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Double Symmetric Multivariate Density Function and its Decomposition

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Abstract: For a T-variate density function, the present paper defines double symmetry, quasi double symmetry of order k ($< T$) and marginal double symmetry of order k, and gives the theorem that the density function is T-variate double symmetry if and only if it is quasi double symmetry and marginal double symmetry of order k. The theorem is also illustrated for the multivariate density functions.

Keywords: Decomposition, double symmetry, marginal double symmetry, normal distribution, odds-ratio, quasi double symmetry

1 Introduction

For square contingency tables, it is known that the symmetry model holds if and only if both the quasi symmetry and marginal homogeneity models hold (for example, see Caussinus, 1965; Tomizawa and Tahata, 2007). For multi-way contingency tables, Bhapkar and Darroch (1990) defined the complete symmetry, quasi symmetry and marginal symmetry models, and showed that the complete symmetry model holds if and only if both the quasi symmetry and marginal symmetry models hold. Tomizawa et al. (1996) gave a similar decomposition for the bivariate density function instead of cell probabilities (see also Tong, 1990, p. 104). Iki et al. (2012) extended the decomposition into multivariate case.

On the other hand, for multi-way contingency tables, Wall and Lienert (1976) defined the point symmetry model for the cell probabilities. Tomizawa (1985a) proposed the point symmetry, quasi point symmetry and marginal point symmetry models for rectangular contingency tables, and gave the theorem that the point symmetry model holds if and only if both the quasi point symmetry and marginal point symmetry models hold. Also, for multi-way contingency tables, Tahata and Tomizawa (2008) defined the quasi point symmetry and marginal point symmetry models, and showed that the point symmetry model holds if and only if both the quasi point symmetry and marginal point symmetry models hold. Tomizawa and Konuma (1998) gave a similar decomposition for the bivariate density function. Iki and Tomizawa (2014) extended the decomposition into multivariate case.

Moreover, for square contingency tables, Tomizawa (1985b) proposed the double symmetry, quasi double symmetry and marginal double symmetry models, and showed that the double symmetry model holds if and only if both the quasi double symmetry and marginal double symmetry models hold. For multi-way contingency tables, Yamamoto et al. (2012) defined the double symmetry, quasi double symmetry and marginal double symmetry models, and showed that the double symmetry model holds if and only if both the quasi double symmetry and marginal double symmetry models hold.

For symmetry of a multivariate distribution, there are various kinds of symmetry; see Kotz et al. (2006, pp.5338-5341), Fang et al. (1990, Ch. 2), Fang and Zhang (1990, Ch. 5) and Muirhead (2005, pp. 32-34). Now, we are interested in considering the double symmetry for multivariate density function. Moreover, we consider the structures of double symmetry having weaker restriction, and the decomposition of the double symmetry. The decomposition may be useful for knowing the reason, i.e., when the density function is not double symmetry, what structure of double symmetry having weaker restriction is lacking.

In the present paper, we define the double symmetry, quasi double symmetry and marginal double symmetry for the multivariate density function, and decompose the double symmetry into quasi double symmetry and marginal double symmetry. Section 2 defines the three kinds of double symmetry for bivariate density function. Section 3 extends the three

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kinds of double symmetry to multivariate case. Section 4 shows decomposition of double symmetry for the multivariate density function. Section 5 illustrates our decomposition for some distributions.

2 Double symmetry for bivariate density function

Let X_1 and X_2 be two continuous random variables with a density function $f(x_1,x_2)$, where

$$f(x_1,x_2) > 0 \quad \text{for } (x_1,x_2) \in D^2,$$
$$f(x_1,x_2) = 0 \quad \text{for } (x_1,x_2) \notin D^2,$$

with

$$D^2 = \{(x_1,x_2) \mid a < x_i < b;\ i = 1,2\},$$

and where $a = -\infty$ and $b = +\infty$, or a and b are finite. Let (c_1,c_2) denote a given point in domain D^2, where $c_i = (a+b)/2$ if a and b are finite. Let $x_i^* = 2c_i - x_i$ when $X_i = x_i$ for $i = 1,2$. For example, when $X_2 = 10$ with $c_2 = 3$, then $10^* = 2\times 3 - 10 = -4$. Note that (i) x_i^* is the symmetrical value of x_i with respect to c_i, (ii) $(x_i^*)^* = x_i$ and (iii) $c_i^* = c_i$, for $i = 1,2$.

