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Local Fractional Natural Homotopy Perturbation Method for Solving Partial Differential Equations with Local Fractional Derivative

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Abstract: A new analytical method called the local fractional natural homotopy perturbation method (LFNHPM) for solving partial differential equations with local fractional derivative is introduced. The proposed analytical method is a combination of the local fractional homotopy perturbation method (LFHPM) and the local fractional natural transform (LFNTM). In this analytical method, the fractional derivative operators are computed in local fractional sense, and the nonlinear terms are calculated using He's polynomial. Some applications are given to illustrate the simplicity, efficiency, and high accuracy of the proposed method.

Keywords: Local fractional natural homotopy perturbation method, local fractional derivative operator, local fractional partial differential equations.

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1 Introduction, Motivation and Preliminaries

The theory and applications of fractional calculus have a long history in pure and applied mathematics, and were first introduced by Leibniz and L'Hôpital in the year 1695. During the last few decades, fractional calculus was successfully applied in many areas of physical science and engineering such as plasma physics, quantum mechanics, astrophysics, fracture mechanics, chaotic dynamics, optics, and so on. Fractional partial differential equations have been solved using many numerical and analytical methods [1,2,3,4,5,6].
Recently, local fractional derivative and calculus which described the non-differentiable function defined on Cantor sets arising in mathematical physics were discussed [7,8] . The solution of wave equation on Cantor sets using the local fractional variational iteration and decomposition methods was presented [9]. In 2013, the approximate solutions of diffusion equations on Cantor sets were studied [10]. Based on local fractional Sumudu transform, the solution of IVPs on Cantor sets was proposed in 2014 [11]. The local fractional homotopy perturbation method for solving fractional partial differential equations arising in mathematical physics was presented in 2015 [12]. Recently, in the year 2017, a hybrid computational approach for solving Klein-Gordon equations on Cantor sets was introduced [13]. More detail about the local fractional derivatives are referred to [14,15,16,17,18,19,20,21,22,23,24,25,26,27,28].
In this article, we investigate the solutions of local fractional partial differential equations based on local fractional homotopy perturbation and natural transform. The local fractional natural homotopy perturbation method is a coupling of the homotopy perturbation method [29,30] with natural transform method [31,32,33], and this gives series solutions which converge rapidly within few iterations. In Table 1 we present some important properties of local fractional calculus.

The organization of our manuscript is given below. In Section 2 we discuss the local fractional derivative and local fractional natural transform. Section 3 deals with the local fractional natural homotopy perturbation method. In Section 4 we explain the results of our applications. Section 5 contains our conclusion.

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2 Local Fractional Derivative and Local Fractional Natural Transform

The local fractional natural transform of the function $v(t)$ of order α is defined by the following integral:

$$^{LF}N_\alpha[v(t)] = V_\alpha(s,u) = \frac{1}{\Gamma(1+\alpha)}\int_0^\infty E_\alpha\left(\frac{-s^\alpha t^\alpha}{u^\alpha}\right)\frac{v(t)}{u^\alpha}(dt)^\alpha;\ \ 0<\alpha\leq 1. \tag{1}$$

And the inverse local fractional natural transform is defined by:

$$^{LF}N_\alpha^{-1}[V_\alpha(s,u)] = v(t) = \frac{1}{(2\pi i)^\alpha}\int_{\gamma-i\infty}^{\gamma+i\infty} E_\alpha\left(\frac{s^\alpha t^\alpha}{u^\alpha}\right)V_\alpha(s,u)\,(ds)^\alpha,\ \ 0<\alpha\leq 1, \tag{2}$$

where s^α and u^α are the local natural transform variables, and γ is a real constant and the integral in Eq. (2) which is taken along $s^\alpha = \gamma$ in the complex plane $s^\alpha = x^\alpha + i^\alpha y^\alpha$.

Some properties of the local fractional natural transform method are given below.

Proposition 1.: *Local fractional natural transform of local fractional derivative is defined by:*

$$^{LF}N_\alpha\left[v^{(n\alpha)}(t)\right] = \frac{s^{n\alpha}}{u^{n\alpha}}V_\alpha(s,u) - \sum_{k=0}^{n-1}\frac{s^{(n-k-1)\alpha}}{u^{(n-k)\alpha}}v^{(k\alpha)}(0). \tag{3}$$

