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# On Weighted Ailamujia Distribution and Its Applications to Lifetime Data

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**Abstract:** This paper deals with the introduction of generalized version of Ailamujia distribution called weighted Ailamujia distribution. In this paper, the different structural properties of the newly developed model have been studied and derived. The parameters of the proposed distribution have been estimated through the maximum likelihood technique and method of moments. Further, a likelihood ratio test of the weighted model has been obtained. In addition to this, the model under study has also been applied to the real life data sets for illustration.

**Keywords:** Weighted distribution, Ailamujia distribution, Reliability analysis, Maximum likelihood estimation, Order statistics, Likelihood ratio test, Real life data sets.

## 1 Introduction

Ailamujia distribution is a newly proposed lifetime model formulated by Lv et al. (2002) for several engineering applications. They studied its various characteristics including mean, variance, median and maximum likelihood estimate. This distribution was also studied by Pan et al. (2009) for interval estimation and hypothesis testing based on sample of small size. The Bayesian estimation of Ailamujia Distribution was obtained by Long (2015) under Type II censoring using the three different priors based on missing data. The minimax estimation of the parameter of Ailamujia model was evaluated by Li (2016) under a non informative prior using the three loss functions. The Ailamujia distribution is a versatile distribution to model the repair time and guarantee the distribution delay time. If X is the lifetime of a product and follows Ailamujia distribution, its probability density function and cumulative density function are given respectively as follows:

$$f(x,\theta)=4x\theta^2e^{-2\theta x} \;\; ;x\geq 0,\theta\geq 0 \tag{1.1}$$

$$F(x,\theta)=1-(1+2\theta x)e^{-2\theta x} \quad ; \quad x\geq 0,\theta\geq 0. \tag{1.2}$$

The main objective of this paper is to study a more flexible extension of Ailamujia distribution by introducing a weight function and check its superiority over its different sub models. Further, the different structural properties of the model have been discussed. The paper is organized as follows: In section 2, the weighted Ailamujia distribution is introduced and the pdf, cdf of the new model are presented. Section 3 deals with the special cases derived from the proposed model. Section 4 and 5 are devoted to discuss the reliability analysis and various statistical properties of the proposed model. The Shannon's entropy and order statistics of the new weighted distribution are discussed in section 6 and 7 respectively. Further, in the sections 8 and 9, method of estimation and likelihood ratio test are provided respectively. Section 10 considers three real data sets to illustrate the importance of the proposed model along with the concluding remarks.

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## 2 Weighted Ailamujia Distribution

The concept of weighted distributions was given by Fisher (1934) to model the ascertainment bias. This concept was later on developed by Rao (1965) in a unified manner while modeling the statistical data when the standard distributions were not appropriate to record these observations with equal probabilities. As a result, weighted models were formulated in such situations to record the observations according to some weighted function. The weighted distribution reduces to length biased distribution when the weight function considers only the length of the units. The concept of length biased sampling was first introduced by Cox (1962) and Zelen (1974). More generally, when the sampling mechanism selects units with probability proportional to some measure of the unit size, resulting distribution is called size-biased. There are various good sources which provide the detailed description of weighted distributions. Different authors have reviewed and studied the various weighted probability models and illustrated their applications in different fields. Weighted distributions are applied in various research areas related to reliability, biomedicine, ecology and branching processes. Reshi et.al (2014) discussed the characterizations and estimations of Size Biased Generalized Rayleigh Distribution while as Afaq et. al (2016) introduced the Length-Biased Weighted Lomax Distribution and studied its statistical properties and applications. Fatima and Ahmad (2017) proposed the weighted Inverse Rayleigh distribution and derived its structural properties. Sofi and Ahmad (2017) defined and estimated the parameters of the Weighted Erlang Distribution using R software.

Assume X is a non negative random variable with probability density function (pdf) $f(x)$ Let $w(x)$ be the weight function which is a non negative function, then the probability density function of the weighted random variable $X_w$ is given by:

$$f_w(x) = \frac{w(x) f(x)}{E(w(x))}, \quad x > 0$$

where $w(x)$ be a non-negative weight function and $E(w(x)) = \int w(x) f(x) dx < \infty$.

