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The Intersection Form of Four-Manifolds and Exceptional Group Symmetries

Simon Davis[*]

Research Foundation of Southern California, 8837 Villa La Jolla Drive #13595, La Jolla, CA 92039, USA

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Abstract: The nontrivial component of the intersection form of nonsmooth four-manifolds is proportional to the Cartan matrix of E_8. Since the background geometry of all elementary particle field theories is a smooth four-manifold, this intersection form will be introduced through the embedding of an infinite-genus surface with a nonsmooth structure in the neighbourhood of the ideal boundary.

Keywords: infinite-genus surface, ideal booundary, intersection form, exceptional group

1 Introduction

The classification of diffeomorphism structures on four-manifolds includes the intersection form of the homology basis. While the conventional form is proportional to a symplectic matrix, there exists manifolds such that it is equal to

$$k \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} + \ell E_8$$

A nonsmooth structure is required of the manifold and the signature must equal 8 $(mod 16)$. The Euclidean path integral in quantum gravity is defined over a space of connected, smooth Riemannian spaces with a boundary, representing the hypersurface at a given time of experimental measurement. The existence of a nonsmooth structure would prevent the analytic formulation of the action in the weighting factor and affect integration near the extrema.

A method for introducing this class of manifolds within the domain of integration of a path integral is apparent in string theory. The expansion of the scattering matrix is defined conventionally over the set of Riemann surfaces of finite genus. It may be extended to surfaces with Dirichlet boundaries and effectively closed surfaces of infinite genus. The ends of these surfaces may be nonplanar and, furthermore, the structure is not necessarily smooth. The embedding of the an infinite-genus surface in a four-manifold would be sufficient to cause the intersection matrix to have the exotic form.

The relation between the intersection form and the subsequent symmetries of a field theory defined on manifolds in this class will be elaborated. It will be demonstrated that a consistency condition on the gauge transformation arises in the neighbourhood of the end of an infinite-genus surface related to the E_8 intersection form. A connection with the phenomenology of gauge theories in four dimensions is established.

2 Metric Structures on Surfaces of Infinite Genus

Infinite-genus surfaces are noncompact, and the structure of the ends differs from that of a finite-genus surface with an extended boundary. It may be proven, for example, that there is no smooth bijection from the end of an infinite-genus surface to a region in the complex plane.

Lemma 1. Infinite-genus surfaces have nonplanar ends.

Proof. A vanishing first homotopy group of the one-point compactification of the end is sufficient for planarity. This characteristic is valid, for example, when the ends are semi-infinite cylinders attached to a finite-genus surface, which can be mapped conformally to a punctured disk. The structure of an end of an infinite-genus surface is considerably more complex. When the handles are compactified such that there is an accumulation point in a finite region in an embedding space, any neighbourhood

[*] Corresponding author e-mail: sbdavis@resfdnsca.org

of non-zero radius in the i intrinsic metric contains a region of the surface that consists of an arbitrary number of handles.

The restriction of an analytic, bounded function f of an end E representing an ideal boundary point to $E' \subset E$ is a proper map from E' to a neighbourhood in $\mathbb{C}$. There is no differentiable bijective mapping from E' to a planar region if the degree of the function is greater than or equal to 2 in this neighbourhood of a point on the ideal boundary.

It has been proven that functions on the ideal boundary of an infinite-genus surface must equal the same value a minimum of two times. By a theorem on algebraic functions on the complex plane, it follows that the lower bound for the degree of this function on the end is 2. Therefore, the end of an infinite-genus surface is nonplanar.

Lemma 2. The metric structure of the infinite-genus surface is smooth if it belongs to O_G and not smooth when the capacity of the ideal boundary is nonvanishing.

Proof. The effectively closed infinite-genus surfaces are the class that represent a direct limit of the set of closed finite-genus surfaces, since the boundary is null. The metric structure is defined within a conformal equivalence class. Under a conformal transformation, the accumulating handles may be mapped to a sequence that is equally spaced. The smoothness of the metric would be valid on any n^{th} order approximation E_n with $\partial E_n \subset \Sigma$ and $lim_{n\to\infty}\partial E_n = \beta$, where β is the ideal boundary.

When the capacity of the ideal boundary is nonvanishing, the Cantor set of ends would have non-zero Hausdorff dimension. Suppose that the ends are arranged to be located on the equator of the sphere. The cardinality of the set of ends will be $2^{\aleph_0}$, which is equal to that of the continuum of the real line. It follow that the set of accumuulation points of the handles on the would be dense in the equator. There would be no neighbourhood of a point on this circle that that can be mapped diffeomorphically to a simply connected region in the complex plane. Therefore, the metric structure would not be smooth in this neighbourhood.

The embedding of infinite-genus surfaces in four-dimensional manifolds depends initially on the representation as a boundary of the quotient of a three-dimensional hyperbolic space. The uniformizing Fuchsian group may be embedded in $PSL(3;\mathbb{Z})$. Although the embedding is not differentiable, it is continuous. The hyperbolic manifold then may be immersed in a four-dimensional space.

3 Exceptional Group Symmetries on the Four-Manifold

A four-manifold would have a nonsmooth metric structure in the neighbourhood of an ideal boundary of an infinite-genus surface with non-zero capacity and a signature satisfying the congruence condition $\sigma \equiv 8\ (mod\ 16)$ [2]. The signature is an integral over the entire space. If it is a smooth manifold except in the vicinity of the infinite-genus surface, the contribution to this integral would satisfy the conventional congruence. Consider then the the remaining part of the integral for the signature.

