Journal of Statistics Applications & Probability
An International Journal

http://dx.doi.org/10.18576/jsap/070110

Laplace Transform Order for Unknown Age Class of Life Distribution with Some Applications

S. E. Abu-Youssef[1] *and M. E. Bakr* [2,*]

[1] Department of Mathematics; Faculty of Science; Al-Azhar University; Nasr City (11884),Cairo, Egypt.
[2] Department of Basic Science; Egyptian Academy for Engineering and Modern Technology, Cairo; Egypt.

Received: 15 Aug. 2017, Revised: 21 Dec. 2017, Accepted: 27 Dec. 2017
Published online: 1 Mar. 2018

Abstract: In this work a new class of life distribution, namely used better than aged in Laplace transform order (UBAL) class of life distribution is introduced, Relations of this aging to other well-known aging and their applications to a shock model are discussed. Preservations of this aging concept under some reliability operations are also given. Testing exponentiality versus (UBAL) class of life distribution is proposed. Pitman's asymptotic efficiencies of the test are calculated and compared with other tests. The percentiles of this test statistic are tabulated.

Keywords: UBA class of life distribution; Laplace transform; Shock model application; Life testing.

1 Introduction

Needing to the reliability theory is very important because, it gives a common scientific language between the Scientists working in various fields of aging studies. In relation to various aging characteristics statisticians divided the life distributions into classes such as increasing failure rate (IFR), increasing failure rate average (IFRA), new better than used (NBU), new better than used in convex ordering (NBUC), new better than used in expectation (NBUE), harmonic new better than used in expectation (HNBUE), new better than used in Laplace transform order (NBUL), used better than aged (UBA), (UBACT) and a lot of other classes. Cline [12] and others studied the connection between the class of age-smooth distributions and the class of distribution with sub-exponential tails which have many applications in queuing theory random walk and infinite divisibility.

Such aging classes are derived via several notions of comparison between random variables. So we introduce a new aging notion derived from the Laplace transform order. Before we go into the details, let us quickly review some common notions of stochastic orderings and aging notions considered in this paper.

If X and Y are two random variables with distributions F and G (survivals $\overline{F}$ and $\overline{G}$), respectively, then we say that X is smaller than Y in the:

a)Usual stochastic order, denoted by $X \leq_{st} Y$ if

$$\overline{F}(x) \leq \overline{G}(x) \qquad \text{for all} \quad x.$$

b)Increasing convex order, denoted by $X \leq_{icx} Y$ if

$$\int_x^\infty \overline{F}(u)du \leq \int_x^\infty \overline{G}(u)du.$$

c)Increasing concave order, denoted by $X \leq_{icv} Y$ if

$$\int_0^x \overline{F}(u)du \leq \int_0^x \overline{G}(u)du.$$

* Corresponding author e-mail: mah8800@hotmail.com

Another importing ordering that has come to use in reliability and life testing is the following:

A random variable X is smaller than a random variable Y with respect to Laplace transform order (denoted by $X \leq_{Lt} Y$) if, and only, if

$$\int_0^\infty e^{-sx}\mathrm{dF(x)} \geq \int_0^\infty e^{-sx}\mathrm{dG\,(x)}, \quad s \geq 0. \tag{1}$$

It is easy to check that (1) is equivalent to

$$\int_0^\infty e^{-sx}\overline{\mathrm{F}}(\mathrm{x})\mathrm{dx} \leq \int_0^\infty e^{-sx}\overline{\mathrm{G}}\,(\mathrm{x})\mathrm{dx}. \tag{2}$$

Applications, properties and interpretations of the Laplace transform order in the statistical theory of reliability, and in economics can be found in Denuit [13], Klefsjo [16], and Ahmed and Kayid [6].

On the other hand, in the context of lifetime distributions, some of the above orderings have been used to give characterizations and new definitions of aging classes. By aging, we mean the phenomenon whereby an older system has a shorter remaining lifetime, in some statistical sense, than a younger one (Bryson and Siddiqui [11]).

We aim in this paper to introduce a new aging notion derived from the Laplace transform order, namely used better than aged Laplace transform order (UBAL) class of life distribution. Definition and relationships are given in section 2. In section 3, we discussed some closure properties to (UBAL) such as convolution and formation of a coherent system. In section 4, applications of this aging to a shock model is given. Based on goodness of fit approach our test is constructed in section 5. Monte Carlo null distribution critical values are simulated and tabulated in Table 1 for sample sizes n = 5(5)100 using Mathematica 8 programme in section six. Finally, Pitman asymptotic efficiencies for linear failure rate (LFR), Weibull and Makeham distributions, which are belong to the UBAL class, are calculated in section seven.

