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A New Bivariate Modified Weibull Distribution and its Extended Distribution

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Abstract: This paper introduced a new Bivariate Modified Weibull (BMW) distribution. It is a Marshall-Olkin type. Also, we introduce an extension of it and we call it bivariate Modified Weibull Geometric distribution. Marginal and conditional distribution functions are studied. Also, joint hazard rate function and maximum likelihood estimators (MLEs) for the parameters are presented. An application of BMW and BMWG distributions to an UEFA Champion′s League data set is provided and the profiles of the log-likelihood function of parameters of BMWD and BMWGD are plotted.

Keywords: Reliability, Geometric distribution, Modified Weibull distribution, maximum likelihood function, Survival Functions.

1 Introduction

Recently, Lai et al. [1] proposed Modified Weibull distribution. His new distribution capable of designing a bathtub-shaped hazard-rate function. Another advantage of this distribution is that it reduces to the Weibull, exponential and type I extreme-value distribution (called a log-gamma distribution)[2] and some other distributions.

Our aim in this paper is to proposed a new bivariate Modified Weibull (BMW) distribution, whose marginal are MW distributions. It is a Marshall-Olkin type. Many authors used this method to introduce a new bivariate distributions, see for example [3], [4], [5], [10], [6], [7]. Also, we introduce an extension of the Bivariate Modified Weibull distribution and we titled this distribution by Bivariate Modified Weibull-Geometric (BMWG) distribution, it is obtained by using a method similar to that used in [8], [9]. This paper is arranged as follows, a new bivariate Modified Weibull (BMW) distribution is given in Section 2. Also, various properties including the joint survival function, the joint cumulative distribution function, joint probability density function (pdf) and marginal pdf are investigated in this Section. Some reliability studies are obtained in Section 3. Section 4 is devoted to the MLEs of the parameters of the BMW distribution. In Section 5, an application of the BMW distribution to a UEFA Champion′s League data set are provided and the profiles of the log-likelihood function of parameters of BMWD are ploted. In Section 6, we introduce a new Modified Weibull-Geometric (BMWG) distribution. Moreover, various properties of BMWGD including the joint survival function, the joint probability density function, marginal probability density functions are computed in Section 6. The MLEs of the undermined parameters of the BMWG distribution are devoted in Section 7.In Section 8, an application of the BMWG distribution to a UEFA Champion′s League data set are provided. Finally, the results of this paper are concluded in Section 9.

2 Bivariate Modified Weibull distribution

In this section, we discuss the BMW distribution. First We discuss the joint survival function and derive their corresponding joint probability densities for this distribution. Let X be a random variable has Modified Weibull (MW) distribution with parameters α, β and $\lambda > 0$, with survival function given by the following form

$$\overline{F}(x) = e^{-\alpha x^{\beta} e^{\lambda x}},\ x \geq 0, \tag{1}$$

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and the corresponding pdf given by

$$f(x) = \alpha x^{\beta-1}(\beta+\lambda x)\, e^{\lambda x - \alpha x^{\beta} e^{\lambda x}},\ x \geq 0. \tag{2}$$

2.1 The Joint Survival Function of BMWD

Suppose that $U_1 \sim MW(\alpha_1,\beta,\lambda)$, $U_2 \sim MW(\alpha_2,\beta,\lambda)$ and $U_3 \sim MW(\alpha_3,\beta,\lambda)$ are independent random variables. Let $X_1 = \min\{U_1,U_3\}$ and $X_2 = \min\{U_2,\ U_3\}$. Then, we get the bivariate vector $(X_1,X_2) \sim BMW(\alpha_1,\alpha_2,\alpha_3,\beta,\lambda)$.

In the next lemma, We discuss the joint survival function for the random variables X_1 and X_2.

Lemma 1. *The joint survival function of X_1 and X_2 is given by*

$$\overline{F}_{BMW}(x_1,x_2) = \left(e^{-x_1^{\beta} e^{\lambda x_1}}\right)^{\alpha_1} \left(e^{-x_2^{\beta} e^{\lambda x_2}}\right)^{\alpha_2} \left(e^{-z^{\beta} e^{\lambda z}}\right)^{\alpha_3} \tag{3}$$

where $z = \max(x_1,x_2)$.

Proof. $\overline{F}_{BMW}(x_1,x_2) = P(X_1 > x_1, X_2 > x_2)$
$= P(\min\{U_1,U_3\} > x_1, \min\{U_2,U_3\} > x_2)$
$= P(U_1 > x_1, U_2 > x_2, U_3 > \max(x_1,x_2))$.
Where, $U_i\ (i=1,2,3)$ are independent random variables. Then, we get the following
$F_{BMW}(x_1,x_2) = P(U_1 > x_1)P(U_2 > x_2) \times P(U_3 > \max(x_1,x_2))$
$= \overline{F}_{MW}(x_1;\alpha_1,\beta,\lambda)\overline{F}_{MW}(x_2;\alpha_2,\beta,\lambda) \times \overline{F}_{MW}(z;\alpha_3,\beta,\lambda)$
$= \left(e^{-x_1^{\beta} e^{\lambda x_1}}\right)^{\alpha_1} \left(e^{-x_2^{\beta} e^{\lambda x_2}}\right)^{\alpha_2} \left(e^{-z^{\beta} e^{\lambda z}}\right)^{\alpha_3}$.

2.2 Joint Probability Density Function of BMWD

This subsection defined the joint pdf of the random variables X_1 and X_2 in the next theorem.

Theorem 1. *Given the joint survival function of (X_1,X_2) as in (3) then, the joint pdf of (X_1,X_2) is written in this way*

$$f_{BMW}(x_1,x_2) = \begin{cases} f_1(x_1,x_2) & if\ x_2 < x_1 \\ f_2(x_1,x_2) & if\ x_1 < x_2 \\ f_3(x) & .\ if\ x_1 = x_2 = x \end{cases} \tag{4}$$

where

$$\begin{aligned} f_1(x_1,x_2) &= f_{MW}(x_1;\alpha_1+\alpha_3,\beta,\lambda)\, f_{MW}(x_2;\alpha_2,\beta,\lambda) \\ &= (\alpha_1+\alpha_3)\,\alpha_2 x_1^{\beta-1}(\beta+\lambda x_1)\, e^{\lambda x_1 - (\alpha_1+\alpha_3)x_1^{\beta} e^{\lambda x_1}} \times x_2^{\beta-1}(\beta+\lambda x_2)\, e^{\lambda x_2 - \alpha_2 x_2^{\beta} e^{\lambda x_2}} \end{aligned} \tag{5}$$

$$\begin{aligned} f_2(x_1,x_2) &= f_{MW}(x_1;\alpha_1,\beta,\lambda)\, f_{MW}(x_2;\alpha_2+\alpha_3,\beta,\lambda) \\ &= (\alpha_2+\alpha_3)\,\alpha_1 x_1^{\beta-1}(\beta+\lambda x_1)\, e^{\lambda x_1 - \alpha_1 x_1^{\beta} e^{\lambda x_1}} \times x_2^{\beta-1}(\beta+\lambda x_2)\, e^{\lambda x_2 - (\alpha_2+\alpha_3)x_2^{\beta} e^{\lambda x_2}} \end{aligned} \tag{6}$$

$$\begin{aligned} f_3(x) &= \frac{\alpha_3}{\alpha_1+\alpha_2+\alpha_3} f_{MW}(x;\alpha_1+\alpha_2+\alpha_3,\beta,\lambda) \\ &= \alpha_3 x^{\beta-1}(\beta+\lambda x)\, e^{\lambda x - (\alpha_1+\alpha_2+\alpha_3)x^{\beta} e^{\lambda x}}. \end{aligned} \tag{7}$$

Proof. Suppose that $x_2 < x_1$. Then, $\overline{F}_{BMW}(x_1,x_2)$ in (3) becomes

$$\overline{F}_{BMW}(x_1,x_2) = \left(e^{-x_1^{\beta} e^{\lambda x_1}}\right)^{\alpha_1+\alpha_3} \left(e^{-x_2^{\beta} e^{\lambda x_2}}\right)^{\alpha_2}.$$

Then, upon differentiating this function w.r.t. x_1 and x_2 we get the following expression of $f_1(x_1,x_2)$ given in (5). By the same way we obtain $f_2(x_1,x_2)$ when $x_1 < x_2$. But $f_3(x)$ cannot be deduced in the same manner. So that, we utilize the following identity to derive $f_3(x)$