We shall define the double symmetry (denoted by DS^2) of density function with respect to the point (c_1,c_2) by

$$f(x_1,x_2) = f(x_2,x_1) = f(x_1^*,x_2^*) = f(x_2^*,x_1^*),$$

for every $(x_1,x_2) \in D^2$.

Let $f_{X_1}(x_1)$ and $f_{X_2}(x_2)$ be the marginal density functions of X_1 and X_2, respectively. For the density function $f(x_1,x_2)$, we shall define the marginal double symmetry (denoted by MDS^2) by

$$f_{X_1}(x) = f_{X_2}(x) = f_{X_1}(x^*) = f_{X_2}(x^*),$$

for every $x \in (a,b)$.

We can express the density function as

$$f(x_1,x_2) = \mu\alpha(x_1)\beta(x_2)\gamma(x_1,x_2), \tag{1}$$

where $(x_1,x_2) \in D^2$, and

$$\alpha(c_1) = \beta(c_2) = \gamma(c_1,x_2) = \gamma(x_1,c_2) = 1.$$

The terms α and β correspond to main effects of the variable X_1 and X_2, respectively, γ to interaction effects of X_1 and X_2. We see

$$\mu = f(c_1,c_2), \quad \alpha(x_1) = \frac{f(x_1,c_2)}{f(c_1,c_2)}, \quad \beta(x_2) = \frac{f(c_1,x_2)}{f(c_1,c_2)}, \quad \gamma(x_1,x_2) = \frac{f(x_1,x_2)f(c_1,c_2)}{f(x_1,c_2)f(c_1,x_2)}.$$

The terms $\alpha(x_1)$ and $\beta(x_2)$ indicates the odds of density function with respect to X_1-values with $X_2 = c_2$ and X_2-values with $X_1 = c_1$, respectively. Note that

$$\begin{aligned}\gamma(x_1,x_2) &= \left(\frac{f(x_1,x_2)}{f(x_1,c_2)}\right) \Big/ \left(\frac{f(c_1,x_2)}{f(c_1,c_2)}\right) \\ &= \left(\frac{f(x_1,x_2)}{f(c_1,x_2)}\right) \Big/ \left(\frac{f(x_1,c_2)}{f(c_1,c_2)}\right).\end{aligned}$$

Thus, $\gamma(x_1,x_2)$ indicates the odds-ratio of density function with respect to (X_1,X_2)-values.

The density function is DS^2 if and only if it is expressed as the form (1) with

$$\begin{cases}\alpha(x_1) = \beta(x_1) = \alpha(x_1^*) = \beta(x_1^*), \\ \gamma(x_1,x_2) = \gamma(x_2,x_1) = \gamma(x_1^*,x_2^*) = \gamma(x_2^*,x_1^*).\end{cases}$$

We shall define the quasi double symmetry (denoted by QDS^2) by (1) with

$$\gamma(x_1,x_2) = \gamma(x_2,x_1) = \gamma(x_1^*,x_2^*) = \gamma(x_2^*,x_1^*).$$

3 Double symmetry for multivariate density function

Let $X_1,\dots,X_T$ be T continuous random variables with a density function $f(x_1,\dots,x_T)$, where $f(x_1,\dots,x_T)>0$ for $(x_1,\dots,x_T)\in D^T$ and D^T is defined in a similar way to D^2. Let $(c_1,\dots,c_T)$ denote a given point in D^T, where $c_i=(a+b)/2$ if a and b are finite. Let $x_i^*=2c_i-x_i$ when $X_i=x_i$ for $i=1,\dots,T$. Also, let $(\pi_1,\dots,\pi_T)$ be each permutation of $(1,\dots,T)$. For the density function $f(x_1,\dots,x_T)$, we shall define the double symmetry (denoted by DS^T) with respect to the point $(c_1,\dots,c_T)$ by

