\right]\left[1-(1-e^{-\theta_2})^{-1}\left(1-e^{-\theta_2\left(1-e^{-\beta_2(1-e^{-\lambda_2})y}\right)^{\alpha_2}}\right)\right]},$$

$\theta_1, \alpha_1, \beta_1, \lambda_1, \theta_2, \alpha_2, \beta_2, \lambda_2, x, y > 0, |\phi| < 1.$

The joint PDF is obtained by finding

$$f_{XY}(x,y) = \frac{\partial^2 F_{XY}(x,y)}{\partial x \partial y}.$$

7 Conclusion

This study proposed the Poisson exponentiated Erlang truncated exponential distribution and studied its statistical properties. The method of maximum likelihood was used to estimate the parameters of the new distribution. Simulation studies were performed to assess the finite sample properties for the estimators of the parameters and the results revealed that the estimators of parameters were stable. The application of the distribution was demonstrated using real data set and the empirical results obtained showed the PEETE distribution is a better model compared with competing models in terms of goodness-of-fit. Finally, the bivariate Poisson exponentiated Erlang-truncated exponential distribution was proposed. We recommend that further studies should be carried out by comparing the maximum likelihood method with other techniques for estimating model parameters in order to identify which of them is most appropriate for estimating the parameters of the PEETE distribution.

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Appendix

Elements of the observed information matrix.

$$\frac{\partial^2 \ell}{\partial \theta^2} = \frac{ne^{-2\theta}}{(1-e^{-\theta})^2} + \frac{ne^{-\theta}}{1-e^{-\theta}} - \frac{n}{\theta^2},$$

$$\frac{\partial^2 \ell}{\partial \theta \partial \alpha} = -\sum_{i=1}^{n} \left(1 - e^{-\beta\left(1-e^{-\lambda}\right)x_i}\right)^{\alpha} \log\left(1 - e^{-\beta\left(1-e^{-\lambda}\right)x_i}\right),$$

$$\frac{\partial^2 \ell}{\partial \theta \partial \beta} = -\alpha(1-e^{-\lambda}) \sum_{i=1}^{n} x_i e^{-\beta\left(1-e^{-\lambda}\right)x_i} \left(1 - e^{-\beta\left(1-e^{-\lambda}\right)x_i}\right)^{\alpha-1},$$

$$\frac{\partial^2 \ell}{\partial \theta \partial \lambda} = -\alpha\beta \sum_{i=1}^{n} x_i e^{-\lambda-\beta\left(1-e^{-\lambda}\right)x_i} \left(1 - e^{-\beta\left(1-e^{-\lambda}\right)x_i}\right)^{\alpha-1},$$

$$\frac{\partial^2 \ell}{\partial \alpha^2} = -\frac{n}{\alpha^2} - \theta \sum_{i=1}^{n} \left(1 - e^{-\beta\left(1-e^{-\lambda}\right)x_i}\right)^{\alpha} \log\left(1 - e^{-\beta\left(1-e^{-\lambda}\right)x_i}\right)^2,$$