$\int_0^\infty \int_{x_2}^\infty f_1(x_1,x_2)\,dx_1dx_2 + \int_0^\infty \int_{x_1}^\infty f_2(x_1,x_2)\,dx_2dx_1 + \int_0^\infty f_3(x)\,dx = 1$
let
$I_1 = \int_0^\infty \int_{x_2}^\infty f_1(x_1,x_2)\,dx_1dx_2$ and $I_2 = \int_0^\infty \int_{x_1}^\infty f_2(x_1,x_2)\,dx_2dx_1$
then

$$\begin{aligned} I_1 &= \int_0^\infty \int_{x_2}^\infty (\alpha_1+\alpha_3)\,\alpha_2 x_1^{\beta-1}(\beta+\lambda x_1)\,e^{\lambda x_1-(\alpha_1+\alpha_3)x_1^\beta e^{\lambda x_1}} \times x_2^{\beta-1}(\beta+\lambda x_2)\,e^{\lambda x_2-\alpha_2 x_2^\beta e^{\lambda x_2}}\,dx_1dx_2 \\ &= \int_0^\infty \alpha_2 x_2^{\beta-1}(\beta+\lambda x_2)\,e^{\lambda x_2-(\alpha_1+\alpha_2+\alpha_3)x_2^\beta e^{\lambda x_2}}\,dx_2. \end{aligned} \tag{8}$$

Similarly

$$I_2 = \int_0^\infty \alpha_1 x_1^{\beta-1}(\beta+\lambda x_1)\,e^{\lambda x_1-(\alpha_1+\alpha_2+\alpha_3)x_1^\beta e^{\lambda x_1}}\,dx_1. \tag{9}$$

From (8) and (9), we get

$$\int_0^\infty f_3(x)\,dx = 1 - I_1 - I_2$$

$$\begin{aligned} &= \int_0^\infty (\alpha_1+\alpha_2+\alpha_3)\,x^{\beta-1}(\beta+\lambda x)\,e^{\lambda x-(\alpha_1+\alpha_2+\alpha_3)x^\beta e^{\lambda x}}\,dx \\ &\quad -\int_0^\infty \alpha_2 x^{\beta-1}(\beta+\lambda x)\,e^{\lambda x-(\alpha_1+\alpha_2+\alpha_3)x^\beta e^{\lambda x}}\,dx \\ &\quad -\int_0^\infty \alpha_1 x^{\beta-1}(\beta+\lambda x)\,e^{\lambda x-(\alpha_1+\alpha_2+\alpha_3)x^\beta e^{\lambda x}}\,dx \end{aligned}$$

Then

$$\begin{aligned} f_3(x) &= \alpha_3 x^{\beta-1}(\beta+\lambda x)\,e^{\lambda x-(\alpha_1+\alpha_2+\alpha_3)x^\beta e^{\lambda x}} \\ &= \frac{\alpha_3}{\alpha_1+\alpha_2+\alpha_3} f_{MW}(x;\alpha_1+\alpha_2+\alpha_3,\beta,\lambda). \end{aligned}$$

2.3 Marginal Probability Density Functions of BMWD

Next theorem introduces the marginal pdf of X_1 and X_2.

Theorem 2. *The marginal pdf for X_i $(i=1,2)$ has the following form*

$$\begin{aligned} f_{X_i}(x_i) &= (\alpha_i+\alpha_3)\,x_i^{\beta-1}(\beta+\lambda x_i)\times e^{\lambda x_i-(\alpha_i+\alpha_3)x_i^\beta e^{\lambda x_i}} \\ &= f_{MW}(x_i;\alpha_i+\alpha_3,\beta,\lambda), \end{aligned} \tag{10}$$

where $x_i > 0$, $i=1,2$.

Proof. Suppose that the marginal survival function of X_i, denoted by $\overline{F}(x_i)$, as following:

$$\begin{aligned} \overline{F}(x_i) &= P(X_i > x_i) \\ &= P(\min\{U_i,U_3\} > x_i) \\ &= P(U_i > x_i, U_3 > x_i) \end{aligned}$$

Since, U_i $(i=1,2)$ and U_3 are mutually independent random variables, then

$$\begin{aligned} \overline{F}(x_i) &= P(U_i > x_i)\,P(U_3 > x_i) \\ &= \overline{F}_{MW}(x_i;\alpha_i+\alpha_3,\beta,\lambda) \\ &= e^{-(\alpha_i+\alpha_3)x_i^\beta e^{\lambda x_i}}. \end{aligned} \tag{11}$$

Since, $f_{X_i}(x_i) = \frac{-d\overline{F}(x_i)}{dx_i}$ then, we obtain the formula given in (10).

2.4 The Joint Cumulative Distribution Function of BMWD

Theorem 3. *The joint cumulative distribution function of* (X_1, X_2) *is given by*

$$F_{X_1,X_2}(x_1,x_2) = \begin{cases} F_1(x_1,x_2) & if\ x_2 < x_1 \\ F_2(x_1,x_2) & if\ x_1 < x_2 \\ F_3(x_1,x_2) & if\ x_1 = x_2 = x \end{cases} \tag{12}$$

where

$$F_1(x_1,x_2) = 1 - e^{-(\alpha_2+\alpha_3)x_2^{\beta}e^{\lambda x_2}} - e^{-(\alpha_1+\alpha_3)x_1^{\beta}e^{\lambda x_1}}\left(1 - e^{-\alpha_2 x_2^{\beta}e^{\lambda x_2}}\right)$$

$$F_2(x_1,x_2) = 1 - e^{-(\alpha_1+\alpha_3)x_1^{\beta}e^{\lambda x_1}} - e^{-(\alpha_2+\alpha_3)x_2^{\beta}e^{\lambda x_2}}\left(1 - e^{-\alpha_1 x_1^{\beta}e^{\lambda x_1}}\right)$$

and

$$F_3(x_1,x_2) = 1 - e^{-\alpha_3 x^{\beta}e^{\lambda x}} \times \left(e^{-\alpha_1 x^{\beta}e^{\lambda x}} + e^{-\alpha_2 x^{\beta}e^{\lambda x}} - e^{-(\alpha_1+\alpha_2)x^{\beta}e^{\lambda x}}\right).$$

Proof. This can be easily deduced by using the following relation:

$$F_{X_1,X_2}(x_1,x_2) = 1 - \overline{F}_{X_1}(x_1) - \overline{F}_{X_2}(x_2) + \overline{F}_{X_1,X_2}(x_1,x_2).$$

2.5 Conditional Probability Density Functions

knowing the marginal pdf for the random variables X_1 and X_2 we can get the formula for the conditional pdf as illustrated in the next theorem.

Theorem 4. *The conditional probability density functions of* X_i*, given* $X_j = x_j$*,* $f(x_i|x_j)$ *,*$i, j = 1,2$*;* $i \neq j$*, has the following formula*

$$f_{X_i|X_j}(x_i|x_j) = \begin{cases} f^{(1)}_{X_i|X_j}(x_i|x_j) & if\ x_j < x_i, \\ f^{(2)}_{X_i|X_j}(x_i|x_j) & if\ x_i < x_j, \\ f^{(3)}_{X_i|X_j}(x_i|x_j) & if\ x_i = x_j = x, \end{cases}$$

where

$$f^{(1)}_{X_i|X_j}(x_i|x_j) = \left((\alpha_i+\alpha_3)\,\alpha_j x_i^{\beta-1}(\beta+\lambda x_i) \times e^{\lambda x_i - (\alpha_i+\alpha_3)x_i^{\beta}e^{\lambda x_i}}\right) \div (\alpha_j+\alpha_3)\, e^{-\alpha_3 x_i^{\beta}e^{\lambda x_i}},$$

$$f^{(2)}_{X_i|X_j}(x_i|x_j) = \alpha_i x_i^{\beta-1}(\beta+\lambda x_i) \times e^{\lambda x_i - \alpha_i x_i^{\beta}e^{\lambda x_i}},$$

$$f^{(3)}_{X_i|X_j}(x_i|x_j) = \frac{\alpha_3}{\alpha_i+\alpha_3} e^{-\alpha_i x_i^{\beta}e^{\lambda x_i}}.$$

Proof. The proof follows immediately by substituting the joint pdf of (X_1, X_2) discussed in (5), (6) and (7) and the marginal probability density function of X_i $(i = 1,2)$ given in (10), using the next formula

$$f_{X_i|X_j}(x_i|x_j) = \frac{f_{X_i,X_j}(x_i,x_j)}{f_{X_j}(x_j)},\ i = 1,2.$$

3 Reliability Studies of BMWD

This section discussed the joint hazard rate function of (X_1, X_2), also we computed the cdf of the random variables $U = \max\{X_1, X_2\}$ and $V = \min\{X_1, X_2\}$.