$$\begin{aligned} f(x_1,\dots,x_T) &= f(x_{\pi_1},\dots,x_{\pi_T}) \\ &= f(x_1^*,\dots,x_T^*), \end{aligned}$$

for every $(x_1,\dots,x_T)\in D^T$. Also, for $k=1,\dots,T-1$, we shall define the marginal double symmetry of order k (denoted by MDS_k^T) by

$$\begin{aligned} f_{X_{i_1}\dots X_{i_k}}(x_{i_1},\dots,x_{i_k}) &= f_{X_{i_1}\dots X_{i_k}}(x_{\pi_{i_1}},\dots,x_{\pi_{i_k}}) \\ &= f_{X_{j_1}\dots X_{j_k}}(x_{i_1},\dots,x_{i_k}) \\ &= f_{X_{i_1}\dots X_{i_k}}(x_{i_1}^*,\dots,x_{i_k}^*), \end{aligned}$$

for $1\le i_1<\dots<i_k\le T$ and $1\le j_1<\dots<j_k\le T$, where $f_{X_{i_1}\dots X_{i_k}}$ is the marginal density function of $(X_{i_1},\dots,X_{i_k})$. We note that MDS_{k+1}^T implies MDS_k^T $(k=1,\dots,T-2)$.

We can express the density function as

$$\begin{aligned} f(x_1,\dots,x_T) &= \mu\left[\prod_{i_1=1}^{T}\alpha_{i_1}(x_{i_1})\right]\left[\prod\prod_{1\le i_1<i_2\le T}\alpha_{i_1i_2}(x_{i_1},x_{i_2})\right]\times\cdots \\ &\times\left[\prod\cdots\prod_{1\le i_1<\dots<i_{T-1}\le T}\alpha_{i_1\dots i_{T-1}}(x_{i_1},\dots,x_{i_{T-1}})\right]\alpha_{1\dots T}(x_1,\dots,x_T), \end{aligned} \tag{2}$$

where $(x_1,\dots,x_T)\in D^T$, and

$$\{\alpha_i(c_i)=\alpha_{i_1i_2}(c_{i_1},x_{i_2})=\dots=\alpha_{1\dots T}(x_1,\dots,x_{T-1},c_T)=1\}.$$

Then, the density function $f(x_1,\dots,x_T)$ being DS^T is also expressed as (2) with

$$\begin{aligned} \alpha_{i_1\dots i_m}(x_{i_1},\dots,x_{i_m}) &= \alpha_{i_1\dots i_m}(x_{\pi_{i_1}},\dots,x_{\pi_{i_m}}) \\ &= \alpha_{j_1\dots j_m}(x_{i_1},\dots,x_{i_m}) \\ &= \alpha_{i_1\dots i_m}(x_{i_1}^*,\dots,x_{i_m}^*), \end{aligned}$$

for $m=1,\dots,T$, $1\le i_1<\dots<i_m\le T$ and $1\le j_1<\dots<j_m\le T$.

For $k=1,\dots,T-1$, we shall define the quasi double symmetry of order k (denoted by QDS_k^T) by (2) with

$$\begin{aligned} \alpha_{i_1\dots i_m}(x_{i_1},\dots,x_{i_m}) &= \alpha_{i_1\dots i_m}(x_{\pi_{i_1}},\dots,x_{\pi_{i_m}}) \\ &= \alpha_{j_1\dots j_m}(x_{i_1},\dots,x_{i_m}) \\ &= \alpha_{i_1\dots i_m}(x_{i_1}^*,\dots,x_{i_m}^*), \end{aligned}$$

for $m=k+1,\dots,T$, $1\le i_1<\dots<i_m\le T$ and $1\le j_1<\dots<j_m\le T$. We note that QDS_k^T implies QDS_{k+1}^T $(k=1,\dots,T-2)$.