2 Definitions and Preliminaries

In reliability theory, aging life is usually characterized by a nonnegative continuous random variable $X \geq 0$ representing equipment life with distribution function F and survival function $\overline{F}\,(t) = 1 - F(t)$ such that $F\,(0-) = 0$. One of the most important approaches to the study of aging is based on the concept of the residual life. For any random variable X, let $X_t = [X - t \mid X > t]$, $\quad t \in \{x : F\,(x) < 1)$, denote a random variable whose distribution is the same as the conditional distribution of $X - t$ given that $X > t$ and has survival function

$$\overline{F}_t\,(x) = \begin{cases} \frac{\overline{F}(x+t)}{\overline{F}(t)} & \overline{F}\,(t) > 0 \\ 0 & \overline{F}\,(t) = 0 \end{cases}.$$

When X is the lifetime of a device which has a finite mean $\mu = E\,(X) = \int_0^\infty \overline{F}\,(u)\,du$, the mean of X_t is called mean residual life (MRL) and is given by

$$\mu\,(t) = E\,(X_t) = \frac{\int_t^\infty \overline{F}\,(u)\,du}{\overline{F}\,(t)}. \tag{3}$$

Further, the hazard rate of X is defined by

$$h\,(t) = -\frac{d}{dt}\ln\overline{F}\,(t) \; = \frac{f\,(t)}{\overline{F}\,(t)},\; t \geq 0,\; \overline{F}\,(t) > 0,$$

where $f(t) = F'(t)$ is the probability density of X assuming it exist. Note that if $\lim_{t\to\infty} h(t) \; = h(\infty)$ exists and is positive, then cf. Willmot and cai [22]

$$\mu\,(\infty) = \lim_{t\to\infty} \mu\,(t) \; = \frac{1}{h(\infty)}.$$

Two classes of life distributions were introduced by Alzaid [7] which are used better than aged (UBA) and used better than aged in expectation (UBAE) classes of life distribution.

Precisely we have the following definitions:

Definition 1. *The df F is said to be used better than aged (UBA) if $0 < \mu\,(\infty) < \infty$ and for all $x,t \geq 0$,*

$$\overline{F}\,(x+t) \geq \overline{F}\,(t)\,e^{-x/\mu(\infty)}, \qquad x,t \geq 0 \tag{4}$$

Definition 2. *The distribution function F is said to be used better than aged in expectation (UBAE) if* $0 < \mu(\infty) < \infty$ *and for all* $x,t \geq 0$,

$$\mu(t) \geq \mu(\infty) \tag{5}$$

Remark that, F is UBA (UBAE) if and only if X_t converges in distribution to a random variable X_A (say) exponentially distributed with failure rate γ and

$$X_t \leq_{st} X_A, \ (E(X_t) \leq_{st} E(X_A)).$$

According to the above definitions we can deduce the following new definition for used better than aged in the Laplace transforms order as follows.

Definition 3. *The distribution function F is said to be used better than aged in the Laplace transform order (UBAL) if* $0 < \mu(\infty) < \infty$ *and for all* $x,t \geq 0$,

$$\int_0^\infty e^{-sx}\overline{\mathrm{F}}(x+t)\mathrm{d}x \geq \frac{\mu(\infty)}{1+s\mu(\infty)}\overline{F}(t), \quad \mathrm{s} \geq 0, \tag{6}$$

It is obvious that (6) is equivalent to $X_t \leq_{Lt} X_A$ for all t ≥ 0.

To introduce the definition of the discrete UBAL, let X be a discrete non-negative random variable such that $P(X = k) = p_k, \ k = 0, 1, 2, \ldots$. Let $\overline{\mathrm{P}}_k = P(X > k), \ k \geq 1, \overline{\mathrm{P}}_0 = 1$ denote the corresponding survival function.

The discrete non-negative random variable X is said to be discrete used better than aged in Laplace transform order (discrete UBAL) if, and only, if

$$\sum_{k=0}^{\infty} \overline{\mathrm{P}}_{k+i} z^k \geq \overline{\mathrm{P}}_i \sum_{k=0}^{\infty} z^k, \quad for\ all\ 0 \leq z \leq 1\ and\ i = 0, 1, \ldots.$$

Now,

$$X \leq_{st} X_A \Rightarrow X \leq_{Lt} X_A.$$

Then, we have the following implication:

$$\begin{array}{ccccc} \text{IFR} & \subset & \text{UBA} & \subset & \text{UBAL} \\ & & \bigcup & & \\ & & \text{UBAE} & & \end{array}$$

See Abu-Youssef and Bakr [1,2].

Applications, properties and interpretations of the Laplace transform order in the statistical theory of reliability, and in economics can be found in Denuit [13], Klefsjo [16], and Ahmed and Kayid [6].