3.1 Joint Hazard Rate Function of BMWD

Theorem 5. *The joint hazard rate function of* (X_1, X_2) *has the following formula*

$$h_{X_1,X_2}(x_1,x_2) = \begin{cases} h_1(x_1,x_2) & if\ x_2 < x_1 \\ h_2(x_1,x_2) & if\ x_1 < x_2 \\ h_3(x_1,x_2) & if\ x_1 = x_2 = x \end{cases}$$

such that
$h_1(x_1,x_2) = (\alpha_1+\alpha_3)\,\alpha_2 x_1^{\beta-1} x_2^{\beta-1}(\beta+\lambda x_1)\times(\beta+\lambda x_2)\,e^{\lambda(x_1+x_2)}$,
$h_2(x_1,x_2) = (\alpha_2+\alpha_3)\,\alpha_1 x_1^{\beta-1} x_2^{\beta-1}(\beta+\lambda x_1)\times(\beta+\lambda x_2)\,e^{\lambda(x_1+x_2)}$,
and
$h_3(x,x) = \alpha_3 x^{\beta-1}(\beta+\lambda x)\,e^{\lambda x}$.

*Proof.*This can be easily deduced by using $h_{X_1,X_2}(x_1,x_2) = \frac{f_{X_1,X_2}(x_1,x_2)}{\overline{F}_{X_1,X_2}(x_1,x_2)}$.

Lemma 2.*The cdfs of the random variables $U = \max\{X_1,X_2\}$ and $V = \min\{X_1,X_2\}$ are given by*
$F_U(u) = 1-e^{-\alpha_3 u^\beta e^{\lambda u}}\left(e^{-\alpha_1 u^\beta e^{\lambda u}} + e^{-\alpha_2 u^\beta e^{\lambda u}} - e^{-(\alpha_1+\alpha_2)u^\beta e^{\lambda u}}\right)$, $F_V(v) = 1-e^{-(\alpha_1+\alpha_2+\alpha_3)v^\beta e^{\lambda v}}$.

*Proof.*The cdf for the random variable $U = \max\{X_1,X_2\}$ is deduced as following
$F_U(u) = P[U\le u]$
$= P[\max\{X_1,X_2\}\le u]$
$= P[X_1\le u,\ X_2\le u]$
$= \overline{F}(u,u)$
$= 1-e^{-\alpha_3 u^\beta e^{\lambda u}}\left(e^{-\alpha_1 u^\beta e^{\lambda u}} + e^{-\alpha_2 u^\beta e^{\lambda u}} - e^{-(\alpha_1+\alpha_2)u^\beta e^{\lambda u}}\right)$.
The cdf for the random variables $V = \min\{X_1,X_2\}$ is deduced as following
$F_V(v) = P[V\le v]$
$= P[\min\{X_1,X_2\}\le v]$
$= 1-P[\min\{X_1,X_2\}>v]$
$= 1-P[X_1>v,\ X_2>v]$
$= 1-\overline{F}(v,v)$
$= 1-e^{-(\alpha_1+\alpha_2+\alpha_3)v^\beta e^{\lambda v}}$.

4 Maximum Likelihood Estimators of BMWD

Kundu and Gupta [3] applied the maximum likelihood method to estimate the undetermined parameters for the bivariate generalized exponential distribution. In the same manner we use the MLM to estimate the undetermined parameters of the BMW distribution.

Assume $((x_{11},x_{21}),\ \ldots,\ (x_{1n},x_{2n}))$ is a random sample from BMW distribution. Assume we introduced the following notations

$I_1 = \{i;\ x_{1i}>x_{2i}\},\ I_2 = \{i;\ x_{1i}<x_{2i}\},\ I_3 = \{i;\ x_{1i}=x_{2i}=x_i\},\ I = I_1\cup I_2\cup I_3,\ |I_1| = n_1,\ |I_2| = n_2,\ |I_3| = n_3$, and $n_1+n_2+n_3 = n$.

The random sample of size n has the following maximum likelihood function:

$l(\alpha_1,\alpha_2,\alpha_3,\beta,\lambda) = \prod_{i=1}^{n_1} f_1(x_{1i},x_{2i})\prod_{i=1}^{n_2} f_2(x_{1i},x_{2i})\prod_{i=1}^{n_3} f_3(x_i)$

The log-likelihood function can be expressed as

$L(\alpha_1,\alpha_2,\alpha_3,\beta,\lambda) = \ln l(\alpha_1,\alpha_2,\alpha_3,\beta,\lambda)$

$= n_1\ln(\alpha_1+\alpha_3) + n_1\ln(\alpha_2) + (\beta-1)\sum_{i=1}^{n1}\ln(x_{1i}) + \sum_{i=1}^{n_1}\ln(\beta+\lambda x_{1i}) + \sum_{i=1}^{n_1}\left(\lambda x_{1i}-(\alpha_1+\alpha_3)x_{1i}^\beta e^{\lambda x_{1i}}\right) + (\beta-1)\sum_{i=1}^{n1}\ln(x_{2i}) + \sum_{i=1}^{n_1}\ln(\beta+\lambda x_{2i}) + \sum_{i=1}^{n_1}\left(\lambda x_{2i}-\alpha_2 x_{2i}^\beta e^{\lambda x_{2i}}\right) + n_2\ln(\alpha_1) + (\beta-1)\sum_{i=1}^{n_2}\ln(x_{1i}) + \sum_{i=1}^{n_2}\ln(\beta+\lambda x_{1i}) + \sum_{i=1}^{n_2}\left(\lambda x_{1i}-\alpha_1 x_{1i}^\beta e^{\lambda x_{1i}}\right) + n_2\ln(\alpha_2+\alpha_3) + (\beta-1)\sum_{i=1}^{n_2}\ln(x_{2i}) + \sum_{i=1}^{n_2}\ln(\beta+\lambda x_{2i}) + \sum_{i=1}^{n_2}\left(\lambda x_{2i}-(\alpha_2+\alpha_3)x_{2i}^\beta e^{\lambda x_{2i}}\right) + n_3\ln(\alpha_3) + (\beta-1)\sum_{i=1}^{n_3}\ln(x_i) + \sum_{i=1}^{n_3}\ln(\beta+\lambda x_i) + \sum_{i=1}^{n_3}\left(\lambda x_i-(\alpha_1+\alpha_2+\alpha_3)x_i^\beta e^{\lambda x_i}\right)$.

Differentiating the previous function with respect to $\alpha_1,\alpha_2,\alpha_3,\beta$ and λ respectively, and setting the results equal to zero, we have

$$\frac{\partial L}{\partial \alpha_1} = \frac{n_1}{\alpha_1+\alpha_3} + \sum_{i=1}^{n_1} -x_{1i}^{\beta} e^{\lambda x_{1i}} + \frac{n_2}{\alpha_1} + \sum_{i=1}^{n_2} -x_{1i}^{\beta} e^{\lambda x_{1i}} + \sum_{i=1}^{n_3} -x_i^{\beta} e^{\lambda x_i} \tag{13}$$

$$\frac{\partial L}{\partial \alpha_2} = \frac{n_1}{\alpha_2} + \sum_{i=1}^{n_1} -x_{2i}^{\beta} e^{\lambda x_{2i}} + \frac{n_2}{\alpha_2+\alpha_3} + \sum_{i=1}^{n_2} -x_{2i}^{\beta} e^{\lambda x_{2i}} + \sum_{i=1}^{n_3} -x_i^{\beta} e^{\lambda x_i} \tag{14}$$

$$\frac{\partial L}{\partial \alpha_3} = \frac{n_1}{\alpha_1+\alpha_3} + \sum_{i=1}^{n_1} -x_{1i}^{\beta} e^{\lambda x_{1i}} + \frac{n_2}{\alpha_2+\alpha_3} + \sum_{i=1}^{n_2} -x_{2i}^{\beta} e^{\lambda x_{2i}} + \sum_{i=1}^{n_3} -x_i^{\beta} e^{\lambda x_i} \tag{15}$$