When n=1, 2, and 3, we obtained the following results:

$$^{LF}N_\alpha\left[v^{(\alpha)}(t)\right] = \frac{s^\alpha}{u^\alpha}V_\alpha(s,u) - \frac{1}{u^\alpha}v(0),$$

$$^{LF}N_\alpha\left[v^{(2\alpha)}(t)\right] = \frac{s^{2\alpha}}{u^{2\alpha}}V_\alpha(s,u) - \frac{s^\alpha}{u^{2\alpha}}v(0) - \frac{1}{u^\alpha}v^{(\alpha)}(0),$$

$$^{LF}N_\alpha\left[v^{(3\alpha)}(t)\right] = \frac{s^{3\alpha}}{u^{3\alpha}}V_\alpha(s,u) - \frac{s^{2\alpha}}{u^{3\alpha}}v(0) - \frac{s^\alpha}{u^{2\alpha}}v^{(\alpha)}(0) - \frac{1}{u^\alpha}v^{(2\alpha)}(0).$$

Property 2: Linearity property of the local fractional natural transform is defined by:
$^{LF}N_\alpha[\gamma f(t) \pm \beta g(t)] = \gamma^{LF}N_\alpha[f(t)] \pm \beta^{LF}N_\alpha[g(t)] = \gamma F_\alpha(s,u) \pm \beta G_\alpha(s,u)$,
where, $F_\alpha(s,u)$ and $G_\alpha(s,u)$ are the local fractional natural transforms of the functions $f(t)$ and $g(t)$, respectively. More properties are presented in table 1.

Definition 2: The local fractional derivative of the function $v(t)$ of order α at $t = t_0$ is defined by [7,8]:

$$v^{(\alpha)}(t) = \frac{d^\alpha v}{dt^\alpha}|_{t=t_0} = \frac{\Delta^\alpha(v(t)-v(t_0))}{(t-t_0)^\alpha}, \tag{4}$$

where,

$$\Delta^\alpha(v(t)-v(t_0)) \cong \Gamma(1+\alpha)[v(t)-v(t_0)]. \tag{5}$$

Moreover, the local fractional derivatives of higher order are defined as [7,8]:

$$D_t^{(n\alpha)}(t) = v^{(n\alpha)}(t) = \overbrace{D_t^{(\alpha)}\cdots D_t^{(\alpha)}}^{n\ times}v(t), \tag{6}$$

$$\frac{\partial^{n\alpha}v(t,x)}{\partial t^{n\alpha}} = \overbrace{\frac{\partial^\alpha}{\partial t^\alpha}\cdots\frac{\partial^\alpha}{\partial t^\alpha}}^{n\ times}v(t,x). \tag{7}$$

Definition 3: The local fractional integral of the function $v(t)$ of order α in the interval $[\gamma,\beta]$ is defined by [7,8]:

$$_\gamma I_\beta^{(\alpha)} = \frac{1}{\Gamma(1+\alpha)}\int_\gamma^\beta v(\tau)(d\tau)^\alpha = \frac{1}{\Gamma(1+\alpha)}\lim_{\Delta\to 0}\sum_{i=0}^{N-1}v(\tau_i)(\Delta_\tau)^\alpha, \tag{8}$$

where $\Delta_{\tau i} = \tau_{i+1} - \tau_i$, $\Delta_\tau = max\Delta_{\tau 0}, \Delta_{\tau 1}, \Delta_{\tau 2}, \cdots$ $[\tau_i, \tau_{i+1}]$, $\tau_0 = \gamma$, $\tau_N = \beta$ is a partition of the interval $[\gamma, \beta]$.

Table 1. Some useful results

${}^{LF}N_\alpha\left[t^\alpha\right] = \frac{u^\alpha}{s^{2\alpha}}$	${}^{LF}N_\alpha\left[\sin_\alpha(t^\alpha)\right] = \frac{u^\alpha}{s^{2\alpha}+u^{2\alpha}}$
${}^{LF}N_\alpha\left[\cos_\alpha(t^\alpha)\right] = \frac{s^\alpha}{s^{2\alpha}+u^{2\alpha}}$	${}^{LF}N_\alpha\left[\cosh_\alpha(t^\alpha)\right] = \frac{s^\alpha}{s^{2\alpha}-u^{2\alpha}}$
${}_0I_t^{(\alpha)} \frac{t^{n\alpha}}{\Gamma(1+n\alpha)} = \frac{t^{(n+1)\alpha}}{\Gamma(1+(n+1)\alpha)}$	$\frac{d^\alpha}{dt^\alpha} \frac{t^{n\alpha}}{\Gamma(1+n\alpha)} = \frac{t^{(n-1)\alpha}}{\Gamma(1+(n-1)\alpha)}$
$E_\alpha(t^\alpha) = \sum_{n=0}^{+\infty} \frac{t^{n\alpha}}{\Gamma(1+n\alpha)},\ 0 < \alpha \leq 1$	$\frac{d^\alpha}{dt^\alpha} E_\alpha(t^\alpha) = E_\alpha(t^\alpha)$
$\sin_\alpha(t^\alpha) = \sum_{n=0}^{+\infty}(-1)^n \frac{t^{2n\alpha}}{\Gamma(1+(2n+1)\alpha)},\ 0 < \alpha \leq 1$	$\frac{d^\alpha}{dt^\alpha} \sin_\alpha(t^\alpha) = \cos_\alpha(t^\alpha)$
$\cos_\alpha(t^\alpha) = \sum_{n=0}^{+\infty}(-1)^n \frac{t^{(2n+1)\alpha}}{\Gamma(1+(2n+1)\alpha)},\ 0 < \alpha \leq 1$	$\frac{d^\alpha}{dt^\alpha} \cos_\alpha(t^\alpha) = -\sin_\alpha(t^\alpha)$