Some interpretations for the UBAL class are as follows:

– One simple interpretation of $\int_0^\infty e^{-sx}\overline{\mathrm{F}}(x)\mathrm{d}x$ is that the mean life of a series system of two statistically independent components, one having exponential survival function and the other having survival function $\overline{\mathrm{F}}$. Consider now two series systems, say system A and system B. System A has a used component of age t with survival function $\overline{\mathrm{F}}_t$ while system B has an aged component with exponential survival function.
  Thus, $F \in$ UBAL implies that the mean life of a system B is not larger than that of system A.

– A machine has survival function $\overline{\mathrm{F}}$ and produces one unit of output per hour when functioning. The present value of one unit produced at time t is $1.e^{-st}$, where s is the discount rate. Then the expected present value of total output produced during the life of the machine is

$$\int_0^\infty e^{-sx}\overline{\mathrm{F}}(x)\mathrm{d}x.$$

Thus, $F \in$ UBAL implies that a used machine of age t governed by survival function $\overline{\mathrm{F}}_t$ produces a greater expected total present value than does an aged machine governed by exponential survival function.

3 Preservation Results

As an important reliability operations, convolution, mixture and formation of coherent system of a certain class of life distribution is often paid much attention. It has been shown that UBAL are closed under these operations.

3.1 Convolution

In the next theorem we establish the closure property for UBAL under convolution.

Theorem 1. *UBAL class of life distribution is closed under convolution operation.*

Proof. Suppose F_1 and F_2 are UBAL, then we have

$$\begin{aligned}\int_0^\infty e^{-sx}\overline{\mathrm{F}}(x+t)\,dx &= \int_0^\infty\int_0^\infty e^{-sx}\overline{\mathrm{F}}_1(x+t-u)\,d\mathrm{F}_2(u)\,dx\\ &= \int_0^\infty\int_0^\infty e^{-sx}\overline{\mathrm{F}}_1(x+t-u)\,dx\,d\mathrm{F}_2(u)\\ &\geq \int_0^\infty \frac{\mu(\infty)}{1+s\mu(\infty)}\overline{\mathrm{F}}_1(t-u)\mathrm{dF}_2(u)\,dx\\ &= \frac{\mu(\infty)}{1+s\mu(\infty)}\overline{\mathrm{F}}(t).\end{aligned}$$

Which proved that the UBAL is closed under convolution operation.
Where

$$\overline{\mathrm{F}}(x+t) = \int_0^\infty \overline{\mathrm{F}}_1(x+t-u)\,d\mathrm{F}_2(u).$$

3.2 Formation of coherent systems using independent components

A system is called coherent if:

1. Every component is relevant.
2. The structure function, which represents the performance of the system in terms with performance of the component is increasing.

Design engineers give greater importance to coherency in building systems.
For more details about coherent system see Barlow and proschan [8].
In the next theorem we establish the closure property of the UBAL class under the formation of a coherent system operation.

Theorem 2. *A series system of n independent UBAL components is UBAL.*

Proof. Let $X_1, X_2, \ldots, X_n$ be independent UBAL then we have

$$\begin{aligned}\int_0^\infty e^{-sx}\frac{p(min(X_1,\ldots,X_n)\geq y+t)}{p(min(X_1,\ldots,X_n)\geq t)}dx &= \prod_{i=1}^n\int_0^\infty e^{-sx}\frac{p(X_i\geq y+t)}{p(X_i\geq t)}dx\\ &= \prod_{i=1}^n\int_0^\infty e^{-sx}\frac{\overline{F}_i(y+t)}{\overline{F}_i(t)}dx\\ &\geq \prod_{i=1}^n\int_0^\infty e^{-sx}e^{-x/\mu_i(\infty)}dx\end{aligned}$$

Since F_i is $UBAL$

$$\begin{aligned}&= \int_0^\infty e^{-sx}e^{-x/(\sum_{i=1}^n\mu_i(\infty))}dx\\ &= \int_0^\infty e^{-\frac{(1+s(\sum_{i=1}^n\mu_i(\infty))}{\sum_{i=1}^n\mu_i(\infty)}x}dx\\ &= \frac{\sum_{i=1}^n\mu_i(\infty)}{1+s(\sum_{i=1}^n\mu_i(\infty))}.\end{aligned}$$

This implies that the series system $\mathrm{X}_1, \mathrm{X}_2, \ldots, \mathrm{X}_n$ is UBAL.

4 Applications:

4.1 Shock model application

Suppose that a device is subject to shocks. Let $N(t)$ be the number of shocks in time interval $(0,t]$. The kth shock arrives at time T_k . Let $X_k = T_{k+1} - T_k$ be the time between the kth and $(k+l)st$ shocks. We assume that $X_1, X_2, \ldots$ are mutually independent and identically distributed according to F.

Let

$$a_k(t) = p(N(t) = k), \qquad k = 1,\ 2, \ldots$$

and let $\overline{P}_k$ be the probability of the device surviving k shocks.