Theorem 1. The equality of the integral of the product of the Riemann curvature tensor and its dual divided by k with the required signature yields a condition on the ideal boundary of a surface of infinite genus embedded in a four-dimensional manifold with a nonsmooth structure.

Proof. The Riemann curvature tensor has one independent component equal to the Ricci scalar in two dimensions. On a hyperbolic surface, R may be set equal to -1 except on the ideal boundary. At each element of the Cantor set, it tends to a delta function and the integral is transformed to a sum over the this set. Since the area element tends to zero on this boundary, the contribution of the sum is finite. The integral over the three dimensions in the four-manifold normal to the ideal boundary would equal $Vol(S^3) = 4\pi^2$. Furthermore, it would be determined by the harmonic measure of the ideal boundary.

The harmonic measure is the solution to the problem $\triangle w = 0$ with $w|_\alpha = 0$ and $w|_\beta = 1$, where α is the interior boundary of an end E and β is the ideal boundary. The area of the end would be

$$\int_E dw\wedge *dw = \int_E [d(w\wedge *dw) - w\wedge d*dw]. \qquad (3.1)$$

Since w is harmonic, $d*dw = 0$, and

$$\int_E d(w\wedge *dw) = -\int_\alpha w\wedge *dw + \int_\beta w\wedge *dw. \qquad (3.2)$$

when the α is a clockwise contour and β is a counterclockwise contour. This integral equals

$$\int_\beta *dw, \qquad (3.3)$$

which can be non-zero based on the normal derivative of w at the ideal boundary. Finiteness of the sum over the elements of the Cantor set would occur if there is a cancellation of the phase of the normal derivative. Then, the condition derived from the signature would be

$$\frac{4\pi^2}{k}\int_\beta *dw \equiv 8\ (mod\ 16). \qquad (3.4)$$

Then the four-manifold would be an E_8 homology manifold.

Since the intersection form of the manifold is equal to the Cartan matrix of E_8, any gauge group symmetry, viewed in terms of the induced change in the potential from a passive transformation of the coordinates, must keep invariant this characteristic of the manifold. It follows that an E_8 group invariance is induced.

Intersection theory on the moduli space of a Riemann surface also provides a formulation of topological two-dimensional gravity with closed string observables being identified with powers of the first Chern class of a line bundle [3]. This symmetry is larger than that allowed by a natural bundle on a four-manifold. The product of the tangent bundle of three-manifold and the time coordinate admits a G_2 structure [4]. Similarly, the product of the tangent bundle of a four-manifold and the normal coordinate would admit an $SO(9)$ structure. This dimension is not large enough to include the several of the exceptional groups.

With an induced E_8 symmetry, the invariances of the theory may be reduced to E_6. The E_6 grand unified theories are known to have several theoretical predictions that coincide closely with experiment. The value of the Weinberg angle in the derived electroweak Lagrangian is accurate to a higher degree of precision [5] than other models.

The group E_6 does arise as a symmetry of the heterotic string effective action after compactification over a space with SU(3) holonomy [6]. A supergravity action may be constructed from vertex operators of the ten-dimensional string theory such that an E_6 invariant charges remain after fixing several of the lattice group parameters [7]. This technique therefore makes essential use of the fundamental string theory for the formulation of the field theory in four dimensions. It corroborates the existence of an exceptional group symmetry under these conditions.

4 Conclusion

The distinction between the two types of infinite-genus surfaces is essential to the embedding in a four-dimensional manifold. It is found that a surface a parabolic surface would have a differentiable metric, while it is not possible to define a smooth metric on surface with an ideal boundary of non-zero capacity. The absence of a smooth structure at the end of the surface requires an embedding a four-manifold that is not smooth. The intersection forms of the four-manifold that do not have a globally smooth structure include the Cartan matrix of E_8. More generally, the intersection form is

$$k \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} + \ell E_8$$

The condition on the signature of E_8 manifolds that can be translated to a restriction on the integral of a curvature combination to a neighbourhood of the ideal boundary. Given the constancy of the curvature on a Riemann surface with a Poincare metric, the integral would be reduced to a line integral over the ideal boundary. The embedding of the infinite-genus surface in an E_8 homology manifold then would yield a condition on the harmonic measure of the ideal boundary.

The symmetry of the field theory formulated on the space-time then must be compatible with the intersection form. The phenomenology of exceptional groups in grand unified theories is especially successful for the theoretical explanation of certain parameters in the theory of elementary particle interactions [5]. It would be expected that the exceptional group G_2 and the classical Lie groups in would govern the phenomenology of physically realistic gauge field theories. Therefore, the reduction of the group symmetry from E_8 to these subgroups in the neighbourhood of the ideal boundary of an infinite-genus in addition to the complement in the four-dimensional manifold.

The relation of the classical groups to rational singularities can be interpreted in terms of the group invariances allowed by the ends of a surface of infinite genus [8]. The accumulation point of the handles at the end of a O_G surface would define a rational singularity which is known to have an intersection form of a classical group [9]. Consequently, group invariances in the neighbourhood of the ideal boundaries of surfaces in the class O_G would be classical while the symmetries in the neighbourhood of the ideal boundary of surfaces with non-zero capacity can include exceptional groups.

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