3 Local Fractional Natural Homotopy Perturbation Method

Let us consider the following nonlinear operator with local fractional derivative of the form:

$$L_\alpha v(x,t) + F_\alpha(v(x,t)) + M_\alpha(v(x,t)) = g_\alpha(x,t), \tag{9}$$

where, $L_\alpha = \frac{\partial^{2\alpha}}{\partial t^{2\alpha}}$ denotes the linear local fractional differential operator, F_α denotes linear fractional derivative operator of order less than L_α, $M_\alpha(v(x,t))$ denotes the nonlinear operator, and $g_\alpha(x,t)$ is the non-differentiable source term.

Applying the local fractional natural transform (denoted in this paper by ${}^{LF}N_\alpha$) on both sides of Eq. (9), we get:

$$ {}^{LF}N_\alpha\left[L_\alpha v(x,t)\right] + {}^{LF}N_\alpha\left[F_\alpha(v(x,t)) + M_\alpha(v(x,t))\right] = {}^{LF}N_\alpha\left[g_\alpha(x,t)\right]. \tag{10}$$

Using the derivative of the local fractional Natural transform on Eq. (10), we get:

$$V_\alpha(x,s,u) = \frac{1}{s^\alpha}v(x,0) + \frac{u^\alpha}{s^{2\alpha}}v(x,0) + \frac{u^{2\alpha}}{s^{2\alpha}}\left({}^{LF}N_\alpha\left[g_\alpha(x,t)\right]\right) - \frac{u^{2\alpha}}{s^{2\alpha}}\left({}^{LF}N_\alpha\left[F_\alpha(v(x,t)) + M_\alpha(v(x,t))\right]\right). \tag{11}$$

Taking the inverse local fractional natural transform of Eq. (11), we obtain:

$$v(x,t) = G_\alpha(x,t) - {}^{LF}N_\alpha^{-1}\left[\frac{u^{2\alpha}}{s^{2\alpha}}\left({}^{LF}N_\alpha\left[F_\alpha(v(x,t)) + M_\alpha(v(x,t))\right]\right]\right], \tag{12}$$

where,

$$G_\alpha(x,t) = v(x,0) + \frac{t^\alpha}{\Gamma(1+\alpha)}v^{(\alpha)}(x,0) + {}^{LF}N_\alpha^{-1}\left[\frac{u^{2\alpha}}{s^{2\alpha}}\left({}^{LF}N_\alpha\left[g(x,t)\right]\right)\right].$$

Now we apply the local fractional homotopy perturbation method of the form:

$$v(x,t) = \sum_{n=0}^{\infty} p^{n\alpha} v_n(x,t). \tag{13}$$

According to the local fractional homotopy perturbation method, $p \in [0,1]$ is an embedding small parameter.
The nonlinear term $M_\alpha(v(x,t))$ is written as:

$$M_\alpha(v(x,t)) = \sum_{n=0}^{\infty} p^{n\alpha} H_n(v). \tag{14}$$

where $H_n(v)$ is the local fractional He's polynomial and can be calculated using the following formula:

$$H_n(v_1, v_2, \cdots, v_n) = \frac{1}{\Gamma(n\alpha+1)} \frac{\partial^{n\alpha}}{\partial p^{n\alpha}} \left[M_\alpha \left(\sum_{j=0}^{n} p^{\alpha j} v_j \right) \right]_{p=0}, n = 0, 1, 2, \cdots$$

By substituting Eq. (13) and Eq. (14) into Eq. (12), we get:

$$\sum_{n=0}^{\infty} p^{n\alpha} v_n(x,t) = G_\alpha(x,t) - p^\alpha \left({}^{LF}N_\alpha^{-1} \left[\frac{u^{2\alpha}}{s^{2\alpha}} \left({}^{LF}N_\alpha \left[\sum_{n=0}^{\infty} p^{n\alpha} F_\alpha(v_n(x,t)) + \sum_{n=0}^{\infty} p^{n\alpha} H_n(v) \right] \right) \right] \right). \tag{15}$$