Then the survival probability of the system until time t is

$$\overline{H}(t) = \sum_{k=0}^{\infty} a_k(t)\overline{P}_k \quad .$$

Theorem 3. *F is UBAL implies H is UBAL.*

Proof.: Observe that $\overline{H}(t)$ can be written in the form

$$\overline{H}(t) = \sum_{k=1}^{\infty} \overline{F}_k(t)\ p_k$$

Where $p_k = \overline{P}_{k-1} - \overline{P}_k,\ k = 1,\ 23,\ \ldots$ and F_k is the distribution function of T_k, and

$$\int_0^{\infty} e^{-sx}\overline{H}(x+t)\,dx\ = \sum_{k=1}^{\infty}\int_0^{\infty} e^{-sx}\overline{F}_k(x+t)\ p_k dx$$

$$\geq \frac{\mu(\infty)}{1+s\mu(\infty)}\sum_{k=1}^{\infty}\overline{F}_k(t)\ p_k$$

Since F is UBAL

$$= \frac{\mu(\infty)}{1+s\mu(\infty)}\overline{H}(t),$$

then H is **UBAL.**

5 Testing Against UBAL

This section is concerned with the construction of the proposed statistic as a U-statistic and discussing its asymptotic normality.

Here, We hope to test the null hypothesis H_0 : F is exponential, against H_1 : F is UBAL, and is not exponential. Non-parametric testing for classes of life distributions has been considered by many authors (see Hendi et al. [15]; Mahmoud et al., [19,20]; Abu-Youssef and Bakr [1,2]; Abu-Youssef et al [3,4].

According to Eq. (6) We may use the following as a measure of departure from H_0.

$$\begin{aligned}\delta(\mathrm{s}) &= E\ \left[\int_0^{\infty} e^{-sx}\overline{\mathrm{F}}(x+t)\mathrm{d}x - \frac{\mu(\infty)}{1+s\mu(\infty)}\overline{F}(t)\right] \\ &= \int_0^{\infty}\left[\int_0^{\infty} e^{-sx}\overline{\mathrm{F}}(x+t)\mathrm{d}x - \frac{\mu(\infty)}{1+s\mu(\infty)}\overline{F}(t)\right]\ \mathrm{dF}_0(t),\end{aligned}$$

The following theorem is essential for the development of our test statistic.

Theorem 4. *Let* X *be the UBAL random variable with distribution function* F*; then based on the Goodness of fit approach technique,*

$$\delta(\mathrm{s}) = \frac{1}{(1-\mathrm{s})}\left[\frac{1}{\mathrm{s}}(1-\ \varphi) + \frac{1+\mu(\infty)}{(1+s\mu(\infty))}\left(\int_0^{\infty}\mathrm{e}^{-\mathrm{x}}\mathrm{dF}(\mathrm{x}) - 1\right)\right] \tag{7}$$

where $\varphi(\mathrm{s}) = \int_0^{\infty}\mathrm{e}^{-\mathrm{sx}}\mathrm{dF}(\mathrm{x})$.

*Proof.*Since

$$\delta(\mathrm{s}) = \int_0^\infty [\int_0^\infty e^{-sx}\overline{\mathrm{F}}(x+t)\mathrm{d}x - \frac{\mu(\infty)}{1+s\mu(\infty)}\overline{F}(t)]\ \mathrm{dF}_0(t).$$

We can take $\mathrm{F}_0(x) = 1-e^{-x}$, $x \geq 0$, then

$$\delta(\mathrm{s}) = \int_0^\infty \int_0^\infty \mathrm{e}^{-\mathrm{t}-\mathrm{su}}\overline{\mathrm{F}}(\mathrm{u}+\mathrm{t})\mathrm{dudt} - \frac{\mu(\infty)}{1+s\mu(\infty)}\int_0^\infty \overline{\mathrm{F}}(\mathrm{t})\ \mathrm{e}^{-\mathrm{t}}\ \mathrm{dt}$$

$$= I_1 - I_2.$$

Where,

$$\begin{aligned} I_1 &= \int_0^\infty \int_0^\infty \mathrm{e}^{-\mathrm{t}}\mathrm{e}^{-\mathrm{su}}\overline{\mathrm{F}}(\mathrm{u}+\mathrm{t})\mathrm{dudt} \\ &= \int_0^\infty \int_{\mathrm{t}}^\infty \mathrm{e}^{-\mathrm{t}}\mathrm{e}^{-\mathrm{s(x-t)}}\overline{\mathrm{F}}(\mathrm{x})\mathrm{dxdt} \\ &= \int_0^\infty \int_0^{\mathrm{t}} \mathrm{e}^{-\mathrm{t}}\mathrm{e}^{-\mathrm{s(t-x)}}\overline{\mathrm{F}}(\mathrm{t})\mathrm{dxdt} \\ &= \frac{1}{s}\int_0^\infty \mathrm{e}^{-\mathrm{t}}(1-\mathrm{e}^{-\mathrm{st}})\overline{\mathrm{F}}(\mathrm{t})\,\mathrm{dt} \\ &= \frac{1}{1-\mathrm{s}}\left[\frac{1}{\mathrm{s}}(1-\varphi(\mathrm{s}))-1+\int_0^\infty \mathrm{e}^{-\mathrm{t}}\mathrm{dF}(\mathrm{t})\right]. \end{aligned} \tag{8}$$