In this paper, we have considered the weight function as $w(x) = x^c$ to obtain the weighted Ailamujia model. The probability density of weighted Ailamujia distribution is given as:

$$f_w(x,\theta) = \frac{x^c f(x,\theta)}{E[x^c]}$$

$$f_w(x,\theta) = \frac{(2\theta)^{c+2} x^{c+1} e^{-2\theta x}}{\Gamma(c+2)} \qquad ; x > 0, \theta, c > 0. \tag{2.1}$$

where $E(x^c) = 4\theta^2 \int_0^\infty x^{c+1} e^{-2\theta x} dx = 4\theta^2 \frac{\Gamma(c+2)}{(2\theta)^{c+2}}$.

The corresponding c.d. f of weighted Ailamujia distribution is obtained as:

$$F_w(x,\theta) = \int_0^x \frac{(2\theta)^{c+2} x^{c+1} e^{-2\theta x}}{\Gamma(c+2)} dx$$

$$F_w(x,\theta) = \frac{\gamma(c+2, 2\theta x)}{\Gamma(c+2)}, \qquad x > 0, \theta, c > 0. \tag{2.2}$$

where $\theta$ and $c$ are positive parameters and $\gamma(s,x)=\int_0^x t^{s-1}e^{-t}dt$ is a lower incomplete gamma function.

The graphs of probability density function and cumulative distribution function are plotted for different values of parameters $\theta$ and c given in Figure 1 and 2 respectively

