Progress in Fractional Differentiation and Applications
An International Journal

http://dx.doi.org/10.18576/pfda/040201

Boundary Value Problems for Fractional Differential Equations with Integral and Anti-Periodic Conditions in a Banach Space

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Received: 2 Sep. 2017, Revised: 18 Dec. 2017, Accepted: 2 Jan. 2018
Published online: 1 Apr. 2018

Abstract: The authors study the existence of solutions to a class of fractional differential equations with anti-periodic and integral boundary conditions involving the Caputo fractional derivative of order $r \in (0,1]$. The proof is based on Mönch's fixed point theorem.

Keywords: Boundary value problems, fractional differential equations, integral conditions, anti-periodic conditions, Mönch's theorem.

1 Introduction

This paper deals with the existence of solutions to the boundary value problem (BVP for short) for fractional order differential equations

$$^cD^r y(t) = f(t, y(t)), \quad \text{a.e. } t \in J = [0,T], \ 0 < r \leq 1, \tag{1}$$

$$y(T) + y(0) = b\int_0^T y(s)ds, \ \ bT \neq 2, \tag{2}$$

where $^cD^r$ is the Caputo fractional derivative, $(\mathbb{E}, |\cdot|)$ is a Banach space, $f : J \times \mathbb{E} \to \mathbb{E}$ is a given function, and b is a constant.

Differential equations of fractional order have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. There are numerous applications in viscoelasticity, electrochemistry, control theory, porous media, electromagnetism, etc. There has been a significant development in fractional differential equations in recent years, and we refer the reader to the monographs of Hilfer [1], Kilbas *et al.* [2], Momani *et al.* [3], Podlubny [4], and the papers of Agarwal *et al.* [5], and Benchohra *et al.* [6,7,8] for further discussion.

Applied problems require the definitions of fractional derivatives to allow the utilization of physically interpretable initial data that contain $y(0)$, $y'(0)$, etc., and there is a similar requirement for the boundary conditions. Caputo's fractional derivative satisfies these demands. For additional details concerning the geometric and physical interpretations of fractional derivatives of Riemann-Liouville and Caputo type, see [4].

Anti-periodic, integral, and nonlocal boundary value problems constitute an important class of problems that are receiving considerable attention in recent years. Anti-periodic boundary conditions occur in mathematical modelling of many physical processes; see, for example, the monographs of Ahmed *et al.* [9] and Chen *et al.* [10]. As examples of this research, the authors in [9] used Banach's fixed point theorem to investigate existence and uniqueness of solutions for integro-differential equations of fractional order $\alpha \in (1,2]$ with anti-periodic boundary conditions.

In this paper, we present existence results for the problem (1)–(2) using a method involving a measure of noncompactness and a fixed point theorem of Mönch type. This approach was mainly initiated in the monograph of Banas and Goebel [11] and subsequently developed and used in many papers; see, for example, Banas and Sadarangani [12], Guo *et al.* [13], Lakshmikantham and Leela [14], Mönch [15], and Szufla [16].

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Section 2 of this paper contains some preliminary facts needed to prove our main result which appears in Section 3. The last section in the paper contains an example illustrating our main theorem.

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts that will be used in the remainder of this paper.

Let $C(J,\mathbb{E})$ be the Banach space of all continuous functions from J into $\mathbb{E}$ with the norm

$$\|y\|_\infty = \sup\{|y(t)| : 0 \le t \le T\},$$

and let $L^1(J,\mathbb{E})$ denote the Banach space of functions $y : J \to \mathbb{E}$ that are Bochner integrable with norm

$$\|y\|_{L^1} = \int_0^T |y(t)|dt.$$

We take $L^\infty(J,\mathbb{E})$ to be the Banach space of bounded measurable functions $y : J \to \mathbb{E}$ equipped with the norm

$$\|y\|_{L^\infty} = \inf\{c > 0 : \|y(t)\| \le c \ \text{ a.e. } t \in J\}.$$

Definition 1.[2,17] *The fractional integral of order $r \in \mathbb{R}_+$ of the function $h \in L^1([a,b],\mathbb{E})$ is defined by*

$$I_a^r h(t) = \int_a^t \frac{(t-s)^{r-1}}{\Gamma(r)} h(s)ds,$$

where Γ is the gamma function. If $a = 0$, we write $I^r h(t) = h(t) * \varphi_r(t)$, where $\varphi_r(t) = \dfrac{t^{r-1}}{\Gamma(r)}$ for $t > 0$, $\varphi_r(t) = 0$ for $t \le 0$, $\varphi_r \to \delta(t)$ as $r \to 0$, and δ is the delta function.