And,

$$I_2 = \frac{\mu(\infty)}{1+s\mu(\infty)}\int_0^\infty \overline{\mathrm{F}}(\mathrm{t})\ \ \mathrm{dF}_0(\mathrm{t}) = \frac{\mu(\infty)}{1+s\mu(\infty)}\left[1-\int_0^\infty \mathrm{e}^{-\mathrm{t}}\mathrm{dF}(\mathrm{t})\right]. \tag{9}$$

From equations, (8) and (9), we obtain (7).

Let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution function F.

For generality, we assume $\mu(\infty)$ is known and equal one. The empirical estimator $\widehat{\delta}(\mathrm{s})$ of our test statistic can be obtained as follows:

$$\widehat{\delta}_n(\mathrm{s}) = \frac{1}{n(1-s)}\sum_{\mathrm{i}}\left\{\frac{1}{s}\left(1-\mathrm{e}^{-\mathrm{sX_i}}\right)+\frac{2}{(1+s)}(\mathrm{e}^{-\mathrm{X_i}}-1)\right\}.$$

To make the test is invariant, let

$$\widehat{\Delta}_n(\mathrm{s}) = \frac{\widehat{\delta}_n(\mathrm{s})}{\overline{\mathrm{X}}}.$$

then,

Let us rewrite $\widehat{\delta}$ as follows,

$$\widehat{\Delta}_n(\mathrm{s}) = \frac{1}{\overline{\mathrm{X}}n}\sum_{\mathrm{i}}\phi(\mathrm{X_i})$$

where

$$\phi(\mathrm{X_i}) = \frac{1}{(1-s)}\{\frac{1}{s}\left(1-\mathrm{e}^{-\mathrm{sX_i}}\right)+\frac{2}{(1+s)}\left(\mathrm{e}^{-\mathrm{X_i}}-1\right)\}.$$

To find the limiting distribution of $\widehat{\delta}(\mathrm{s})$ we resort to the U-statistic theory and practice (Lee [17]).

Set

$$\phi(\mathrm{X_1}) = \frac{1}{(1-s)}\{\frac{1}{s}\left(1-\mathrm{e}^{-\mathrm{sX_1}}\right)+\frac{2}{(1+s)}\left(\mathrm{e}^{-\mathrm{X_1}}-1\right)\}.$$

Then, $\widehat{\Delta}_n(\mathrm{s})$ is equivalent to U-statistic given by:

$$U_n = \binom{1}{n}\sum_{\mathrm{i}}\phi(\mathrm{X_i}).$$

The following theorem summarizes the asymptotic normality of $\widehat{\delta}_n(\mathrm{s})$.

Table 1: The Upper Percentile Points of $\widehat{\delta}(5)$ with 10000 Replications.

n	90%	95%	99%
5	0.006021	0.0070851	0.0085853
10	0.004619	0.0057107	0.0074085
15	0.003828	0.0046726	0.006104
20	0.003680	0.0045432	0.0057393
25	0.003424	0.0041355	0.0052877
30	0.002935	0.0035352	0.0047174
35	0.002756	0.003277	0.004525
40	0.002546	0.0032092	0.0044086
45	0.002419	0.0031397	0.0043173
50	0.002323	0.0030042	0.0042904
55	0.002235	0.0028723	0.0039834
60	0.002149	0.0027979	0.0038899
65	0.002048	0.0027145	0.0037649
70	0.001937	0.0027041	0.0035067
75	0.001817	0.0024491	0.0032827
80	0.001718	0.0024184	0.0030806
85	0.001627	0.0024076	0.0029668
90	0.001534	0.0022396	0.0028221
95	0.001434	0.0021586	0.0027326
100	0.001352	0.0021096	0.0026798

Theorem 5. *(i)As* $\mathrm{n}\to\infty$, $(\widehat{\delta}_n(\mathrm{s})-\delta(\mathrm{s}))$ *is asymptotically normal with mean 0 and variance* $\sigma^2(\mathrm{s})$ *where,*

$$\sigma^2(\mathrm{s})=\mathrm{Var}\left[\widehat{\delta}_n(\mathrm{s})\right]=\mathrm{E}(\frac{1}{(1-s)}\left\{\frac{1}{s}\left(1-\mathrm{e}^{-\mathrm{sx}}\right)+\frac{2}{(1+s)}\left(\mathrm{e}^{-\mathrm{x}}-1\right)\right\})^2$$

(ii)Under H_0, the variance $\sigma_0^2(\mathrm{s})=\frac{2\mathrm{s}^2-8\mathrm{s}-6}{3\mathrm{s}(2\mathrm{s}+1)(\mathrm{s}-1)^2(s+1)^2}$.