$$\begin{aligned}\frac{\partial L}{\partial \beta} = & \sum_{i=1}^{n_1} \ln x_{1i} + \sum_{i=1}^{n_1} \frac{1}{\beta+\lambda x_{1i}} + \sum_{i=1}^{n_1} -(\alpha_1+\alpha_3)\, x_{1i}^{\beta} e^{\lambda x_{1i}} \ln x_{1i} \\ & + \sum_{i=1}^{n_1} \ln x_{2i} + \sum_{i=1}^{n_1} \frac{1}{\beta+\lambda x_{2i}} + \sum_{i=1}^{n_1} -\alpha_2 x_{2i}^{\beta} e^{\lambda x_{2i}} \ln x_{2i} \\ & + \sum_{i=1}^{n_2} \ln x_{1i} + \sum_{i=1}^{n_2} \frac{1}{\beta+\lambda x_{1i}} + \sum_{i=1}^{n_2} -\alpha_1 x_{1i}^{\beta} e^{\lambda x_{1i}} \ln x_{1i} \\ & + \sum_{i=1}^{n_2} \ln x_{2i} + \sum_{i=1}^{n_2} \frac{1}{\beta+\lambda x_{2i}} + \sum_{i=1}^{n_2} -(\alpha_2+\alpha_3)\, x_{2i}^{\beta} e^{\lambda x_{2i}} \ln x_{2i} \\ & + \sum_{i=1}^{n_3} \ln x_i + \sum_{i=1}^{n_3} \frac{1}{\beta+\lambda x_i} + \sum_{i=1}^{n_3} -(\alpha_1+\alpha_2+\alpha_3)\, x_i^{\beta} e^{\lambda x_i} \ln x_i \end{aligned} \tag{16}$$

$$\begin{aligned}\frac{\partial L}{\partial \lambda} = & \sum_{i=1}^{n_1} \frac{x_{1i}}{\beta+\lambda x_{1i}} + \sum_{i=1}^{n_1} \left(x_{1i} - (\alpha_1+\alpha_3)\, x_{1i}^{\beta+1} e^{\lambda x_{1i}}\right) \\ & + \sum_{i=1}^{n_1} \frac{x_{2i}}{\beta+\lambda x_{2i}} + \sum_{i=1}^{n_1} \left(x_{2i} - \alpha_2 x_{2i}^{\beta+1} e^{\lambda x_{2i}}\right) + \sum_{i=1}^{n_2} \frac{x_{1i}}{\beta+\lambda x_{1i}} \\ & + \sum_{i=1}^{n_2} \left(x_{1i} - \alpha_1 x_{1i}^{\beta+1} e^{\lambda x_{1i}}\right) + \sum_{i=1}^{n_3} \left(x_i - (\alpha_1+\alpha_2+\alpha_3)\, x_i^{\beta+1} e^{\lambda x_i}\right) \\ & + \sum_{i=1}^{n_2} \left(x_{2i} - (\alpha_2+\alpha_3)\, x_{2i}^{\beta+1} e^{\lambda x_{2i}}\right) + \sum_{i=1}^{n_2} \frac{x_{2i}}{\beta+\lambda x_{2i}} + \sum_{i=1}^{n_3} \frac{x_i}{\beta+\lambda x_i} \end{aligned} \tag{17}$$

Solving Equations (13) - (17) we obtained the maximum likelihood estimators denoted by $\widehat{\alpha}_1$, $\widehat{\alpha}_2$, $\widehat{\alpha}_3$, $\widehat{\beta}$ and $\widehat{\lambda}$ for the unknown parameters α_1, α_2, α_3, β and λ respectively by using Mathcad software.

5 Data Analysis of BMWD

A set of real data is used to compare the fits of the Marshall-Olkin Bivariate Exponential (MOD) distribution, Bivariate Generalized Exponential (BVGE) distribution, Bivariate Generalized Linear Failure Rate (BGLFR) distribution and Bivariate Modified Weibull (BMW) distribution. The data set (see Table 1) was first analyzed in [11] It is a bivariate data set, and the variables X_1 and X_2 are defined [11] From their definitions, we know that both variables take the following manner: (i) $X_1 < X_2$, (ii) $X_1 = X_2$, (iii) $X_1 > X_2$.

Table 1 UEFA Champion′s League Data

X_1	X_2	X_1	X_2	X_1	X_2
26	20	66	62	51	28
63	18	25	9	76	64
19	19	41	3	64	15
66	85	16	75	26	48
40	40	18	18	16	16
49	49	22	14	44	13
8	8	42	42	25	14
69	71	36	52	55	11
39	39	34	34	49	49
82	48	53	39	24	24
72	72	54	7	44	30
42	3	27	47	2	2
28	28				

These numerical evaluations are calculated using the Package of Mathcad software. Table 2 provides the MLEs for the discussed model parameters. The model selection executed using the *AIC* (Akaike Information Criterion) and the *BIC* (Bayesian Information Criterion): $AIC = -2L + 2q$, $BIC = L - \frac{q}{2}\ln(n)$.

Where L refers to the log-likelihood function, the number of parameters denoted by q and the sample size denoted by n.

Table 2: The MLEs for UEFA Champion′s League data

Model	MLEs
MO	$\widehat{\lambda}_1 = 0.012, \widehat{\lambda}_2 = 0.014$ $\widehat{\lambda}_3 = 0.022$
BVGE	$\widehat{\alpha}_1 = 1.351, \widehat{\alpha}_2 = 0.465$ $\widehat{\alpha}_3 = 1.153, \widehat{\beta} = 0.039$
BGLFR	$\widehat{\alpha}_1 = 0.492, \widehat{\alpha}_2 = 0.166$ $\widehat{\alpha}_3 = 0.411, \widehat{\beta} = 2.013 \times 10^{-4}$ $\widehat{\beta} = 8.051 \times 10^{-4}$
BMWD	$\widehat{\alpha}_1 = 2.393 \times 10^{-3}, \widehat{\beta} = 1.051$ $\widehat{\alpha}_3 = 5.1 \times 10^{-3}, \widehat{\lambda} = 0.018$ $\widehat{\alpha}_2 = 5.346 \times 10^{-3}$

Table 3: The values 0f L, *AIC* and *BIC*

Model	L	*AIC*	*BIC*
MO	-339.006	684.012	-344.423
BVGE	-296.935	601.870	-304.157
BGLFR	-293.379	596.757	-302.406
BMWD	-287.468	584.936	-296.027

The values of $-L$, *AIC* and *BIC* calculated in (see Table 3) are smaller for the BMW distribution compared with those values for the other distributions in the table, thus we can conclude that the proposed distribution a very competitive distribution to these data.

The profiles of the log-likelihood function of α_1, α_2, α_3, β and λ of BMWD for UEFA Champion′s League data are plotted in Figures from Figure 1 to Figure 5 respectively. These plots of the profiles of the log-likelihood function of α_1, α_2, α_3, β and λ shown that the equations of the likelihood have a unique solution.

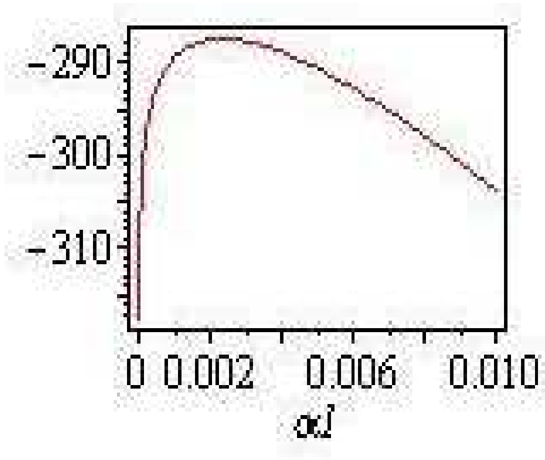

Fig. 1: Variation for the log-likelihood function of α_1 for UEFA Champion′s League data.

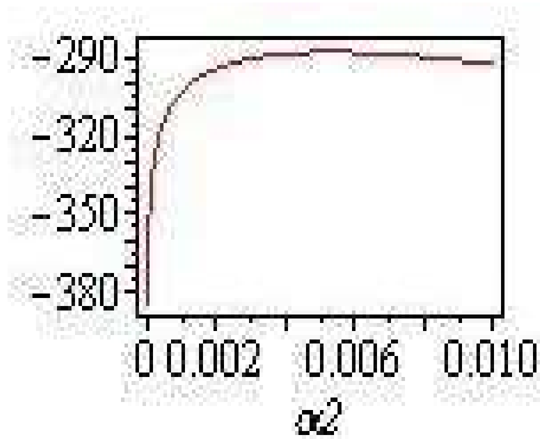

Fig. 2: Variation for the log-likelihood function of α_2 for UEFA Champion′s League data.