Definition 2. [2,17] *For a function h on the interval $[a,b]$, the Riemann-Liouville fractional derivative of h of order r is defined by*

$$(D_{a+}^r h)(t) = \frac{1}{\Gamma(n-r)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-r-1} h(s)ds.$$

Here $n = \lceil r \rceil$ and $\lceil r \rceil$ denotes the smallest integer greater than or equal to r.

Definition 3. [18] *For a function h on the interval $[a,b]$, the Caputo fractional derivative of order r of h is defined by*

$$({}^c D_{a+}^r h)(t) = \frac{1}{\Gamma(n-r)} \int_a^t (t-s)^{n-r-1} h^{(n)}(s)ds,$$

where $n = \lceil r \rceil$.

For a given set V of functions $\upsilon : J \to E$, we set

$$V(t) = \{\upsilon(t) : \upsilon \in V\}, t \in J,$$

and

$$V(J) = \{\upsilon(t) : \upsilon \in V(t), t \in J\}.$$

Definition 4. *A multivalued map $F : J \times \mathbb{E} \to \mathscr{P}(\mathbb{E})$ is said to be Carathéodory if:*

(1)$t \to F(t,u)$ is measurable for each $u \in \mathbb{E}$;
(2)$u \to F(t,u)$ is upper semicontinuous for almost all $t \in J$.

For convenience, we recall the definitions of the Kuratowski measure of noncompactness and summarize the main properties of this measure.

Definition 5. [19,11] *Let $\mathbb{E}$ be a Banach space and let $\Omega_{\mathbb{E}}$ denote the bounded subsets of $\mathbb{E}$. The Kuratowski measure of noncompactness is the map $\alpha : \Omega_{\mathbb{E}} \to [0,\infty)$ defined by*

$$\alpha(B) = \inf\{\varepsilon > 0 : B \subset \bigcup_{j=1}^m B_j \text{ and } diam(B_j) \le \varepsilon\}.$$

Properties: The Kuratowski measure of noncompactness satisfies the following properties (for additional details see [19,11]).

(1) $\alpha(B) = 0$ if and only if $\overline{B}$ is compact (B is relatively compact).
(2) $\alpha(B) = \alpha(\overline{B})$.
(3) $A \subset B$ implies $\alpha(A) \leq \alpha(B)$.
(4) $\alpha(A+B) \leq \alpha(A) + \alpha(B)$.
(5) $\alpha(cB) = |c|\,\alpha(B),\ c \in \mathbb{R}$.
(6) $\alpha(conB) = \alpha(B)$.

Here $\overline{B}$ and $conB$ denote the closure and the convex hull of the bounded set B, respectively.

Let us now recall Mönch's fixed point theorem and an important lemma.

Lemma 1. [15] *Let D be a bounded, closed, and convex subset of a Banach space $\mathbb{E}$ such that $0 \in D$, and let N be a continuous mapping from D into itself. If the implication*

$$V = \left\{ \begin{array}{c} \overline{co}N(V) \\ or \\ N(V) \cup \{0\} \end{array} \right\} \ implies \ \ \alpha(V) = 0$$

holds for every subset V of D, then N has a fixed point.

Lemma 2. [16] *Let D be a bounded, closed and convex subset of a Banach space $C(J,\mathbb{E})$, G be a continuous function on $J \times J$, and let $f : J \times \mathbb{E} \to \mathbb{E}$ satisfy the Carathéodory conditions. Assume there exists $p \in L^1(J,\mathbb{R}_+)$ such that, for each $t \in J$ and each bounded set $B \subset \mathbb{E}$,*

$$\lim_{k\to 0^+} \alpha(f(J_{t,k} \times B)) \leq p(t)\alpha(B),\ where\ J_{t,k} = [t-k,t] \cap J.$$

If V is an equicontinuous subset of D, then

$$\alpha\left(\left\{\int_J G(s,t) f(s,y(s))ds : y \in V\right\}\right) \leq \int_J \|G(t,s)\| p(s)\alpha(V(s))ds.$$

3 Main Results

We begin this section with some lemmas about fractional equations.

Lemma 3. *Let $r \geq 0$. The differential equation*

$$^cD^r h(t) = 0 \tag{3}$$

has solutions $h(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1}t^{n-1}$, $c_i \in \mathbb{R}$, $i = 0,\ldots,n-1$, $n = \lceil r \rceil$.

Lemma 4. *Let $r \geq 0$. Then*

$$I^r\, {}^cD^r h(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1}t^{n-1} + h(t), \tag{4}$$

where $c_i \in \mathbb{R}$, $i = 0,\ldots,n-1$, $n = \lceil r \rceil$.