4 Decomposition of multivariate density function

For the multivariate density function, permutation symmetry (denoted S^T) is defined by Tong (1990, p. 104). For a fixed k $(k=1,\dots,T-1)$, Iki et al. (2012) defined quasi symmetry of order k (denoted by QS_k^T) and marginal symmetry of order k (denoted by MS_k^T). Also, Iki and Tomizawa (2014) defined the point symmetry (denoted by PS^T), quasi point symmetry of order k (denoted by QPS_k^T), and marginal point symmetry of order k (denoted by MPS_k^T). We see that (i) DS^T indicates the structure of both S^T and PS^T, (ii) QDS_k^T indicates the structure of both QS_k^T and QPS_k^T, and (iii) MDS_k^T indicates the structure of both MS_k^T and MPS_k^T. Then, we obtain obviously following lemmas.

Lemma 3.1. The multivariate density function $f(x_1,\dots,x_T)$ is DS^T if and only if it is both S^T and PS^T.

Lemma 3.2. For a fixed k $(k = 1,\dots,T-1)$, the multivariate density function $f(x_1,\dots,x_T)$ is QDS_k^T if and only if it is both QS_k^T and QPS_k^T.

Lemma 3.3. For a fixed k $(k = 1,\dots,T-1)$, the multivariate density function $f(x_1,\dots,x_T)$ is MDS_k^T if and only if it is both MS_k^T and MPS_k^T.

Moreover, Iki et al. (2012) and Iki and Tomizawa (2014) give the Lemmas 3.4 and 3.5, respectively, as follows.

Lemma 3.4. For a fixed k $(k = 1,\dots,T-1)$, the multivariate density function $f(x_1,\dots,x_T)$ is S^T if and only if it is both QS_k^T and MS_k^T.

Lemma 3.5. For a fixed k $(k = 1,\dots,T-1)$, the multivariate density function is PS^T if and only if it is both QPS_k^T and MPS_k^T.

From Lemmas 3.1 to 3.5, we obtain the following theorem.

Theorem 3.1. For a fixed k $(k = 1,\dots,T-1)$, the multivariate density function $f(x_1,\dots,x_T)$ is DS^T if and only if it is both QDS_k^T and MDS_k^T.

5 Double symmetry of some distributions

Example 1. Consider a T-dimensional random vector $X = (X_1,\dots,X_T)'$ having a normal distribution with mean vector $\mu = (\mu_1,\dots,\mu_T)'$ and covariance matrix Σ. The density function is

$$f(x_1,\dots,x_T) = \frac{1}{(2\pi)^{\frac{T}{2}}|\Sigma|^{\frac{1}{2}}}\exp\Big\{-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)\Big\}. \tag{3}$$

Denote Σ^{-1} by $A = (a_{ij})$ with $a_{ij} = a_{ji}$. Then the density function can be expressed as

$$f(x_1,\dots,x_T) = C\exp\Big\{-\frac{1}{2}H\Big\},$$

where C is positive constant and

$$H = \sum_{s=1}^{T} a_{ss}x_s^2 + \sum_{s\neq t} a_{st}x_sx_t - 2\sum_{s=1}^{T}\sum_{t=1}^{T} a_{st}\mu_s x_t.$$

For an arbitrary given point $(c_1,\dots,c_T)$, we set $\widetilde{x}_i = x_i - c_i$ and $\widetilde{\mu}_i = \mu_i - c_i$ $(i = 1,\dots,T)$. Then noting that $x_i - \mu_i = \widetilde{x}_i - \widetilde{\mu}_i$ $(i = 1,\dots,T)$, we see

$$f(x_1,\dots,x_T) = \widetilde{C}\exp\Big\{-\frac{1}{2}\widetilde{H}\Big\},$$

where $\widetilde{C}$ is positive constant and

$$\widetilde{H} = \sum_{s=1}^{T} a_{ss}\widetilde{x}_s^2 + \sum_{s\neq t} a_{st}\widetilde{x}_s\widetilde{x}_t - 2\sum_{s=1}^{T}\sum_{t=1}^{T} a_{st}\widetilde{\mu}_s\widetilde{x}_t.$$