Comparing the coefficients of like powers of p^α in Eq. (15), we have the following approximations:

$$\begin{aligned}
p^{0\alpha} &: v_0(x,t) = G_\alpha(x,t), \\
p^{1\alpha} &: v_1(x,t) = -{}^{LF}N_\alpha^{-1} \left[\frac{u^{2\alpha}}{s^{2\alpha}} \left({}^{LF}N_\alpha \left[F_\alpha(v_0(x,t)) + H_0(v) \right] \right) \right], \\
p^{2\alpha} &: v_2(x,t) = -{}^{LF}N_\alpha^{-1} \left[\frac{u^{2\alpha}}{s^{2\alpha}} \left({}^{LF}N_\alpha \left[F_\alpha(v_1(x,t)) + H_1(v) \right] \right) \right], \\
p^{3\alpha} &: v_3(x,t) = -{}^{LF}N_\alpha^{-1} \left[\frac{u^{2\alpha}}{s^{2\alpha}} \left({}^{LF}N_\alpha \left[F_\alpha(v_2(x,t)) + H_2(v) \right] \right) \right], \\
&\vdots
\end{aligned}$$

and so on.
Setting p and α to be equal to 1, the series solution of Eq. (9) is given by:

$$v(x,t) = \lim_{N \to \infty} \sum_{n=0}^{N} v_n(x,t). \tag{16}$$

Convergence of the series solutions: The series solutions in Eq. (16) converge in most cases. Since the proposed method gives solution in iterative form, then the Banach's fixed point theorem can be applied to study the convergence of the series solutions. Presently, the proposed analytical method can only be applied to solve problems with initial conditions. Thus, the case of problems with boundary conditions remains an open problem.

4 Applications

In this section, the applications of the local fractional natural homotopy perturbation method are clearly illustrated to show its efficiency and high accuracy.

*Example 1.*Consider the following local fractional partial differential equation of the form [12]:

$$\frac{\partial^\alpha v(x,t)}{\partial t^\alpha} - \frac{\partial^{2\alpha} v(x,t)}{\partial x^{2\alpha}} = 0,\ t > 0,\ x \in \mathbb{R},\ 0 < \alpha < 1, \tag{17}$$

subject to the initial condition

$$v(x,0) = E_\alpha(x^\alpha). \tag{18}$$

Applying the local fractional natural transform to Eq. (17) subject to the given initial condition, we get:

$$V_\alpha(x,s,u) = \frac{1}{s^\alpha} E_\alpha(x^\alpha) + \frac{u^\alpha}{s^\alpha} \left({}^{LF}N_\alpha \left[\frac{\partial^{2\alpha} v(x,t)}{\partial x^{2\alpha}} \right] \right). \tag{19}$$

Taking the inverse local fractional natural transform of Eq. (19), we deduce:

$$v(x,t) = E_\alpha(x^\alpha) + {}^{LF}N_\alpha^{-1} \left[\frac{u^\alpha}{s^\alpha} \left({}^{LF}N_\alpha \left[\frac{\partial^{2\alpha} v(x,t)}{\partial x^{2\alpha}} \right] \right) \right]. \tag{20}$$

Now we apply the local fractional homotopy perturbation method of the form:

$$v(x,t)=\sum_{n=0}^{\infty}p^{n\alpha}v_n(x,t).$$

Then Eq. (20) becomes:

$$\sum_{n=0}^{\infty}p^{n\alpha}v_n(x,t)=E_\alpha(x^\alpha)+p^\alpha\left({}^{LF}N_\alpha^{-1}\left[\frac{u^\alpha}{s^\alpha}\left({}^{LF}N_\alpha\left[\sum_{n=0}^{\infty}p^{n\alpha}\frac{\partial^{2\alpha}v_n(x,t)}{\partial x^{2\alpha}}\right]\right)\right]\right). \tag{21}$$

Comparing the coefficients of like powers of p^α in Eq. (21), the following approximations are obtained:

$$\begin{aligned}
p^{0\alpha}:\ v_0(x,t)&=E_\alpha(x^\alpha),\\
p^{1\alpha}:\ v_1(x,t)&={}^{LF}N_\alpha^{-1}\left[\frac{u^\alpha}{s^\alpha}\left({}^{LF}N_\alpha\left[\frac{\partial^{2\alpha}v_0(x,t)}{\partial x^{2\alpha}}\right]\right)\right]\\
&=\frac{t^\alpha}{\Gamma(1+\alpha)}E_\alpha(x^\alpha),\\
p^{2\alpha}:\ v_2(x,t)&={}^{LF}N_\alpha^{-1}\left[\frac{u^\alpha}{s^\alpha}\left({}^{LF}N_\alpha\left[\frac{\partial^{2\alpha}v_1(x,t)}{\partial x^{2\alpha}}\right]\right)\right]\\
&=\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}E_\alpha(x^\alpha),\\
p^{3\alpha}:\ v_3(x,t)&={}^{LF}N_\alpha^{-1}\left[\frac{u^\alpha}{s^\alpha}\left({}^{LF}N_\alpha\left[\frac{\partial^{2\alpha}v_2(x,t)}{\partial x^{2\alpha}}\right]\right)\right]\\
&=\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}E_\alpha(x^\alpha),\\
&\ \vdots
\end{aligned}$$

and so on.
Then the series solution of Eq. (17)-Eq. (18) is given by:

$$\begin{aligned}
v(x,t)&=\sum_{n=0}^{\infty}v_n(x,t)\\
&=v_0(x,t)+v_1(x,t)+v_2(x,t)+\cdots\\
&=E_\alpha(x^\alpha)\left(1+\frac{t^\alpha}{\Gamma(1+\alpha)}+\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}+\cdots\right)\\
&=E_\alpha(x^\alpha)\sum_{n=0}^{\infty}\frac{t^{n\alpha}}{\Gamma(1+n\alpha)}=E_\alpha(t^\alpha)E_\alpha(x^\alpha).
\end{aligned}$$

Thus, the exact solution of Eq. (17)-Eq. (18) is given by:

$$v(x,t)=E_\alpha(t^\alpha)E_\alpha(x^\alpha). \tag{22}$$

The exact solution is in close agreement with the result obtained in [12].

*Example 2.*Consider the following local fractional partial differential equation of the form [27]:

$$\frac{\partial^\alpha v(x,t)}{\partial t^\alpha}+\frac{\partial^{2\alpha}v(x,t)}{\partial x^{2\alpha}}-v(x,t)=0,\ t>0,\ x\in\mathbb{R},\ 0<\alpha<1, \tag{23}$$

subject to the initial condition

$$v(x,0)=\sin_\alpha(x^\alpha). \tag{24}$$

Applying the local fractional natural transform to Eq. (23) subject to the given initial condition, we get:

$$V_\alpha(x,s,u) = \frac{1}{s^\alpha}\sin_\alpha(x^\alpha) + \frac{u^\alpha}{s^\alpha}\left({}^{LF}N_\alpha\left[v(x,t) - \frac{\partial^{2\alpha}v(x,t)}{\partial x^{2\alpha}}\right]\right). \tag{25}$$

Taking the inverse local fractional natural transform of Eq. (25), we have:

$$v(x,t) = \sin_\alpha(x^\alpha) + {}^{LF}N_\alpha^{-1}\left[\frac{u^\alpha}{s^\alpha}\left({}^{LF}N_\alpha\left[v(x,t) - \frac{\partial^{2\alpha}v(x,t)}{\partial x^{2\alpha}}\right]\right)\right]. \tag{26}$$

Now we apply the local fractional homotopy perturbation method of the form:

$$v(x,t) = \sum_{n=0}^{\infty} p^{n\alpha} v_n(x,t). \tag{27}$$

Then Eq. (26) becomes:

$$\sum_{n=0}^{\infty} p^{n\alpha} v_n(x,t) = \sin_\alpha(x^\alpha) + p^\alpha\left({}^{LF}N_\alpha^{-1}\left[\frac{u^\alpha}{s^\alpha}\left({}^{LF}N_\alpha\left[\sum_{n=0}^{\infty} p^{n\alpha} v_n(x,t) - \sum_{n=0}^{\infty} p^{n\alpha}\frac{\partial^{2\alpha}v_n(x,t)}{\partial x^{2\alpha}}\right]\right)\right]\right). \tag{28}$$

Comparing the coefficient of like powers of p^α in Eq. (28), the following approximations are obtained:

$$\begin{aligned}
p^{0\alpha}:\ v_0(x,t) &= \sin_\alpha(x^\alpha),\\
p^{1\alpha}:\ v_1(x,t) &= {}^{LF}N_\alpha^{-1}\left[\frac{u^\alpha}{s^\alpha}\left({}^{LF}N_\alpha\left[v_0(x,t) - \frac{\partial^{2\alpha}v_0(x,t)}{\partial x^{2\alpha}}\right]\right)\right]\\
&= \frac{2t^\alpha}{\Gamma(1+\alpha)}\sin_\alpha(x^\alpha),\\
p^{2\alpha}:\ v_2(x,t) &= {}^{LF}N_\alpha^{-1}\left[\frac{u^\alpha}{s^\alpha}\left({}^{LF}N_\alpha\left[v_1(x,t) - \frac{\partial^{2\alpha}v_1(x,t)}{\partial x^{2\alpha}}\right]\right)\right]\\
&= \frac{4t^{2\alpha}}{\Gamma(1+2\alpha)}\sin_\alpha(x^\alpha),\\
p^{3\alpha}:\ v_3(x,t) &= {}^{LF}N_\alpha^{-1}\left[\frac{u^\alpha}{s^\alpha}\left({}^{LF}N_\alpha\left[v_2(x,t) - \frac{\partial^{2\alpha}v_2(x,t)}{\partial x^{2\alpha}}\right]\right)\right]\\
&= \frac{8t^{3\alpha}}{\Gamma(1+3\alpha)}\sin_\alpha(x^\alpha),\\
&\ \ \vdots
\end{aligned}$$

and so on.
Then the series solution of Eq. (23)-Eq. (24) is given by:

$$\begin{aligned}
v(x,t) &= \sum_{n=0}^{\infty} v_n(x,t)\\
&= v_0(x,t) + v_1(x,t) + v_2(x,t) + \cdots\\
&= \sin_\alpha(x^\alpha)\left(1 + \frac{2t^\alpha}{\Gamma(1+\alpha)} + \frac{(2t^\alpha)^2}{\Gamma(1+2\alpha)} + \frac{(2t^\alpha)^3}{\Gamma(1+3\alpha)} + \cdots\right)\\
&= \sin_\alpha(x^\alpha)\sum_{n=0}^{\infty}\frac{(2t^\alpha)^n}{\Gamma(1+n\alpha)} = \sin_\alpha(x^\alpha)E_\alpha(2t^\alpha).
\end{aligned}$$

Thus, the exact solution of Eq. (23)-Eq. (24) is given by:

$$v(x,t) = \sin_\alpha(x^\alpha)E_\alpha(2t^\alpha). \tag{29}$$

The exact solution is in close agreement with the result obtained in [27].

*Example 3.*Consider the following local fractional partial differential equation of the form [17,27]:

$$\frac{\partial^{2\alpha}v(x,t)}{\partial t^{2\alpha}}-\frac{\partial^{2\alpha}v(x,t)}{\partial x^{2\alpha}}=0,\ t>0,\ x\in\mathbb{R},\ 0<\alpha<1, \tag{30}$$

subject to the initial conditions

$$v(x,0)=0,\ \frac{\partial^{\alpha}v(x,0)}{\partial t^{\alpha}}=E_{\alpha}(x^{\alpha}). \tag{31}$$

Applying the local fractional natural transform to Eq. (30) subject to the given initial conditions, we obtain:

$$V_{\alpha}(x,s,u)=\frac{u^{\alpha}}{s^{\alpha+1}}E_{\alpha}(x^{\alpha})+\frac{u^{2\alpha}}{s^{2\alpha}}\left({}^{LF}N_{\alpha}\left[\frac{\partial^{2\alpha}v(x,t)}{\partial x^{2\alpha}}\right]\right). \tag{32}$$

Taking the inverse local fractional natural transform of Eq. (32), we get:

$$v(x,t)=\frac{t^{\alpha}}{\Gamma(1+\alpha)}E_{\alpha}(x^{\alpha})+{}^{LF}N_{\alpha}^{-1}\left[\frac{u^{2\alpha}}{s^{2\alpha}}\left({}^{LF}N_{\alpha}\left[\frac{\partial^{2\alpha}v(x,t)}{\partial x^{2\alpha}}\right]\right)\right]. \tag{33}$$

Now we apply the local fractional homotopy perturbation method of the form:

$$v(x,t)=\sum_{n=0}^{\infty}p^{n\alpha}v_n(x,t). \tag{34}$$

Then Eq. (33) becomes:

$$\sum_{n=0}^{\infty}p^{n\alpha}v_n(x,t)=\frac{t^{\alpha}}{\Gamma(1+\alpha)}E_{\alpha}(x^{\alpha})+p^{\alpha}\left({}^{LF}N_{\alpha}^{-1}\left[\frac{u^{2\alpha}}{s^{2\alpha}}\left({}^{LF}N_{\alpha}\left[\sum_{n=0}^{\infty}p^{n\alpha}\frac{\partial^{2\alpha}v_n(x,t)}{\partial x^{2\alpha}}\right]\right)\right]\right). \tag{35}$$