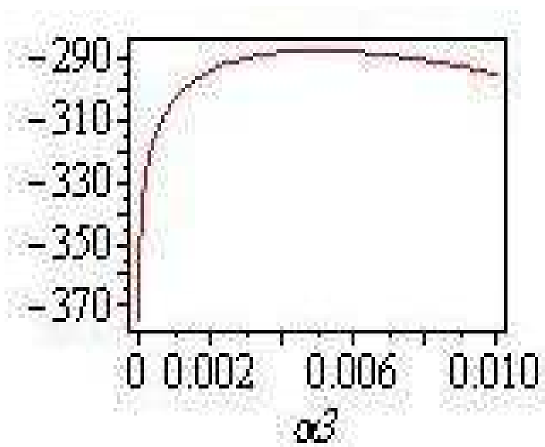

Fig. 3: Variation for the log-likelihood function of α_3 for UEFA Champion′s League data.

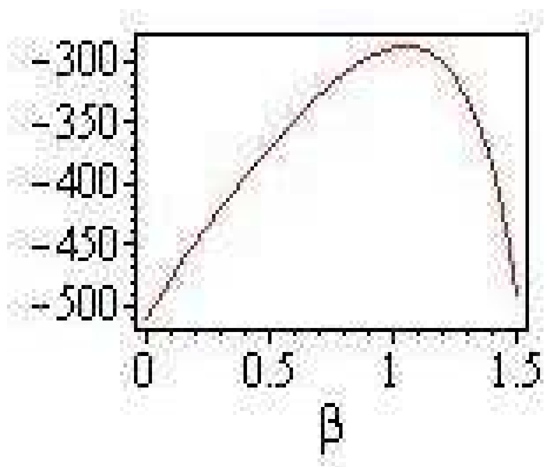

Fig. 4: Variation for the log-likelihood function of β for UEFA Champion′s League data.

6 Bivariate Modified Weibull Geometric distribution

This section proposed the Bivariate Modified Weibull Geometric (BMWG) distribution.We first discuss the joint survival function and derive the corresponding joint pdf and marginal pdf of this distribution.

6.1 The Joint Survival Function of BMWGD

Suppose that $\{(X_{1n}, X_{2n});\ n = 1, 2, ...\}$ is a series of i.i.d. non-negative bivariate r.v. with common joint distribution function $F(.,.)$ and N is a geometric r.v. independent of $\{(X_{1n}, X_{2n});\ n = 1, 2, ...\}$. Suppose the next bivariate r.v. (X_1, X_2) :

$X_1 = \min(X_{11}, ..., X_{1N})$ and $X_2 = \min(X_{21}, ..., X_{2N})$.

Then, the bivariate random variable (X_1, X_2) has a bivariate geometric distribution if its joint survival function with the following form:

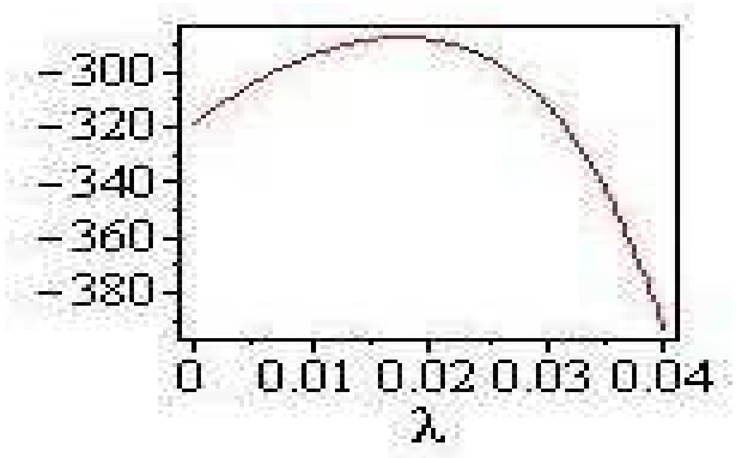

Fig. 5: Variation for the log-likelihood function of λ for UEFA Champion′s League data.

$$\begin{aligned}\overline{G}(x_1,x_2) &= \sum_{n=1}^{\infty} \overline{F}^n(x_1,x_2)\,\theta\,(1-\theta)^{n-1} \\ &= \frac{\theta \overline{F}(x_1,x_2)}{1-(1-\theta)\overline{F}(x_1,x_2)},\end{aligned} \tag{18}$$

where $0 < \theta \leq 1$.

If $\overline{F}(x_1,x_2)$ in (18) is the joint survival function of BMW$(\alpha_1,\alpha_2,\alpha_3,\beta,\lambda)$, then, the joint survival function of (X_1,X_2) becomes

$$\overline{G}(x_1,x_2) = \begin{cases} \overline{G}_1(x_1,x_2), ...x_1 \geq x_2 \\ \overline{G}_2(x_1,x_2), ...x_1 < x_2, \end{cases} \tag{19}$$

where

$$\overline{G}_1(x_1,x_2) = \frac{\theta e^{-(\alpha_1+\alpha_3)x_1^{\beta}e^{\lambda x_1}-\alpha_2 x_2^{\beta}e^{\lambda x_2}}}{1-(1-\theta)e^{-(\alpha_1+\alpha_3)x_1^{\beta}e^{\lambda x_1}-\alpha_2 x_2^{\beta}e^{\lambda x_2}}},\ x_1 \geq x_2$$

$$\overline{G}_2(x_1,x_2) = \frac{\theta e^{-\alpha_1 x_1^{\beta}e^{\lambda x_1}-(\alpha_2+\alpha_3)x_2^{\beta}e^{\lambda x_2}}}{1-(1-\theta)e^{-\alpha_1 x_1^{\beta}e^{\lambda x_1}-(\alpha_2+\alpha_3)x_2^{\beta}e^{\lambda x_2}}},\ x_1 < x_2.$$

Hence the vector of random variables (X_1,X_2) has a bivariate Modified Weibull geometric distribution with parameters $\alpha_1,\alpha_2,\alpha_3,\beta,\lambda$ and θ and it will be written as $(X_1,X_2) \sim BMWG(\alpha_1,\alpha_2,\alpha_3,\beta,\lambda,\theta)$.

6.2 The Joint Probability Density Function of BMWGD

The joint pdf of the random variables X_1 and X_2 for the Bivariate Modified Weibull Geometric distribution illustrated in the following theorem.

Theorem 6.*Consider the joint survival function of (X_1,X_2) given as in (19) then, the joint pdf of (X_1,X_2) is obtained as*

$$g_{BMWG}(x_1,x_2) = \begin{cases} g_1(x_1,x_2) & if\ x_2 < x_1 \\ g_2(x_1,x_2) & if\ x_1 < x_2 \\ g_3(x) & .\ if\ x_1 = x_2 = x \end{cases} \tag{20}$$

where

$$g_1(x_1,x_2) = k_1(x_1,x_2)\,\theta\alpha_2(\alpha_1+\alpha_3)\,e^{\lambda(x_1+x_2)} \times \left(x_1^{\beta}\lambda+\beta x_1^{\beta-1}\right)\left(x_2^{\beta}\lambda+\beta x_2^{\beta-1}\right)e^{-\alpha_2 x_2^{\beta}e^{\lambda x_2}} \times e^{-(\alpha_1+\alpha_3)x_1^{\beta}e^{\lambda x_1}}$$

$g_2(x_1,x_2) = k_2(x_1,x_2)\,\theta\alpha_1(\alpha_2+\alpha_3)\,e^{\lambda(x_1+x_2)} \times \left(x_1^{\beta}\lambda+\beta x_1^{\beta-1}\right)\left(x_2^{\beta}\lambda+\beta x_2^{\beta-1}\right)e^{-\alpha_1 x_1^{\beta}e^{\lambda x_1}} \times e^{-(\alpha_2+\alpha_3)x_2^{\beta}e^{\lambda x_2}}$
$g_3(x_1,x_2) = k_3(x_1,x_2)\,\theta\alpha_3 e^{\lambda x}\left(x^{\beta}\lambda+\beta x^{\beta-1}\right)\times e^{-(\alpha_1+\alpha_2+\alpha_3)x^{\beta}e^{\lambda x}}$
and
$k_1(x_1,x_2) = \frac{1+(1-\theta)e^{-(\alpha_1+\alpha_3)x_1^{\beta}e^{\lambda x_1}-\alpha_2 x_2^{\beta}e^{\lambda x_2}}}{\left(1-(1-\theta)e^{-(\alpha_1+\alpha_3)x_1^{\beta}e^{\lambda x_1}-\alpha_2 x_2^{\beta}e^{\lambda x_2}}\right)^3},$
$k_2(x_1,x_2) = \frac{1+(1-\theta)e^{-\alpha_1 x_1^{\beta}e^{\lambda x_1}-(\alpha_2+\alpha_3)x_2^{\beta}e^{\lambda x_2}}}{\left(1-(1-\theta)e^{-\alpha_1 x_1^{\beta}e^{\lambda x_1}-(\alpha_2+\alpha_3)x_2^{\beta}e^{\lambda x_2}}\right)^3},$
$k_3(x_1,x_2) = \left(1-(1-\theta)e^{-(\alpha_1+\alpha_2+\alpha_3)x^{\beta}e^{\lambda x}}\right)^{-2}.$

*Proof.*This can be easily deduced by using the same method as in (4).