Thus

$$\begin{aligned}
\alpha_i(x_i) &= \frac{f(c_1,\dots,c_{i-1},x_i,c_{i+1},\dots,c_T)}{f(c_1,\dots,c_T)}\\
&= \exp\Big\{-\frac{1}{2}\big(a_{ii}\widetilde{x}_i^2 - 2\sum_{s=1}^{T} a_{si}\widetilde{\mu}_s\widetilde{x}_i\big)\Big\} \quad (i = 1,\dots,T),\\
\alpha_{ij}(x_i,x_j) &= \frac{f(c_1,\dots,c_{i-1},x_i,c_{i+1},\dots,c_{j-1},x_j,c_{j+1},\dots,c_T)f(c_1,\dots,c_T)}{f(c_1,\dots,c_{i-1},x_i,c_{i+1},\dots,c_T)f(c_1,\dots,c_{j-1},x_j,c_{j+1},\dots,c_T)}\\
&= \exp\Big(-\frac{1}{2}a_{ij}\widetilde{x}_i\widetilde{x}_j\Big) \quad (i < j),
\end{aligned}$$

and for $m = 3,\dots,T$,

$$\alpha_{i_1\dots i_m}(x_{i_1},\dots,x_{i_m}) = 1 \quad (1 \le i_1 < \cdots < i_m \le T).$$

First, we shall consider about QDS_k^T $(k=1,\dots,T-1)$ of density function (3). Since $\alpha_{i_1\dots i_m}(x_{i_1},\dots,x_{i_m})=1$ for $m=3,\dots,T$ and $1\le i_1<\cdots<i_m\le T$, the normal density function (3) is QDS_k^T $(k=2,\dots,T-1)$. Noting that $x_i^*=2c_i-x_i$ $(i=1,\dots,T)$, we see

$$\begin{aligned}\alpha_{ij}(x_i^*,x_j^*) &= \exp\Big\{-\frac{1}{2}a_{ij}(x_i^*-c_i)(x_j^*-c_j)\Big\}\\ &= \exp\Big\{-\frac{1}{2}a_{ij}(x_i-c_i)(x_j-c_j)\Big\}\\ &= \alpha_{ij}(x_i,x_j) \quad (i<j).\end{aligned}$$

Thus, the density function $f(x_1,\dots,x_T)$ is QDS_1^T, namely

$$\alpha_{ij}(x_i,x_j)=\alpha_{ij}(x_j,x_i)=\alpha_{kl}(x_i,x_j)=\alpha_{ij}(x_i^*,x_j^*),$$

for $1\le i<j\le T$ and $1\le k<l\le T$, if and only if $\{a_{ij}\ (=a_{ji})\}$ are constant (e.g., equals w) for all $i<j$; namely, Σ^{-1} has the form

$$\Sigma^{-1}=D+wee', \tag{4}$$

where D is the $T\times T$ diagonal matrix, e is the $T\times 1$ vector of 1 elements, and w is scalar. Although the detail is omitted, then Σ has the form

$$\Sigma=D^{-1}+dD^{-1}ee'D^{-1},$$

where d is scalar. Therefore, the density function is QDS_1^T if and only if Σ has the form

$$\Sigma=\begin{pmatrix} b_1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & b_T\end{pmatrix}+d\begin{pmatrix} b_1\\ \vdots\\ b_T\end{pmatrix}\begin{pmatrix} b_1,\dots,b_T\end{pmatrix}. \tag{5}$$

Let $V(X_i)=\sigma_i^2$ $(i=1,\dots,T)$ and let ρ_{ij} be the correlation coefficient of X_i and X_j $(i\ne j)$ with $|\rho_{ij}|<1$. Assume that (i) $\sigma_1^2=\cdots=\sigma_T^2\ (=\sigma^2)$ and $\rho_{ij}=\rho$ $(i<j)$. Then

$$\Sigma=\sigma^2(1-\rho)\Big(E+\frac{\rho}{1-\rho}ee'\Big),$$

where E is the $T\times T$ identity matrix. This satisfies the form (5) of Σ. Therefore the density function (3) with condition (i) is QDS_1^T.