Proof. (i)Using standard U-statistic theory, Lee [17], and direct calculations, we get

$$E\left[\widehat{\delta}_n(\mathrm{s})\right]=E\left(\frac{1}{(1-s)}\left\{\frac{1}{s}\left(1-\mathrm{e}^{-\mathrm{sx}}\right)+\frac{2}{(1+s)}\left(\mathrm{e}^{-\mathrm{x}}-1\right)\right\}\right);$$

$$\sigma^2(\mathrm{s})=\mathrm{Var}\left[\widehat{\delta}_n(\mathrm{s})\right]=\mathrm{E}(\frac{1}{(1-s)}\left\{\frac{1}{s}\left(1-\mathrm{e}^{-\mathrm{sx}}\right)+\frac{2}{(1+s)}\left(\mathrm{e}^{-\mathrm{x}}-1\right)\right\})^2.$$

(ii)Under H_0, the parameter $s=5$ say, and

$$\mu_0=E\left[\widehat{\delta}_n(\mathrm{s})\right]=0;$$

$$\sigma_0^2(\mathrm{s})=\frac{2\mathrm{s}^2-8\mathrm{s}-6}{3\mathrm{s}(2\mathrm{s}+1)(\mathrm{s}-1)^2(s+1)^2}=0.00004$$

6 Monte Carlo null distribution critical points

Based on 10000 generated samples from the standard exponential distribution the Monte Carlo null distribution critical values of our test $\widehat{\delta}(5)$ are simulated and tabulated, where n = 5(5)100 in Table 1. Mathematica 8 programme is used.

From Table 1 and Fig. 1, the critical values decrease as the sample size increases and they increase as the confidence level increases.

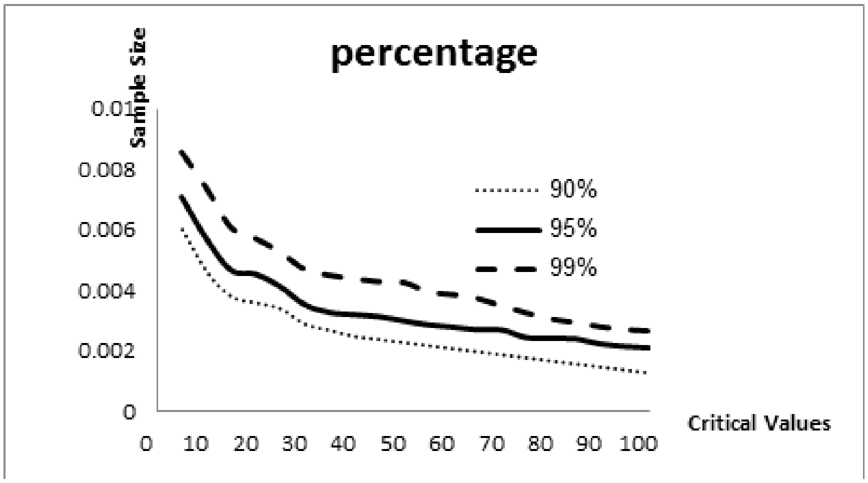

Fig. 1: The Relation Between Sample Size and Critical Values.

7 Pittman asymptotic relative efficiency (ARE)

Since the above test statistic $\widehat{\delta}\,(s)=\frac{\delta}{\bar{X}}$ is new and no other tests are known for this class (UBAL). We may compare our test to the other classes. Here we choose the test $\Delta_{\theta,(1)}$ presented by Mugdadi and Ahmad [21] and $\delta^{(2)}_{F_n}$ presented Mahmoud and Abdul Alim [18] for (NBAFR) class of life distribution. Then comparisons are achieved by using Pitman asymptotic relative efficiency PARE, which is defined as follows:

Let T_{1n} and T_{2n} be two statistics, then PARE of T_{1n} relative to T_{2n} is defined by

$$e\,(T_{1n},T_{2n})=\frac{\mu_1^{\backslash}(\theta_0)}{\sigma_1(\theta_0)}\Big/\frac{\mu_2^{\backslash}(\theta_0)}{\sigma_2(\theta_0)}.$$

Where

$$\mu_i^{\backslash}\,(\theta_0)=\lim_{n\to\infty}\frac{\partial}{\partial\theta}E\,(T_{n_i})\Big|_{\theta\to\theta_0},$$

and

$$\sigma_i^2\,(\theta_0)=\lim_{n\to\infty}var(T_{n_i})\,.$$

Three of the most commonly used alternatives they are:

(i)Linear failure rate family

$$\overline{F}_1\,(x)=e^{-x-\frac{x^2}{2}\theta},\qquad \theta,x\geq 0. \tag{10}$$

(ii)Weibull family:

$$\overline{F}_2\,(x)=e^{-x^{\theta}},\qquad \theta\geq 1,,x\geq 0. \tag{11}$$

(iii)Makeham family:

$$\overline{F}_2\,(x)=e^{-x-\theta(x+e^{-x}-1)},\qquad \theta,x\geq 0. \tag{12}$$

Note that H_0 (the exponential distribution) is attained at $\theta=0$ in (i) and (iii) and $\theta=1$ in (ii). The Pitman's asymptotic efficiency (PAE) of $\widehat{\Delta}\,(s)$ is equal to PAE $\left(\widehat{\delta}\,(s)\right)=\frac{\left|\frac{\partial}{\partial\theta}\delta(s)\right|_{\theta\to\theta_0}}{\sigma_0(s)}$

$$=\frac{1}{\sigma_0\,(s)}\left|\frac{1}{s(1-s)}\int_0^{\infty}e^{-sx}d\overline{F}^{\backslash}_{\theta_0}\,(x)-\frac{2}{(1+s)(1-s)}\int_0^{\infty}e^{-x}d\overline{F}^{\backslash}_{\theta_0}\,(x)\right|$$

Where $\overline{F}^{\backslash}_{\theta_0}\,(x)=\frac{d}{d\theta}\overline{F}_{\theta}(u)\big|_{\theta\to\theta_0}$

This leads to:

(i)PAE in case of the linear failure rate distribution:

$$\mathrm{PAE}\left(\widehat{\delta}\left(\mathrm{s}\right)\right)=\frac{1}{\sigma_0\left(5\right)}\left|\frac{-1}{20}\int_0^\infty e^{-sx}d\left(\frac{-x^2}{2}e^{-x}\right)+\frac{1}{12}\int_0^\infty e^{-x}d\left(\frac{-x^2}{2}e^{-x}\right)\right|=1.43$$

(ii)PAE in case of the Weibull distribution:

$$\mathrm{PAE}\left(\widehat{\delta}\left(\mathrm{s}\right)\right)=\frac{1}{\sigma_0\left(5\right)}\left|\frac{-1}{20}\int_0^\infty e^{-sx}d\left(-x\ln|x|e^{-x}\right)+\frac{1}{12}\int_0^\infty e^{-x}d\left(-x\ln|x|e^{-x}\right)\right|=0.5972$$

(iii)PAE in case of the Makeham distribution.

$$\mathrm{PAE}\left(\widehat{\delta}\left(\mathrm{s}\right)\right)=\frac{1}{\sigma_0\left(5\right)}\left|\frac{-1}{20}\int_0^\infty e^{-sx}d\left((1-x-e^{-x})e^{-x}\right)+\frac{1}{12}\int_0^\infty e^{-x}d\left((1-x-e^{-x})e^{-x}\right)\right|=0.1019$$

Direct calculations of PAE of $\Delta_{\theta,(1)}$, $\delta^2_{F_n}$ and $\widehat{\delta}\left(\mathrm{s}\right)$ are summarized in table (2), the efficiencies in table shows clearly our U-statistic $\widehat{\delta}\left(\mathrm{s}\right)$ perform well for $F_1,\ F_2$ *and* F_3.

Table 2: PAE of $\Delta_{\theta,(1)}$, $\delta^{(2)}_{\mathrm{F_n}}$ and $\widehat{\delta}\left(\mathrm{s}\right)$

Distribution	$\Delta_{\theta,(1)}$	$\delta^{(2)}_{\mathrm{F_n}}$	$\widehat{\delta}\left(\mathrm{s}\right)$
LFR	0.408	0.217	1.42
Weibull	0.170	0.050	0.5972
Makeham	0.0395	0.144	0.1019

In table (3), we give PARE's of $\widehat{\delta}\left(\mathrm{s}\right)$ with respect to $\Delta_{\theta,(1)}$and $\delta^{(2)}_{\mathrm{F_n}}$ whose PAE are mentioned in table 2.

Table 3: PARE of $\widehat{\delta}\left(\mathrm{s}\right)$ with respect to $\Delta_{\theta,(1)}$ and $\delta^{(2)}_{\mathrm{F_n}}$.

Distribution	$\mathrm{e}(\widehat{\delta}\left(\mathrm{s}\right),\Delta_{\theta,(1)})$	$\mathrm{e}\left(\widehat{\delta}\left(\mathrm{s}\right),\delta^{(2)}_{\mathrm{F_n}}\right)$
LFR	3.48	6.54
Weibull	3.51	11.94
Makeham	2.58	0.71

It is clear from table (3) that the statistic $\widehat{\delta}\left(\mathrm{s}\right)$ perform well for $\overline{\mathrm{F}}_1,\ \overline{\mathrm{F}}_2$ *and* $\overline{\mathrm{F}}_3$ and $\overline{\mathrm{F}}_2$ and it is more efficient than both $\Delta_{\theta,(1)}$ and $\delta^{(2)}_{\mathrm{F_n}}$ for all cases mentioned above. Hence our test, which deals the much larger UBA is better and also simpler.