6.3 Marginal Survival Functions of BMWGD

The next theorem discussed the marginal survival functions of X_1 and X_2.

Theorem 7.*The marginal survival functions of X_i $(i=1,2)$, say $\overline{G}_i(x_i)$, is given by*

$$\overline{G}_i(x_i) = \frac{\theta e^{-(\alpha_i+\alpha_3)x_i^{\beta}e^{\lambda x_i}}}{1-(1-\theta)e^{-(\alpha_i+\alpha_3)x_i^{\beta}e^{\lambda x_i}}},\ i=1,2 \tag{21}$$

*Proof.*By using $\overline{G}_1(x_i) = \overline{G}_1(x_i,0)$ and $\overline{G}_2(x_i) = \overline{G}_2(0,x_i)$, we get the proof directly.

7 Maximum Likelihood Estimators of BMWGD

By the same way, we use the method of maximum likelihood to estimate the undetermined parameters of the BMWG distribution.

Suppose $((x_{11},x_{21}),\ \ldots,\ (x_{1n},x_{2n}))$ is a random sample from BMWG distribution.Consider the following notation $I_1=\{i;\ x_{1i}>x_{2i}\},\ I_2=\{i;\ x_{1i}<x_{2i}\},\ I_3=\{i;\ x_{1i}=x_{2i}=x_i\},\ I=I_1\cup I_2\cup I_3,\ |I_1|=n_1,\ |I_2|=n_2,\ |I_3|=n_3$, and $n_1+n_2+n_3=n$.

For the sample of size n the likelihood function is given by:
$l(\alpha_1,\alpha_2,\alpha_3,\beta,\lambda,\theta) = \prod_{i=1}^{n_1} g_1(x_{1i},x_{2i})\prod_{i=1}^{n_2} g_2(x_{1i},x_{2i})\prod_{i=1}^{n_3} g_3(x_i)$

The log-likelihood function can be expressed as
$L(\alpha_1,\alpha_2,\alpha_3,\beta,\lambda,\theta) = \ln l(\alpha_1,\alpha_2,\alpha_3,\beta,\lambda,\theta)$
$= n_1\ln(\theta)+n_1\ln(\alpha_1+\alpha_3)+n_1\ln(\alpha_2)+\sum_{i=1}^{n1}\lambda x_{1i}+\sum_{i=1}^{n1}\lambda x_{2i}+\sum_{i=1}^{n_1}\ln\left(\lambda x_{1i}^{\beta}+\beta x_{1i}^{\beta-1}\right)+\sum_{i=1}^{n_1}\ln\left(\lambda x_{2i}^{\beta}+\beta x_{2i}^{\beta-1}\right)+ n_2\ln\theta + n_2\ln\alpha_1 + \sum_{i=1}^{n_1}\ln\left(1+(1-\theta)e^{-(\alpha_1+\alpha_3)x_1^{\beta}e^{\lambda x_1}-\alpha_2 x_2^{\beta}e^{\lambda x_2}}\right) + \sum_{i=1}^{n_1}\left(-(\alpha_1+\alpha_3)x_1^{\beta}e^{\lambda x_1}-\alpha_2 x_2^{\beta}e^{\lambda x_2}\right) - 3\sum_{i=1}^{n_1}\ln\left(1-(1-\theta)e^{-(\alpha_1+\alpha_3)x_1^{\beta}e^{\lambda x_1}-\alpha_2 x_2^{\beta}e^{\lambda x_2}}\right) + n_2\ln(\alpha_2+\alpha_3) + \sum_{i=1}^{n_2}\lambda x_{1i} + \sum_{i=1}^{n_2}\lambda x_{2i} + \sum_{i=1}^{n_2}\ln\left(\lambda x_{1i}^{\beta}+\beta x_{1i}^{\beta-1}\right) + \sum_{i=1}^{n_2}\ln\left(\lambda x_{2i}^{\beta}+\beta x_{2i}^{\beta-1}\right) + \sum_{i=1}^{n_2}\left(-\alpha_1 x_1^{\beta}e^{\lambda x_1}-(\alpha_2+\alpha_3)x_2^{\beta}e^{\lambda x_2}\right) + \sum_{i=1}^{n_2}\ln\left(1+(1-\theta)e^{-\alpha_1 x_1^{\beta}e^{\lambda x_1}-(\alpha_2+\alpha_3)x_2^{\beta}e^{\lambda x_2}}\right) + n_3\ln\theta + n_3\ln\alpha_3 - 3\sum_{i=1}^{n_2}\ln\left(1-(1-\theta)e^{-\alpha_1 x_1^{\beta}e^{\lambda x_1}-(\alpha_2+\alpha_3)x_2^{\beta}e^{\lambda x_2}}\right) + \sum_{i=1}^{n_3}\lambda x_i + \sum_{i=1}^{n_3}\ln\left(\lambda x_i^{\beta}+\beta x_i^{\beta-1}\right) - \sum_{i=1}^{n_3}(\alpha_1+\alpha_2+\alpha_3)x^{\beta}e^{\lambda x} - 2\sum_{i=1}^{n_3}\ln\left(1-(1-\theta)e^{-(\alpha_1+\alpha_2+\alpha_3)x^{\beta}e^{\lambda x}}\right).$

Differentiating the log-likelihood with respect to $\alpha_1,\alpha_2,\alpha_3,\theta,\lambda$ and β respectively, and setting the results equal to zero, we have

$$\frac{\partial L}{\partial\alpha_1} = \frac{n_1}{\alpha_1+\alpha_3} - 3\sum_{i=1}^{n_1}\frac{x_{1i}^{\beta}e^{\lambda x_{1i}}C_1(x_{1i},x_{2i})}{1-C_1(x_{1i},x_{2i})} + \frac{n_2}{\alpha_1} - \sum_{i=1}^{n_1}x_{1i}^{\beta}e^{\lambda x_{1i}} - \sum_{i=1}^{n_1}\frac{x_{1i}^{\beta}e^{\lambda x_{1i}}C_1(x_{1i},x_{2i})}{1+C_1(x_{1i},x_{2i})} - 3\sum_{i=1}^{n_2}\frac{x_{1i}^{\beta}e^{\lambda x_{1i}}C_2(x_{1i},x_{2i})}{1-C_2(x_{1i},x_{2i})} - \sum_{i=1}^{n_2}x_{1i}^{\beta}e^{\lambda x_{1i}} - \sum_{i=1}^{n_2}\frac{x_{1i}^{\beta}e^{\lambda x_{1i}}C_2(x_{1i},x_{2i})}{1+C_2(x_{1i},x_{2i})} - \sum_{i=1}^{n_3}x_i^{\beta}e^{\lambda x_i} - 2\sum_{i=1}^{n_3}\frac{x_i^{\beta}e^{\lambda x_i}C_3(x_i,x_i)}{1+C_3(x_i,x_i)} \tag{22}$$

$$\frac{\partial L}{\partial \alpha_2}=\frac{n_1}{\alpha_2}-3\sum_{i=1}^{n_1}\frac{x_{2i}^{\beta}e^{\lambda x_{2i}}C_1(x_{1i},x_{2i})}{1-C_1(x_{1i},x_{2i})}+\frac{n_2}{\alpha_2+\alpha_3}-\sum_{i=1}^{n_1}x_{2i}^{\beta}e^{\lambda x_{2i}}-\sum_{i=1}^{n_1}\frac{x_{2i}^{\beta}e^{\lambda x_{2i}}C_1(x_{1i},x_{2i})}{1+C_1(x_{1i},x_{2i})}-3\sum_{i=1}^{n_2}\frac{x_{2i}^{\beta}e^{\lambda x_{2i}}C_2(x_{1i},x_{2i})}{1-C_2(x_{1i},x_{2i})}-\sum_{i=1}^{n_2}x_{2i}^{\beta}e^{\lambda x_{2i}}$$
$$-\sum_{i=1}^{n_2}\frac{x_{2i}^{\beta}e^{\lambda x_{2i}}C_2(x_{1i},x_{2i})}{1+C_2(x_{1i},x_{2i})}-\sum_{i=1}^{n_3}x_i^{\beta}e^{\lambda x_i}-2\sum_{i=1}^{n_3}\frac{x_i^{\beta}e^{\lambda x_i}C_3(x_{1i},x_{2i})}{1+C_3(x_{1i},x_{2i})} \tag{23}$$