$$\begin{aligned}\frac{\partial^2 \ell}{\partial \alpha \partial \beta} = &\sum_{i=1}^{n} \frac{(1-e^{-\lambda})x_i e^{-\beta\left(1-e^{-\lambda}\right)x_i}}{1-e^{-\beta\left(1-e^{-\lambda}\right)x_i}} - \theta(1-e^{-\lambda}) \sum_{i=1}^{n} x_i e^{-\beta\left(1-e^{-\lambda}\right)x_i} \left(1 - e^{-\beta\left(1-e^{-\lambda}\right)x_i}\right)^{\alpha-1} \\ &- \alpha\theta(1-e^{-\lambda}) \sum_{i=1}^{n} x_i e^{-\beta\left(1-e^{-\lambda}\right)x_i} \left(1 - e^{-\beta\left(1-e^{-\lambda}\right)x_i}\right)^{\alpha-1} \log\left(1 - e^{-\beta\left(1-e^{-\lambda}\right)x_i}\right),\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 \ell}{\partial \alpha \partial \lambda} = &\sum_{i=1}^{n} \frac{\beta x_i e^{-\lambda-\beta\left(1-e^{-\lambda}\right)x_i}}{1-e^{-\beta\left(1-e^{-\lambda}\right)x_i}} - \theta\beta \sum_{i=1}^{n} x_i e^{-\lambda-\beta\left(1-e^{-\lambda}\right)x_i} \left(1 - e^{-\beta\left(1-e^{-\lambda}\right)x_i}\right)^{\alpha-1} \\ &- \alpha\beta\theta \sum_{i=1}^{n} x_i e^{-\lambda-\beta\left(1-e^{-\lambda}\right)x_i} \left(1 - e^{-\beta\left(1-e^{-\lambda}\right)x_i}\right)^{\alpha-1} \log\left(1 - e^{-\beta\left(1-e^{-\lambda}\right)x_i}\right),\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 \ell}{\partial \beta^2} = &-\frac{n}{\beta^2} + (\alpha-1) \sum_{i=1}^{n} \left[\frac{\left(1-e^{-\lambda}\right)^2 x_i^2 e^{-2\beta\left(1-e^{-\lambda}\right)x_i}}{\left(1-e^{-\beta\left(1-e^{-\lambda}\right)x_i}\right)^2} - \frac{\left(1-e^{-\lambda}\right)^2 x_i^2 e^{-\beta\left(1-e^{-\lambda}\right)x_i}}{1-e^{-\beta\left(1-e^{-\lambda}\right)x_i}} \right] \\ &+ \alpha\theta(1-e^{-\lambda})^2 \sum_{i=1}^{n} x_i^2 e^{-\beta\left(1-e^{-\lambda}\right)x_i} \left(1 - e^{-\beta\left(1-e^{-\lambda}\right)x_i}\right)^{\alpha-1} \\ &- \alpha\theta(\alpha-1))(1-e^{-\lambda})^2 \sum_{i=1}^{n} x_i^2 e^{-2\beta\left(1-e^{-\lambda}\right)x_i} \left(1 - e^{-\beta\left(1-e^{-\lambda}\right)x_i}\right)^{\alpha-1},\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 L}{\partial\beta\partial\lambda} &= -e^{-\lambda}\sum_{i=1}^{n} x_i + (\alpha-1)\\
&\times\sum_{i=1}^{n}\left[\frac{x_i e^{-\lambda-\beta\left(1-e^{-\lambda}\right)x_i}}{1-e^{-\beta\left(1-e^{-\lambda}\right)x_i}} - \frac{\beta\left(1-e^{-\lambda}\right)x_i^2 e^{-\lambda-2\beta\left(1-e^{-\lambda}\right)x_i}}{\left(1-e^{-\beta\left(1-e^{-\lambda}\right)x_i}\right)^2} - \frac{\beta\left(1-e^{-\lambda}\right)x_i^2 e^{-\lambda-\beta\left(1-e^{-\lambda}\right)x_i}}{1-e^{-\beta\left(1-e^{-\lambda}\right)x_i}}\right]\\
&-\alpha\theta\sum_{i=1}^{n} x_i e^{-\lambda-\beta\left(1-e^{-\lambda}\right)x_i}\left(1-e^{-\beta\left(1-e^{-\lambda}\right)x_i}\right)^{\alpha-1}\\
&+\alpha\beta\theta\left(1-e^{-\lambda}\right)\sum_{i=1}^{n} x_i^2 e^{-\lambda-\beta\left(1-e^{-\lambda}\right)x_i}\left(1-e^{-\beta\left(1-e^{-\lambda}\right)x_i}\right)^{\alpha-1}\\
&-\alpha\beta\theta(\alpha-1)\left(1-e^{-\lambda}\right)\sum_{i=1}^{n} x_i^2 e^{-\lambda-2\beta\left(1-e^{-\lambda}\right)x_i}\left(1-e^{-\beta\left(1-e^{-\lambda}\right)x_i}\right)^{\alpha-2},\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 \ell}{\partial\lambda^2} &= -\frac{ne^{-2\lambda}}{(1-e^{-\lambda})^2} - \frac{ne^{-\lambda}}{1-e^{-\lambda}} + \beta e^{-\lambda}\sum_{i=1}^{n} x_i\\
&+(\alpha-1)\sum_{i=1}^{n}\left[-\frac{\beta^2 x_i^2 e^{-2\lambda-2\beta\left(1-e^{-\lambda}\right)x_i}}{\left(1-e^{-\beta\left(1-e^{-\lambda}\right)x_i}\right)^2} + \frac{\beta x_i\left(-1-e^{-\lambda}\beta x_i\right)e^{-\lambda-\beta\left(1-e^{-\lambda}\right)x_i}}{1-e^{-\beta\left(1-e^{-\lambda}\right)x_i}}\right]\\
&-\alpha\beta^2\theta(\alpha-1)\sum_{i=1}^{n} x_i^2 e^{-2\lambda-2\beta\left(1-e^{-\lambda}\right)x_i}\left(1-e^{-\beta\left(1-e^{-\lambda}\right)x_i}\right)^{\alpha-2}\\
&-\alpha\beta\theta\sum_{i=1}^{n} x_i e^{-\lambda-\beta\left(1-e^{-\lambda}\right)x_i}\left(1-e^{-\beta\left(1-e^{-\lambda}\right)x_i}\right)^{\alpha-1}\left(-1-e^{-\lambda}\beta x_i\right).\end{aligned}$$