Also, assume that (ii) $\sigma_1^2=\cdots=\sigma_T^2\ (=\sigma^2)$. From (5), then QDS_1^T holds if and only if

$$\begin{cases}\sigma^2=b_i+db_i^2 & (i=1,\dots,T),\\ \sigma^2\rho_{ij}=db_ib_j & (i<j),\end{cases}$$

hold, namely, $b_1=\cdots=b_T$ since $|\rho_{ij}|<1$. Therefore the density function with condition (ii) is QDS_1^T if and only if $\rho_{ij}=\rho$ for all $i<j$ hold.

Assume that (iii) $\rho_{ij}=\rho\ (\ne 0)$ for all $i<j$. Then we see

$$\Sigma=\begin{pmatrix}\sigma_1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \sigma_T\end{pmatrix}\big((1-\rho)E+\rho ee'\big)\begin{pmatrix}\sigma_1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \sigma_T\end{pmatrix}.$$

Although the detail is omitted, we can see

$$\Sigma^{-1}=\frac{1}{1-\rho}\left(\begin{pmatrix}\sigma_1^{-2} & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \sigma_T^{-2}\end{pmatrix}+\frac{1}{m}\begin{pmatrix}\sigma_1^{-1}\\ \vdots\\ \sigma_T^{-1}\end{pmatrix}\begin{pmatrix}\sigma_1^{-1},\dots,\sigma_T^{-1}\end{pmatrix}\right),$$

where $m=-(1-\rho)/\rho-T$. Therefore from (4), the density function (3) with condition (iii) is QDS_1^T if and only if $\sigma_1^2=\cdots=\sigma_T^2$ holds.

Assume that (iv) $\rho_{ij}=0$ for all $i<j$. Then the density function (3) is QDS_1^T because $\alpha_{ij}(x_i,x_j)=1$ with $a_{ij}=0$ for $i<j$.

Next, we shall consider about MDS_k^T $(k = 1, \dots, T-1)$ of density function (3). Obviously, the density function (3) is MDS_1^T, namely,

$$f_{X_i}(x_i) = f_{X_j}(x_i) = f_{X_i}(x_i^*),$$

for all $i < j$, if and only if $\mu_1 = \cdots = \mu_T = c_1 = \cdots = c_T$, and $\sigma_1^2 = \cdots = \sigma_T^2$ hold. The density function (3) is MDS_2^T, namely,

$$f_{X_iX_j}(x_i, x_j) = f_{X_iX_j}(x_j, x_i) = f_{X_kX_l}(x_i, x_j) = f_{X_iX_j}(x_i^*, x_j^*),$$

for $1 \le i < j \le T$ and $1 \le k < l \le T$, if and only if $\mu_1 = \cdots = \mu_T = c_1 = \cdots = c_T$, $\sigma_1^2 = \cdots = \sigma_T^2$ and $\rho_{ij} = \rho$ for all $i < j$ hold. Similarly, for each k $(k = 3, \dots, T-1)$, it is MDS_k^T if and only if $\mu_1 = \cdots = \mu_T = c_1 = \cdots = c_T$, $\sigma_1^2 = \cdots = \sigma_T^2$, and $\rho_{ij} = \rho$ for all $i < j$ hold.

Thus, from Theorem 3.1 we can see that the density function (3) with $\mu_1 = \cdots = \mu_T = c_1 = \cdots = c_T$ and $\sigma_1^2 = \cdots = \sigma_T^2$ is DS^T if and only if it is QDS_1^T. Also, from Theorem 3.1, the density function (3) is DS^T if and only if $\mu_1 = \cdots = \mu_T = c_1 = \cdots = c_T$, $\sigma_1^2 = \cdots = \sigma_T^2$ and $\rho_{ij} = \rho$ for all $i < j$ hold.