$$\frac{\partial L}{\partial \alpha_3}=\frac{n_1}{\alpha_1+\alpha_3}-3\sum_{i=1}^{n_1}\frac{x_{1i}^{\beta}e^{\lambda x_{1i}}C_1(x_{1i},x_{2i})}{1-C_1(x_{1i},x_{2i})}+\frac{n_3}{\alpha_3}-\sum_{i=1}^{n_1}x_{1i}^{\beta}e^{\lambda x_{1i}}-\sum_{i=1}^{n_1}\frac{x_{1i}^{\beta}e^{\lambda x_{1i}}C_1(x_{1i},x_{2i})}{1+C_1(x_{1i},x_{2i})}-3\sum_{i=1}^{n_2}\frac{x_{2i}^{\beta}e^{\lambda x_{2i}}C_2(x_{1i},x_{2i})}{1-C_2(x_{1i},x_{2i})}-\sum_{i=1}^{n_2}x_{2i}^{\beta}e^{\lambda x_{2i}}$$
$$-\sum_{i=1}^{n_2}\frac{x_{2i}^{\beta}e^{\lambda x_{2i}}C_2(x_{1i},x_{2i})}{1+C_2(x_{1i},x_{2i})}-\sum_{i=1}^{n_3}x_i^{\beta}e^{\lambda x_i}+\frac{n_2}{\alpha_2+\alpha_3}-2\sum_{i=1}^{n_3}\frac{x_i^{\beta}e^{\lambda x_i}C_3(x_i,x_i)}{1+C_3(x_i,x_i)} \tag{24}$$

$$\frac{\partial L}{\partial \theta}=\frac{n_1}{\theta}+\frac{n_2}{\theta}-3\sum_{i=1}^{n_1}\frac{C_1(x_{1i},x_{2i})/(1-\theta)}{1-C_1(x_{1i},x_{2i})}-\sum_{i=1}^{n_1}\frac{C_1(x_{1i},x_{2i})/(1-\theta)}{1+C_1(x_{1i},x_{2i})}-3\sum_{i=1}^{n_2}\frac{C_2(x_{1i},x_{2i})/(1-\theta)}{1-C_2(x_{1i},x_{2i})}-\sum_{i=1}^{n_2}\frac{C_2(x_{1i},x_{2i})/(1-\theta)}{1+C_2(x_{1i},x_{2i})}$$
$$-\sum_{i=1}^{n_3}\frac{C_3(x_i,x_i)/(1-\theta)}{1+C_3(x_i,x_i)}+\frac{n_3}{\theta} \tag{25}$$

$$\frac{\partial L}{\partial \lambda}=\sum_{i=1}^{n_1}\frac{x_{1i}}{\beta+\lambda x_{1i}}+\sum_{i=1}^{n_1}x_{1i}+\sum_{i=1}^{n_1}\frac{x_{2i}}{\beta+\lambda x_{2i}}-\sum_{i=1}^{n_1}\left((\alpha_1+\alpha_3)x_{1i}^{\beta+1}e^{\lambda x_{1i}}+\alpha_2x_{2i}^{\beta+1}e^{\lambda x_{2i}}\right)+$$
$$3\sum_{i=1}^{n_1}\frac{\left((\alpha_1+\alpha_3)x_{1i}^{\beta+1}e^{\lambda x_{1i}}+\alpha_2x_{2i}^{\beta+1}e^{\lambda x_{2i}}\right)C_1(x_{1i},x_{2i})}{1-C_1(x_{1i},x_{2i})}-\sum_{i=1}^{n_1}\frac{\left((\alpha_1+\alpha_3)x_{1i}^{\beta+1}e^{\lambda x_{1i}}+\alpha_2x_{2i}^{\beta+1}e^{\lambda x_{2i}}\right)C_1(x_{1i},x_{2i})}{1+C_1(x_{1i},x_{2i})}+\sum_{i=1}^{n_1}x_{2i}+\sum_{i=1}^{n_2}\frac{x_{1i}}{\beta+\lambda x_{1i}}+$$
$$\sum_{i=1}^{n_2}\frac{x_{2i}}{\beta+\lambda x_{2i}}+\sum_{i=1}^{n_3}x_i-\sum_{i=1}^{n_1}\left(\alpha_1x_{1i}^{\beta+1}e^{\lambda x_{1i}}+(\alpha_2+\alpha_3)x_{2i}^{\beta+1}e^{\lambda x_{2i}}\right)+\sum_{i=1}^{n_2}x_{1i}+\sum_{i=1}^{n_2}x_{2i}+\sum_{i=1}^{n_3}\frac{x_i}{\beta+\lambda x_i}+$$
$$3\sum_{i=1}^{n_2}\frac{\left(\alpha_1x_{1i}^{\beta+1}e^{\lambda x_{1i}}+(\alpha_2+\alpha_3)x_{2i}^{\beta+1}e^{\lambda x_{2i}}\right)C_2(x_{1i},x_{2i})}{1-C_2(x_{1i},x_{2i})}-\sum_{i=1}^{n_2}\frac{\left(\alpha_1x_{1i}^{\beta+1}e^{\lambda x_{1i}}+(\alpha_2+\alpha_3)x_{2i}^{\beta+1}e^{\lambda x_{2i}}\right)C_2(x_{1i},x_{2i})}{1+C_2(x_{1i},x_{2i})}$$
$$-2\sum_{i=1}^{n3}\frac{(\alpha_1+\alpha_2+\alpha_3)x_i^{\beta+1}e^{\lambda x_i}C_3(x_{1i},x_{2i})}{1-C_3(x_{1i},x_{2i})} \tag{26}$$

$$\frac{\partial L}{\partial \beta}=\sum_{i=1}^{n_1}\frac{1+\lambda x_{1i}\ln x_{1i}+\beta\ln x_{1i}}{\beta+\lambda x_{1i}}+\sum_{i=1}^{n_1}\frac{1+\lambda x_{2i}\ln x_{2i}+\beta\ln x_{2i}}{\beta+\lambda x_{1i}}-3\sum_{i=1}^{n_1}\frac{W_1(x_{1i},x_{2i})C_1(x_{1i},x_{2i})}{1-C_1(x_{1i},x_{2i})}+\sum_{i=1}^{n_1}\frac{W_1(x_{1i},x_{2i})C_1(x_{1i},x_{2i})}{1+C_1(x_{1i},x_{2i})}-$$
$$\sum_{i=1}^{n_1}W_1(x_{1i},x_{2i})+\sum_{i=1}^{n_2}\frac{1+\lambda x_{1i}\ln x_{1i}+\beta\ln x_{1i}}{\beta+\lambda x_{1i}}+\sum_{i=1}^{n_2}\frac{1+\lambda x_{2i}\ln x_{2i}+\beta\ln x_{2i}}{\beta+\lambda x_{1i}}-3\sum_{i=1}^{n_2}\frac{W_2(x_{1i},x_{2i})C_2(x_{1i},x_{2i})}{1-C_2(x_{1i},x_{2i})}+\sum_{i=1}^{n_2}\frac{W_2(x_{1i},x_{2i})C_2(x_{1i},x_{2i})}{1+C_2(x_{1i},x_{2i})}-$$
$$\sum_{i=1}^{n_2}W_2(x_{1i},x_{2i})-\sum_{i=1}^{n_3}\left((\alpha_1+\alpha_2+\alpha_3)x_i^{\beta}\ln x_ie^{\lambda x_i}\right)-2\sum_{i=1}^{n_3}\frac{\left((\alpha_1+\alpha_2+\alpha_3)x_i^{\beta}\ln x_ie^{\lambda x_i}\right)C_3(x_i,x_i)}{1-C_3(x_{1i},x_{2i})}$$
$$+\sum_{i=1}^{n_3}\frac{1+\lambda x_i\ln x_i+\beta\ln x_i}{\beta+\lambda x_i} \tag{27}$$

where

$C_1(x_{1i},x_{2i})=(1-\theta)e^{-(\alpha_1+\alpha_3)x_{1i}^{\beta}e^{\lambda x_{1i}}}\times e^{-\alpha_2x_{2i}^{\beta}e^{\lambda x_{2i}}}$

$C_2(x_{1i},x_{2i})=(1-\theta)e^{-(\alpha_2+\alpha_3)x_{2i}^{\beta}e^{\lambda x_{2i}}}\times e^{-\alpha_1x_{1i}^{\beta}e^{\lambda x_{1i}}}$

$C_3(x_i,x_i)=(1-\theta)e^{-(\alpha_1+\alpha_2+\alpha_3)x_i^{\beta}e^{\lambda x_i}}$

$W_1(x_{1i},x_{2i})=(\alpha_1+\alpha_3)x_{1i}^{\beta}\ln x_{1i}e^{\lambda x_{1i}}+\alpha_2x_{2i}^{\beta}\ln x_{2i}e^{\lambda x_{2i}}$

$W_2(x_{1i},x_{2i})=\alpha_1x_{1i}^{\beta}\ln x_{1i}e^{\lambda x_{1i}}+(\alpha_2+\alpha_3)x_{2i}^{\beta}\ln x_{2i}e^{\lambda x_{2i}}.$

The maximum likelihood estimates $\widehat{\alpha}_1$, $\widehat{\alpha}_2$, $\widehat{\alpha}_3$, $\widehat{\theta}$, $\widehat{\lambda}$ and $\widehat{\beta}$ of the unknown parameters α_1, α_2, α_3, θ and λ respectively, are calculated by solving Equations (22) - (27), using Mathcad software.