R Algorithm

```
EETEDensityFunction
PEETE_PDF <- function(x,theta,alpha,beta,lambda)
{
A <- theta*alpha*beta*(1-exp(-lambda))*((1-exp(-theta))^(-1))
B <- exp(-beta*(1-exp(-lambda))*x)
C <- (1-B)^(alpha-1)
D <- exp(-theta*((1-B)^alpha))
fxn <- A*B*C*D
return(fxn)
}
```

```
PEETE CDF
PEETE_CDF <- function(x,theta,alpha,beta,lambda)
{
CA <- (1-exp(-beta*(1-exp(-lambda))*x))^alpha
B <- 1-exp(-theta)
fxn <- (1-exp(-theta*A))/B
return(fxn)
}
```

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PEETE Hazard Function

$PEETE_{H}azard <- function(x, theta, alpha, beta, lambda)$

$\{$

$A <- theta * alpha * beta * (1 - exp(-lambda))$

$B <- exp(-beta * (1 - exp(-lambda)) * x)$

$C <- (1 - B)^{(}alpha - 1)$

$D <- exp(-theta + theta * ((1 - B)^{a}lpha))$

$fxn <- (A * B * C)/(1 - D)$

$return(fxn)$

$\}$

$PEETEMoment$

$PEETE_{M}oment <- function(theta, alpha, beta, lambda, r)$

$\{$

$func <- function(x, theta, alpha, beta, lambda, r)$

$\{$

$(x^{r}) * (PEETE_{P}DF(x, theta, alpha, beta, lambda))$

$\}$

$results <- integrate(func, lower = 0, upper = Inf, subdivisions = 10000,$

$theta = theta, alpha = alpha, beta = beta, lambda = lambda, r = r)$

$return(results$value)

$\}$

Negative Log-likelihood function of PEETE

$PEETE_{LL} <- function(theta, alpha, beta, lambda)$

$\{$

$A <- theta * alpha * beta * (1 - exp(-lambda)) * ((1 - exp(-theta))^{(} - 1))$

$B <- exp(-beta * (1 - exp(-lambda)) * x)$

$C <- (1 - B)^{(}alpha - 1)$

$D <- exp(-theta * ((1 - B)^{a}lpha))$

$fxn <- - sum(log(A * B * C * D))$

$return(fxn)$

$\}$

Fitting PEETE to Real Data Set

$library(bbmle)$

$fit <- mle2(PEETE_{L}L, start = list(theta = theta, alpha = alpha, beta = beta,$

$lambda = lambda), method = "Nelder - Mead", data = list(x))$

$summary(fit)$

Computing the variance-covariance matrix

$vcov(fit)$

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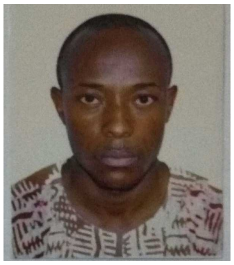

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