Example 2. We consider Sarmanov's (1966) bivariate distributions with beta marginals. Let X_1 and X_2 be bivariate random variables with a density function $f(x_1, x_2)$, defined by

$$f(x_1, x_2) = \begin{cases} f_1(x_1) f_2(x_2) \{1 + \omega(x_1 - \mu_1)(x_2 - \mu_2)\} & (0 < x_i < 1; i = 1, 2), \\ 0 & \text{otherwise}, \end{cases} \tag{6}$$

where

$$f_i(x_i) = \frac{1}{B(a_i, b_i)} x_i^{a_i - 1} (1 - x_i)^{b_i - 1} \quad (i = 1, 2),$$
$$\mu_i = \frac{a_i}{a_i + b_i} \quad (i = 1, 2),$$
$$1 + \omega(x_1 - \mu_1)(x_2 - \mu_2) > 0,$$

and where $B(a_i, b_j)$ is beta function and ω is a real value. Also, $f_1(x_1)$ and $f_2(x_2)$ are the marginal distributions of X_1 and X_2, respectively. We shall consider about the double symmetry of density function (6).

Using the form (1), the density function (6) is expressed as

$$f(x_1, x_2) = \mu \alpha(x_1) \beta(x_2) \gamma(x_1, x_2),$$

where

$$\mu = f_1(c_1) f_2(c_2) \{1 + \omega(c_1 - \mu_1)(c_2 - \mu_2)\},$$
$$\alpha(x_1) = \frac{f_1(x_1)\{1 + \omega(x_1 - \mu_1)(c_2 - \mu_2)\}}{f_1(c_1)\{1 + \omega(c_1 - \mu_1)(c_2 - \mu_2)\}},$$
$$\beta(x_2) = \frac{f_2(x_2)\{1 + \omega(c_1 - \mu_1)(x_2 - \mu_2)\}}{f_2(c_2)\{1 + \omega(c_1 - \mu_1)(c_2 - \mu_2)\}},$$
$$\gamma(x_1, x_2) = \frac{\{1 + \omega(x_1 - \mu_1)(x_2 - \mu_2)\}\{1 + \omega(c_1 - \mu_1)(c_2 - \mu_2)\}}{\{1 + \omega(x_1 - \mu_1)(c_2 - \mu_2)\}\{1 + \omega(c_1 - \mu_1)(x_2 - \mu_2)\}},$$

Since the support of $f_i(x_i)$ is $(0, 1)$ $(i = 1, 2)$, we set $c_i = 1/2$ $(i = 1, 2)$. Then, the density function (6) is QDS^2 if and only if both $a_1 = b_1$ and $a_2 = b_2$ hold. The density function (6) is MDS^2 if and only if $a_1 = a_2 = b_1 = b_2$ hold. Therefore, from Theorem 3.1, we can see that the density function (6) is DS^2 if and only if $a_1 = a_2 = b_1 = b_2$ hold.

6 Concluding remarks

When a density function $f(x_1, \dots, x_T)$ is not double symmetry, Theorem 3.1 may be useful for knowing the reason, i.e., for a fixed k, which structure of quasi double symmetry of order k and marginal double symmetry of order k is lacking. Indeed, for a random vector having normal distribution, when its density function is not DS^T, it is caused by the lack of the structure of MDS_k^T $(k = 2, \dots, T-1)$ because the normal density function is always QDS_k^T $(k = 2, \dots, T-1)$. Namely, the reason why the normal density function is not double symmetry, is caused by the lack of double symmetry for second (or more) order marginal distributions (see Example 1).

7 Discussion

In Section 2, many readers may be interested in considering the domain D^2 as such

$$D^2 = \{(x_1, x_2) | a < x_i < b; i = 1, 2\},$$

where a is finite and $b = +\infty$. However, it seems difficult to consider such domain D^2. Because for such D^2, we cannot denote a adequate point (c_1, c_2). For example, when $(X_1, X_2) = (10, 10)$ with $(c_1, c_2) = (3, 3)$, $(10^*, 10^*) = (-4, -4)$ is not in D^2. Therefore, we cannot define the three kinds of the double symmetry.

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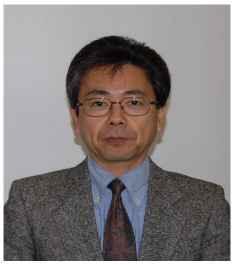

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