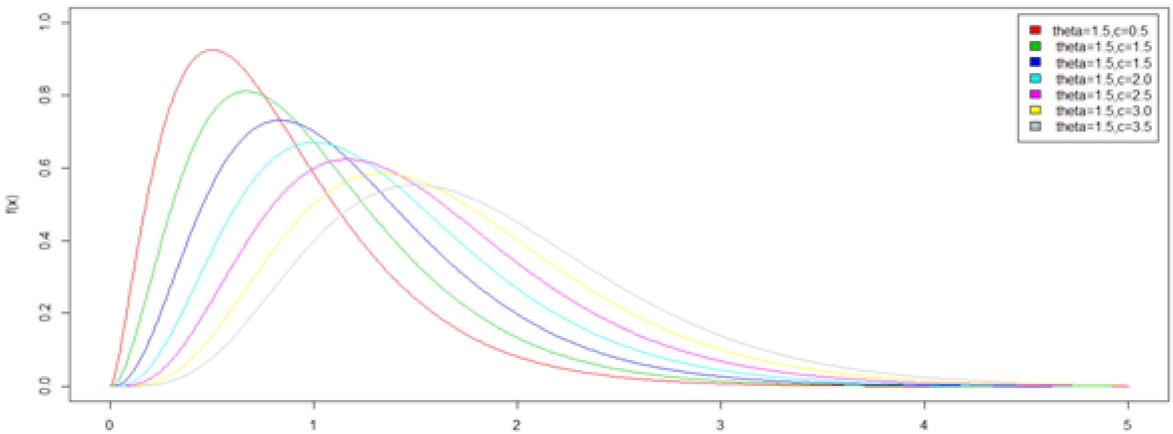


**Figure 1.** Graph of density function

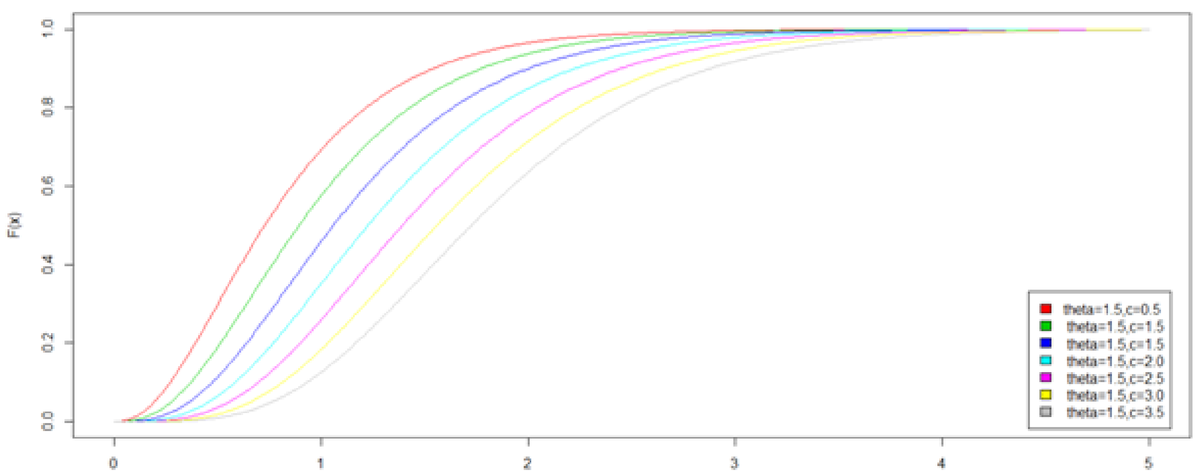


**Figure 2.** Graph of distribution function

Figure 1 gives the description of some of the possible shapes of weighed Ailamujia distribution for different values of the parameters $\theta$ and $c$. Figure 1 illustrates that the density function of weighted Ailamujia distribution is positively skewed, for fixed $\theta$ it becomes more and more flatter as the value of $c$ is increased. Figure 2 shows the graph of distribution function which is an increasing function.

## 3 Special Cases

Case 1: If we put $c=0$, then weighted Ailamujia distribution (2.1) reduces to Ailamujia distribution with probability density function as:

$$f(x)=4x\theta^2\exp(-2\theta x); \quad 0<x<\infty \quad ,\theta>0. \tag{3.1}$$

Case 2: If we put $c = 1$, then the weighted Ailamujia distribution (WAD) (2.1) reduces to Length Biased Ailamujia distribution (LBAD) with probability density function as:

$$f(x) = 4\theta^3 x^2 \exp(-2\theta x) \ ; 0 < x < \infty \ , \ \theta > 0. \tag{3.2}$$

Case 3: If we put $c = 2$, then weighted Ailamujia distribution distribution (2.1) reduces to Area Biased Ailamujia distribution (ABAD) with probability density function as:

$$f(x) = \frac{8\theta^4 x^3 \exp(-2\theta x)}{3}; \quad 0 < x < \infty, \theta > 0. \tag{3.3}$$

## 4 Reliability Analysis

In this section, we have obtained the reliability, hazard rate, reverse hazard rate and Mills ratio of the proposed weighted Ailamujia model.

### *4.1 Reliability Function R(x)*

The reliability function is defined as the probability that a system survives beyond a specified time. It is also referred to as survival or survivor function of the distribution. It can be computed as complement of the cumulative distribution function of the model. The reliability function or the survival function of weighted Ailamujia distribution is calculated as:

$$R_w(x,\theta) = 1 - \frac{1}{\Gamma(c+2)} \gamma(c+2, 2\theta x)$$

$$R_w(x,\theta) = \frac{\Gamma(c+2) - \gamma(c+2, 2\theta x)}{\Gamma(c+2)}$$

$$R_w(x,\theta) = \frac{\Gamma(c+2, 2\theta x)}{\Gamma(c+2)}. \tag{4.1}$$

The graphical representation of the reliability function for the weighted Ailamujia distribution is shown in figure 3