8 Data Analysis of BMWGD

This section used the same real data set (see Table 1) to compare the Bivariate Modified Weibull Geometric (BMWG) distribution with the other distributions in Table 3.

Table 4: The MLEs for UEFA Champion′s League data

Model	MLEs
BMWGD	$\widehat{\alpha}_1 = 1.958\times10^{-3}, \widehat{\beta} = 1.041$
	$\widehat{\alpha}_3 = 4.166\times10^{-3}, \widehat{\lambda} = 0.02$
	$\widehat{\alpha}_2 = 4.542\times10^{-3}, \widehat{\theta} = 0.828$

Table 5: The value 0f *L*, *AIC* and *BIC*.

Model	*L*	*AIC*	*BIC*
BMWGD	-287.4278	586.8556	-298.26

Since the value of $(-L)$ (see Table 3 & Table 5) for the BMWG distribution is less than all the values computed from the other models, then the proposed distribution a very competitive model to these data.

The profiles of the log-likelihood function of α_1, α_2, α_3, β, θ and λ of BMWGD for UEFA Champion′s League data are ploted in in Figures from Figure 6 to Figure 11 respectively. From the plots of the profiles of the log-likelihood function of α_1, α_2, α_3, β, θ and λ ,we observe that the equations of the likelihood have a unique solution.

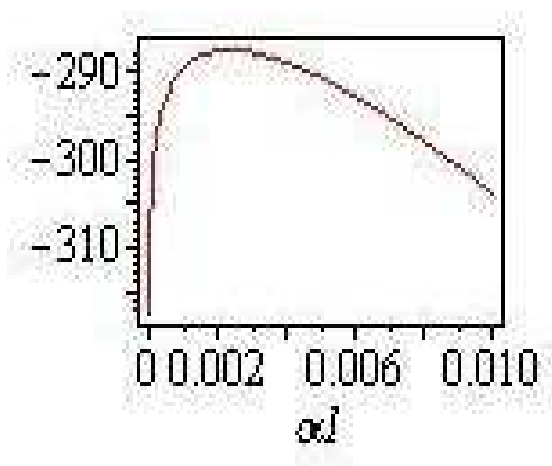

Fig. 6: Variation for the log-likelihood function of α_1 for UEFA Champion′s League data.

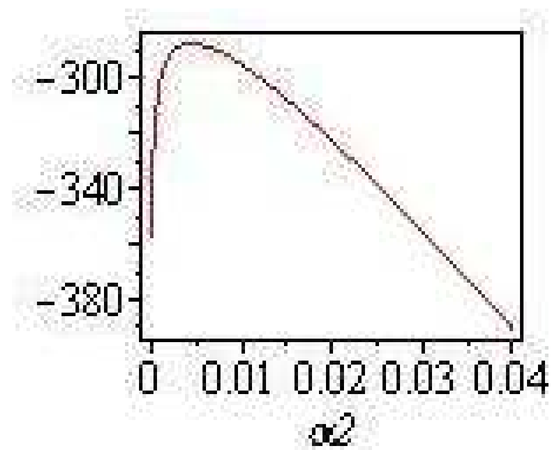

Fig. 7: Variation for the log-likelihood function of α_2 for UEFA Champion′s League data.

9 Conclusions

This paper proposed a new bivariate modified Weibull (BMW) distribution, whose marginals are MW distributions. Also, we introduce an extension of the bivariate Modified Weibull distribution and we call it the bivariate Modified Weibull-Geometric (BMWG) distribution. Some statistical properties of this distributions have been studied and discussed. The maximum likelihood method investigated to estimate the parameters of BMW and BMWG distributions. A real data set is analyzed using the Bivariate Modified Weibull distribution, Marshall-Olkin bivariate exponential distribution, Bivariate Generalized Exponential distribution, Bivariate generalized linear failure rate distribution. Based on the comparisons between all these models, we conclude that, the introduced model is highly competitive in the sense of fitting this real data set. And, its extension (BMWGD) is better than it.

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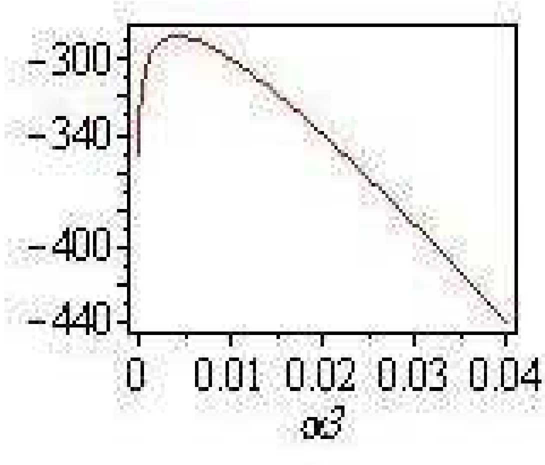

Fig. 8: Variation for the log-likelihood function of α_3 for UEFA Champion′s League data.

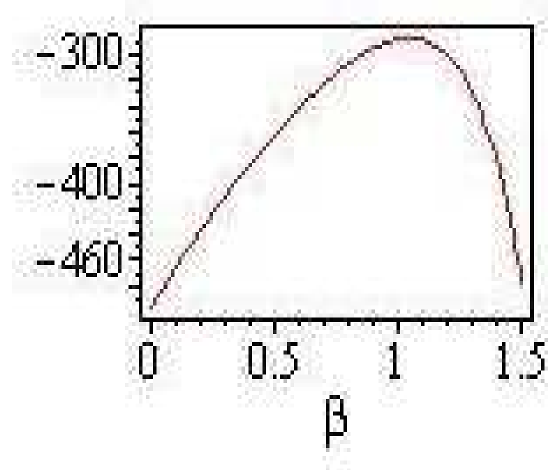

Fig. 9: Variation for the log-likelihood function of β for UEFA Champion′s League data.

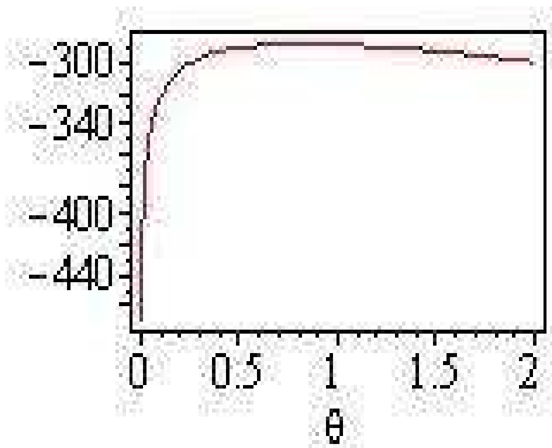

Fig. 10: Variation for the log-likelihood function of θ for UEFA Champion′s League data.

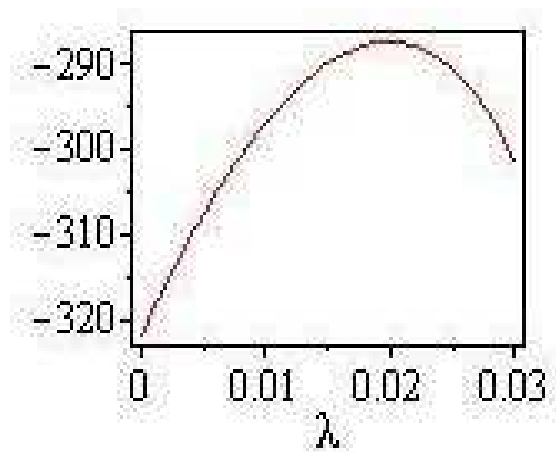

Fig. 11: Variation for the log-likelihood function of λ for UEFA Champion′s League data.

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