Acknowledgments

The authors would like to thank the editors and the referees for their valuable comments and suggestions.

References

[1] Abu-Youssef, S. E. and Bakr, M.E. (2014). Some properties of UBACT class of life distribution. J. of Adva. Res. in Appl. Math. 1-9.

[2] Abu-Youssef, S. E. and Bakr, M.E. (2014). A goodness of fit approach to class of life distribution with unknown age. Int. J. of Comp. App., 30-35.

[3] Abu-Youssef, S. E., Mohammed, B. I. and Bakr, M.E. (2014). Nonparametric test for unknown age class of life distributions. Int. J. of Rel. and Appl., 99-110.

[4] Abu-Youssef, S. E., Mohammed, B. I. and Bakr, M.E. (2015). Nonparametric Test for UBACT Class of Life Distribution Based on U-Statistic. J. of adv. of math., 402-411.

[5] Ahmad, I. A. (2004). Some properties of classes of life distributions with unknown age. J. Statist. Prob., 333-342.

[6] Ahmed H, Kayid M (2004) Preservation properties for the Laplace transform ordering of residual lives. Statist. Pap. 583?590

[7] Alzaid, A. A. (1994). Aging concepts for item of unknown age. Stochastic models, 649-659.

[8] Barlow RE., Proschan F. (1981). Statistical Theory of Reliability and Life Testing Probability Models.To Begin With: Silver-Spring, MD.

[9] Bhattacharjee. M. C. (1986). Tail behavior of age-smooth failure distributions and applications. Reli. and Qua. Cont. Ed. A. P. Basu, Elsevier Science Publishers B. V. North-Holland, 69-85.
[10] Block. H. and Savits. T. (1980). Laplace transforms for classes of life distributions. Ann. Prob., pp. 465:474.
[11] Bryson, M. C. and Siddiqui, M. M. (1969). Some criteria for aging. J. of Ame. Sta. Ass. 1472-1483.
[12] Cline D. B. H. (1987). Convolutions of distribution with exponential and sub-exponential tails. J. Austeal. Math. Soc. Ser. A. 347-365.
[13] Denuit M (2001) Laplace transform ordering of actuarial quantities. Insur Math Econ. 83?102.
[14] Embrechts P. and Goldie C. M. (1980). On Closure and Factorization Properties on subexponential and Relted Distributions. J. Austral. Math. Soe. Sei. A. , pp. 243:256.
[15] Hendi MI, Al-Nachawati H, AL-Ruzaiza AS (1999). A test for exponentiality against new better than used average. J King Saud Univ. 107?121.
[16] Klefsjo B (1983) A useful aging property based on the Laplace transform. J Appl Prob. 615?626.
[17] Lee (1990). U-statistics theory and practice. Marcel Dekker, New York.
[18] Mahmoud, M. A. W. and Abdul Alim, A. N. (2008). A Goodness of Fit Approach to for Testing NBUFR (NWUFR) and NBAFR (NWAFR) Properties, Int. J. Reliab. Appl. 125-140.
[19] Mahmoud M. A. W., EL-arishy SM, Diab LS (2003) Moment inequalities for testing new renewal better than used and renewal new better than used classes of life distributions. Int. J. Rel. Appl. 97?123.
[20] Mahmoud M. A. W., EL-arishy SM, Diab LS (2004). Testing renewal new better than used life distributions based on U-test. Appl. Math. Mod. 784?796.
[21] Mugdadi, A. R. and Ahmad, I. A. (2005). Moment Inequalities Derived from Comparing Life with Its Equlibrium Form. J. Stat. Plann. Inf. 303-317.
[22] Willmot, G.E. and Cai, J. (2000). On classes of lifetime distributions with unknown age. Probab. Eng. Inform. Sci. 473-484.

S. E. Abu-Youssef He is presently employed professor of Statistics and head of the Depart of Mathematics , Al Azhar University. He is referee of several journals in the branch of mathematical statistics . His research interest include Theory of reliability , Statistical inference , ordered statistics and Bayesian analysis . He is published mor than 70 paper in reputed international journals of Statistics.

Mahmoud Ebrahim Bakr is a researcher of Mathematical statistics at Basic Science Department, Egyptian Academy for Engineering and Modern Technology, Cairo, Egypt. He received his M. Sc. in Mathematics in 2015 from Al-Azhar University, Cairo, Egypt. His research interests include: Classes of life distributions and their Hypothesis tests.