Comparing the coefficients of like powers of p^{α} in Eq. (35), the following approximations are obtained:

$$\begin{aligned}
p^{0\alpha}:\ v_0(x,t)&=\frac{t^{\alpha}}{\Gamma(1+\alpha)}E_{\alpha}(x^{\alpha}),\\
p^{1\alpha}:\ v_1(x,t)&={}^{LF}N_{\alpha}^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}\left({}^{LF}N_{\alpha}\left[\frac{\partial^{2\alpha}v_0(x,t)}{\partial x^{2\alpha}}\right]\right)\right]\\
&=\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}E_{\alpha}(x^{\alpha}),
\end{aligned}$$

$$\begin{aligned}
p^{2\alpha}:\ v_2(x,t)&={}^{LF}N_{\alpha}^{-1}\left[\frac{u^{2\alpha}}{s^{2\alpha}}\left({}^{LF}N_{\alpha}\left[\frac{\partial^{2\alpha}v_1(x,t)}{\partial x^{2\alpha}}\right]\right)\right]\\
&=\frac{t^{5\alpha}}{\Gamma(1+5\alpha)}E_{\alpha}(x^{\alpha}),
\end{aligned}$$

$$\begin{aligned}
p^{3\alpha}:\ v_3(x,t)&={}^{LF}N_{\alpha}^{-1}\left[\frac{u^{2\alpha}}{s^{2\alpha}}\left({}^{LF}N_{\alpha}\left[\frac{\partial^{2\alpha}v_2(x,t)}{\partial x^{2\alpha}}\right]\right)\right]\\
&=\frac{t^{7\alpha}}{\Gamma(1+7\alpha)}E_{\alpha}(x^{\alpha}),\\
&\ \vdots
\end{aligned}$$

and so on.
Then the series solution of Eq. (30)-Eq. (31) is given by:

$$\begin{aligned}
v(x,t) &= \sum_{n=0}^{\infty} v_n(x,t) \\
&= v_0(x,t) + v_1(x,t) + v_2(x,t) + \cdots \\
&= E_\alpha(x^\alpha)\left(\frac{t^\alpha}{\Gamma(1+\alpha)} + \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + \frac{t^{5\alpha}}{\Gamma(1+5\alpha)} + \frac{t^{7\alpha}}{\Gamma(1+7\alpha)} + \cdots\right) \\
&= E_\alpha(x^\alpha)\sum_{n=0}^{\infty}\frac{t^{(2n+1)\alpha}}{\Gamma(1+(2n+1)\alpha)} = \sinh_\alpha(t^\alpha)E_\alpha(x^\alpha).
\end{aligned}$$

Thus, the exact solution of Eq. (30)-Eq. (31) is given by:

$$v(x,t) = \sinh_\alpha(t^\alpha)E_\alpha(x^\alpha). \tag{36}$$

The exact solution is in close agreement with the result obtained in [17,27].

*Example 4.*Consider the following nonlinear local fractional partial differential equation of the form:

$$\frac{\partial^\alpha v(x,t)}{\partial t^\alpha} - \frac{\partial^\alpha v(x,t)}{\partial x^\alpha}\frac{\partial^\alpha v(x,t)}{\partial t^\alpha} - v(x,t) = 0,\ t>0,\ x\in\mathbb{R},\ 0<\alpha<1, \tag{37}$$

subject to the initial condition

$$v(x,0) = \cos_\alpha(x^\alpha). \tag{38}$$

Applying the local fractional natural transform to Eq. (37) subject to the given initial condition, we get:

$$V_\alpha(x,s,u) = \frac{1}{s^\alpha}\cos_\alpha(x^\alpha) + \frac{u^\alpha}{s^\alpha}\left({}^{LF}N_\alpha\left[v(x,t) + \frac{\partial^\alpha v(x,t)}{\partial x^\alpha}\frac{\partial^\alpha v(x,t)}{\partial t^\alpha}\right]\right). \tag{39}$$

Taking the inverse local fractional natural transform of Eq. (39), we obtain:

$$v(x,t) = \cos_\alpha(x^\alpha) + {}^{LF}N_\alpha^{-1}\left[\frac{u^\alpha}{s^\alpha}\left({}^{LF}N_\alpha\left[v(x,t) + \frac{\partial^\alpha v(x,t)}{\partial x^\alpha}\frac{\partial^\alpha v(x,t)}{\partial t^\alpha}\right]\right)\right]. \tag{40}$$