Lemma 5. *Let $0 < r \leq 1$, $bT \neq 2$, and let $h : J \to \mathbb{E}$ be continuous. A function y is a solution of the fractional integral equation*

$$y(t) = \int_0^T G(t,s)h(s)ds, \tag{5}$$

where $G(t,s)$ is the Green's function defined by

$$G(t,s) = \begin{cases} \dfrac{(t-s)^{r-1}}{\Gamma(r)} + \dfrac{b(T-s)^r}{(2-Tb)\Gamma(r+1)} - \dfrac{(T-s)^{r-1}}{(2-Tb)\Gamma(r)}, & 0 \leq s \leq t \leq T, \\ \dfrac{b(T-s)^r}{(2-Tb)\Gamma(r+1)} - \dfrac{(T-s)^{r-1}}{(2-Tb)\Gamma(r)}, & 0 \leq t \leq s \leq T, \end{cases} \tag{6}$$

if and only if y is a solution of the fractional BVP

$$^cD^r y(t) = h(t),\ a.e.\ t \in J = [0,T],\ \ 0 < r \leq 1, \tag{7}$$

$$y(T) + y(0) = b\int_0^T y(s)ds,\ \ bT \neq 2. \tag{8}$$

*Proof.*If y satisfies (7), then Lemma 4 implies that

$$y(t)=c_0+\int_0^t \frac{(t-s)^{r-1}}{\Gamma(r)}h(s)ds. \tag{9}$$

By (8),

$$c_0=\frac{b}{2-Tb}\int_0^T \frac{(T-s)^r}{\Gamma(r+1)}h(s)ds-\frac{1}{2-bT}\int_0^T \frac{(T-s)^{r-1}}{\Gamma(r)}h(s)ds, \tag{10}$$

so we obtain equation (5), where G defined in (6). Conversely, it is clear that if y satisfies (5), then (7) and (8) hold.

Theorem 1. *Assume the following conditions hold:*

(H1) The function $f: J\times\mathbb{E}\longrightarrow\mathbb{E}$ is Carathéodory;
(H2) There exists $p\in L^1(J,\mathbb{R}_+)$, such that

$$\|f(t,y)\|\leq p(t)\|y\| \text{ for a.e. } t\in J \text{ and each } y\in\mathbb{E};$$

(H3) For a.e. $t\in J$ and each bounded set $B\subset\mathbb{E}$, we have

$$\lim_{k\to 0^+}\alpha(f(J_{t,k}\times B))\leq p(t)\alpha(B),$$

where α is the Kuratowski measure of compactness and $J_{t,k}=[t-k,t]$.

Then the BVP (1)–(2) has at least one solution in $C(J,B)$ provided that

$$\|I^r(p)\|_{L^1}+\frac{|b|(I^{r+1}p)(T)}{|2-Tb|}+\frac{(I^rp)(T)}{|2-Tb|}<1. \tag{11}$$

Proof. To transform the problem (1)–(2) into a fixed point problem, consider the operator

$$\begin{aligned}N(y)=&\int_0^t \frac{(t-s)^{r-1}}{\Gamma(r)}f(s,y(s))ds+\int_0^T \frac{b(T-s)^r}{(2-Tb)\Gamma(r+1)}f(s,y(s))ds\\&-\int_0^T \frac{(T-s)^{r-1}}{(2-Tb)\Gamma(r)}f(s,y(s))ds.\end{aligned}$$

Clearly, from Lemma 5, the fixed points of N are solutions to the problem (1)–(2).

Let $R>0$ and consider the set

$$D_R=\{y\in C(J,\mathbb{E}):\|y\|_\infty\leq R\}.$$

We shall show that N satisfies the assumptions of Mönch's fixed point theorem (Lemma 1 above). The proof will be given in several steps.

Step 1: *N is continuous.* Let $\{y_n\}$ be a sequence such that $y_n\to y$ in $C(J,\mathbb{E})$. Then, for each $t\in J$,

$$\begin{aligned}|(Ny_n)(t)-(Ny)(t)|\leq&\int_0^t \frac{(t-s)^{r-1}}{\Gamma(r)}|f(s,y_n(s))-f(s,y(s))|ds\\&+\int_0^T \frac{|b|(T-s)^r}{|2-Tb|\Gamma(r+1)}|f(s,y_n(s))-f(s,y(s))|ds\\&+\int_0^T \frac{(T-s)^{r-1}}{|2-Tb|\Gamma(r)}|f(s,y_n(s))-f(s,y(s))|ds.\end{aligned}$$