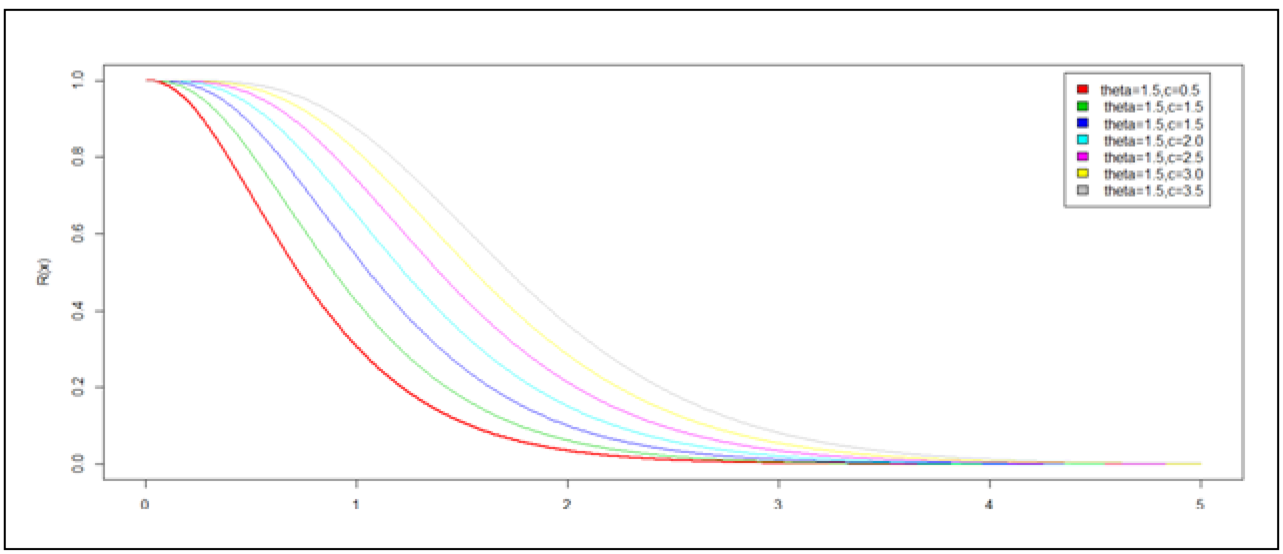


**Figure 3.** Graph of reliability function

### *4.2 Hazard Function*

The hazard function is also known as hazard rate, instantaneous failure rate or force of mortality is given as:

$$H.R = h(x,\theta) = \frac{f_w(x,\theta)}{R_w(x,\theta)} = \frac{(2\theta)^{c+2} x^{c+1} e^{-2\theta x}}{\Gamma(c+2, 2\theta x)}. \tag{4.2}$$

### *4.3 Reverse Hazard Rate and Mills Ratio*

The reverse hazard rate and the Mills ratio of the weighted Ailamujia distribution are respectively given as:

$$R.H.R = h_r(x,\theta) = \frac{f_w(x,\theta)}{F_w(x,\theta)} = \frac{(2\theta)^{c+2} x^{c+1} e^{-2\theta x}}{\gamma(c+2, 2\theta x)}. \tag{4.3}$$

$$\text{Mills ratio} = \frac{1}{h_r(x,\theta)} = \frac{\gamma(c+2, 2\theta x)}{(2\theta)^{c+2} x^{c+1} e^{-2\theta x}}. \tag{4.4}$$

## 5 Statistical properties

In this section, the different structural properties of the proposed weighted Ailamujia model have been evaluated. These include moments, mode, harmonic mean, moment generating function and characteristic function

### *5.1 Moments*

Suppose X is a random variable following weighted Ailamujia distribution with parameter $\theta$, then the rth moment for a given probability distribution is given by

$$\mu_r' = E(X_w^r) = \int_0^\infty x^r f_w(x,\theta) dx$$

$$= \int_0^\infty x^r \frac{(2\theta)^{c+2}}{\Gamma(c+2)} x^{c+1} e^{-2\theta x} dx$$

$$\mu_r' = \frac{(2\theta)^{c+2}}{\Gamma(c+2)} \frac{\Gamma(r+c+2)}{(2\theta)^{r+c+2}}$$

$$\mu_r' = \frac{\Gamma(r+c+2)}{\Gamma(c+2)(2\theta)^r} \tag{5.1}$$

Substituting $r = 1$, 2, 3, 4 we get the first four central moments as follows:

If we put r=1 in eq. (5.1), we get the mean of weighted Ailamujia distribution which is given by:

$$\textbf{Mean} = \mu_1' = \frac{c+2}{2\theta} \tag{5.2}$$

If we put r=2 in eq. (5.1), we have

$$\mu_2' = \frac{(c+3)(c+2)}{4\theta^2}. \tag{5.3}$$

Thus, the variance of weighted Ailamujia distribution is given as:

$$Variance=\mu_2=\frac{(c+2)}{4\theta^2}. \tag{5.4}$$

If we put r=3 in eq. (5.4), we have

$$\mu_3{}'=\frac{(c+4)(c+3)\,(c+2)}{8\theta^3}. \tag{5.5}$$

Then $\mu_3=\mu_3'-3\mu_2'\mu_1'+2\mu_1'^3$.