Now we apply the local fractional homotopy perturbation method of the form:

$$v(x,t) = \sum_{n=0}^{\infty} p^{n\alpha} v_n(x,t). \tag{41}$$

Then Eq. (40) becomes:

$$\sum_{n=0}^{\infty} p^{n\alpha} v_n(x,t) = \cos_\alpha(x^\alpha) + p^\alpha\left({}^{LF}N_\alpha^{-1}\left[\frac{u^\alpha}{s^\alpha}\left({}^{LF}N_\alpha\left[\sum_{n=0}^{\infty} p^{n\alpha} v_n(x,t) + \sum_{n=0}^{\infty} p^{n\alpha} H_n(v)\right]\right)\right]\right), \tag{42}$$

where $H_n(v)$, is the local fractional He's polynomials which represent the nonlinear terms $\frac{\partial^\alpha v(x,t)}{\partial x^\alpha}\frac{\partial^\alpha v(x,t)}{\partial t^\alpha}$.
Some few components of the nonlinear terms $H_n(v)$ are computed below:

$$\begin{aligned}
H_0(v) &= \frac{\partial^\alpha v_0(x,t)}{\partial x^\alpha}\frac{\partial^\alpha v_0(x,t)}{\partial t^\alpha}, \\
H_1(v) &= \frac{\partial^\alpha v_1(x,t)}{\partial x^\alpha}\frac{\partial^\alpha v_0(x,t)}{\partial t^\alpha} + \frac{\partial^\alpha v_0(x,t)}{\partial x^\alpha}\frac{\partial^\alpha v_1(x,t)}{\partial t^\alpha}, \\
H_2(v) &= \frac{\partial^\alpha v_0(x,t)}{\partial x^\alpha}\frac{\partial^\alpha v_2(x,t)}{\partial t^\alpha} + \frac{\partial^\alpha v_1(x,t)}{\partial x^\alpha}\frac{\partial^\alpha v_1(x,t)}{\partial t^\alpha} + \frac{\partial^\alpha v_2(x,t)}{\partial x^\alpha}\frac{\partial^\alpha v_0(x,t)}{\partial t^\alpha}, \\
&\vdots,
\end{aligned}$$

and so on.
Comparing the coefficients of like powers of p^α in Eq. (42), the following approximations are obtained:

$$\begin{aligned}
p^{0\alpha}: v_0(x,t) &= \cos_\alpha(x^\alpha),\\
p^{1\alpha}: v_1(x,t) &= {}^{LF}N_\alpha^{-1}\left[\frac{u^\alpha}{s^\alpha}\left({}^{LF}N_\alpha\left[v_0(x,t)+H_0(v)\right]\right)\right]\\
&= \frac{2t^\alpha}{\Gamma(1+\alpha)}\cos_\alpha(x^\alpha),\\
p^{2\alpha}: v_2(x,t) &= {}^{LF}N_\alpha^{-1}\left[\frac{u^\alpha}{s^\alpha}\left({}^{LF}N_\alpha\left[v_1(x,t)+H_1(v)\right]\right)\right]\\
&= \frac{4t^{2\alpha}}{\Gamma(1+2\alpha)}\cos_\alpha(x^\alpha),\\
p^{3\alpha}: v_3(x,t) &= {}^{LF}N_\alpha^{-1}\left[\frac{u^\alpha}{s^\alpha}\left({}^{LF}N_\alpha\left[v_2(x,t)+H_2(v)\right]\right)\right]\\
&= \frac{8t^{3\alpha}}{\Gamma(1+3\alpha)}\cos_\alpha(x^\alpha),\\
&\vdots
\end{aligned}$$

and so on.
Then the series solution of Eq. (37)-Eq. (38) is given by:

$$\begin{aligned}
v(x,t) &= \sum_{n=0}^{\infty} v_n(x,t)\\
&= v_0(x,t)+v_1(x,t)+v_2(x,t)+\cdots\\
&= \cos_\alpha(x^\alpha)\left(1+\frac{2t^\alpha}{\Gamma(1+\alpha)}+\frac{(2t^\alpha)^2}{\Gamma(1+2\alpha)}+\frac{(2t^\alpha)^3}{\Gamma(1+3\alpha)}+\cdots\right)\\
&= \cos_\alpha(x^\alpha)\sum_{n=0}^{\infty}\frac{(2t^\alpha)^n}{\Gamma(1+n\alpha)} = \cos_\alpha(x^\alpha)E_\alpha(2t^\alpha).
\end{aligned}$$

Thus, the exact solution of Eq. (37)-Eq. (38) is given by:

$$v(x,t) = \cos_\alpha(x^\alpha)E_\alpha(2t^\alpha). \tag{43}$$

5 Conclusion

In this paper, partial differential equations involving local fractional derivatives are studied, using local fractional homotopy perturbation method and local fractional natural transform. The analytical method called LFNHPM reduces the computational size, and is applied directly to differential equations with local fractional operators without any linearization, discretization of variables, transformation, or taking some restrictive assumptions. It gives series solutions which converge rapidly within few iterations. The proposed analytical method is successfully applied to differential equations with local fractional derivatives, and proved to be highly efficient and computational accurate.

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