Let $\rho>0$ be such that

$$\|y_n\|_\infty\leq\rho \text{ and } \|y\|_\infty\leq\rho.$$

Then from (H2), we have

$$|f(s,y_n(s))-f(s,y(s))|\leq 2\rho\, p(s):=\sigma(s),$$

and $\sigma\in L^1(J,\mathbb{R}_+)$. Since f is Carathéodory,

$$\|N(y_n)-N(y)\|_\infty\to 0 \text{ as } n\to\infty.$$

Step 2: *N maps D_R into itself.* For each $y \in D_R$, by (H2) and (11), we have for each $t \in J$,

$$\begin{aligned}|N(y)(t)| &\leq \int_0^t \frac{(t-s)^{r-1}}{\Gamma(r)}|f(s,y(s))|ds \\ &\quad + \int_0^T \frac{|b|(T-s)^r}{|2-Tb|\Gamma(r+1)}|f(s,y(s))|ds + \int_0^T \frac{(T-s)^{r-1}}{|2-Tb|\Gamma(r)}|f(s,y(s))|ds \\ &\leq R\left(\|I^r(p)\|_{L^1} + \frac{|b|(I^{r+1}p)(T)}{|2-Tb|} + \frac{(I^r p)(T)}{|2-Tb|}\right) \\ &\leq R.\end{aligned}$$

Step 3: *$N(D_R)$ is bounded and equicontinuous.* From Step 2, it is clear that $N(D_R) \subset C(J,\mathbb{E})$ is bounded. To show the equicontinuity of $N(D_R)$, let $t_1,\ t_2 \in J$, with $t_1 < t_2$, and $y \in D_R$. We have

$$\begin{aligned}|(Ny)(t_2)-(Ny)(t_1)| &\leq \frac{1}{\Gamma(r)}\int_0^{t_1}\left[(t_2-s)^{r-1}-(t_1-s)^{r-1}\right]|f(s,y(s))|ds \\ &\quad + \frac{1}{\Gamma(r)}\int_{t_1}^{t_2}(t_2-s)^{r-1}|f(s,y(s))|ds \\ &\leq \frac{R}{\Gamma(r)}\left[\int_0^{t_1}[(t_2-s)^{r-1}-(t_1-s)^{r-1}]p(s)ds \right. \\ &\quad \left. + \int_{t_1}^{t_2}(t_2-s)^{r-1}p(s)ds\right].\end{aligned}$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero.

Now let V be a subset of D_R such that $V \subset \overline{co}(N(V)\cup\{0\})$. Since V is bounded and equicontinuous, the function $t \to \vartheta(t) = \alpha(V(t))$ is continuous on J. By (H2)–(H3), Lemma 2, and the properties of the measure α, we have that for each $t \in J$,

$$\begin{aligned}\vartheta(t) &\leq \alpha(N(V)(t)\cup\{0\}) \\ &\leq \alpha(N(V)(t)) \\ &\leq \alpha(V(t))\left((I^r p)(T) + \frac{|b|(I^{r+1}p)(T)}{|2-Tb|} + \frac{(I^r p)(T)}{|2-Tb|}\right).\end{aligned}$$

Hence,

$$\|\vartheta\|_\infty\left(1-\left[\|I^r(p)\|_{L^1} + \frac{|b|(I^{r+1}p)(T)}{|2-Tb|} + \frac{(I^r p)(T)}{|2-Tb|}\right]\right) \leq 0.$$

From (11), it follows that $\|\vartheta\|_\infty = 0$, that is, $\vartheta = 0$ for each $t \in J$, and so $V(t)$ is relatively compact in $\mathbb{E}$. In view of the Ascoli-Arzelà theorem, V is relatively compact in D_R. Applying Lemma 1, we conclude that N has a fixed point that is a solution of the problem (1)–(2).

4 An Example

As an application of our main results, we consider the fractional differential equation

$$^{C}D^{\frac{1}{2}}y(t) = \frac{t\sqrt{\pi}-1}{16}y(t), \text{ for a.e. } (t,y) \in ([0,1],\mathbb{R}_+), \tag{12}$$

$$y(1)+y(0) = \int_0^1 y(s)ds. \tag{13}$$

Here $r = \frac{1}{2}$, $T = 1$, $b = 1$, and

$$f(t,y) = \frac{t\sqrt{\pi}-1}{16}y(t).$$

Then,

$$|f(t,y)| = \left|\frac{t\sqrt{\pi}-1}{16}y(t)\right| \leq \frac{t\sqrt{\pi}}{16}|y|.$$

Choosing $p(t) = \frac{t\sqrt{\pi}}{16}$, we have that

$$\begin{aligned}\left(\|I^r(p)\|_{L^1} + \frac{|b|(I^{r+1}p)(T)}{|2-Tb|} + \frac{(I^r p)(T)}{|2-Tb|}\right) &= (I^{\frac{1}{2}}p)(1) + (I^{\frac{3}{2}}p)(1) + (I^{\frac{1}{2}}p)(1) \\ &= \left(\frac{1}{12} + \frac{1}{30} + \frac{1}{12}\right) = \frac{1}{5} < 1.\end{aligned}$$

Then, by Theorem 1, the problem (12)–(13) has a solution on $[0,1]$.

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