Using the values of eq. (5.2), (5.3) and (5.5), we get

$$\mu_3=\frac{c+2}{4\theta^3}. \tag{5.6}$$

When we put r=4 in eq. (5.1), we get

$$\mu_4{}'=\frac{(c+5)(c+4)(c+3)(c+2)}{16\theta^4}. \tag{5.7}$$

Then, $\mu_4=\mu_4'-4\mu_3'\mu_1'+6\mu_2'\mu_1'^2-3\mu_1'^4$

Substituting the values of eq. (5.2), (5.3), (5.5) and (5.7) we get

$$\mu_4=\frac{3(c+2)(c+4)}{16\theta^4}. \tag{5.8}$$

Standard deviation= $\sigma=\frac{\sqrt{(c+2)}}{2\theta}$.

Coefficient of variation= $C.V=\frac{\sigma}{\mu}=\frac{1}{\sqrt{c+2}}$.

### *5.2 Skewness and Kurtosis of the Weighted Ailamujia Distribution*

*i) Skewness:* It may be defined as the lack of symmetry of tails (about mean) of frequency distribution curve. Karl Pearson gave the formula for measuring the skewness of the distribution in terms of the moments of the frequency distribution as:

$$\beta_1=\frac{\mu_3{}^2}{\mu_2{}^3}. \tag{5.9}$$

After using eq. (5.4) and eq. (5.6) in eq. (5.9), we have

$$\beta_1=\frac{4}{c+2}.$$

and $\gamma_1=\sqrt{\beta_1}=\frac{2}{\sqrt{c+2}}$. (5.10)

*ii) Kurtosis:* It may be defined as the degree of peakedness of the density curve. The formula for obtaining the kurtosis of a distribution in terms of moments is given as:

$$\beta_2 = \frac{\mu_4}{\mu_2^2}. \tag{5.11}$$

After using eq. (5.8) and eq. (5.4) in eq. (5.11), we have

$$\beta_2 = \frac{3(c+4)}{(c+2)}$$

and $$\gamma_2 = \beta_2 - 3 = \frac{6}{(c+2)}. \tag{5.12}$$

### *5.3 Mode of Weighted Ailamujia Distribution*

In order to calculate the mode of the weighted Ailamujia model we take the logarithm of the probability density function of the weighted model:

$$\log f_w(x,\theta) = (c+2)\log 2 + (c+2)\log\theta - \log\Gamma(c+2) + (c+1)\log x - 2\theta x.$$

Differentiating the above equation and equating it zero, we obtain the mode as follows:

$$\frac{d}{dx}\log f_w(x,\theta) = 0$$

$$\frac{c+1}{x} = 2\theta$$

$$\Rightarrow \qquad x = \frac{c+1}{2\theta}. \tag{5.13}$$

### *5.4 Harmonic Mean*

The harmonic mean for the proposed model is computed as:

$$H.M = E\left[\frac{1}{X}\right] = \int_0^\infty \frac{1}{x} f_w(x,\theta)dx$$

$$= \int_0^\infty \frac{1}{x} \frac{(2\theta)^{c+2}}{\Gamma(c+2)} x^{c+1} e^{-2\theta x} dx$$

$$H.M = \frac{2\theta}{c+1}. \tag{5.14}$$

### *5.5 Moment Generating Function and Characteristic Function*

This section deals with the derivation of moment generating function and characteristic function of the weighted Ailamujia distribution. By the definition of moment generating function, we have:

$$M_X(t) = E\left(e^{tx}\right) = \int_0^\infty e^{tx} f_w(x,\theta)dx.$$

Using Taylor series

$$M_X(t)=\int_0^{\infty}\left(1+tx+\frac{(tx)^2}{2!}+\cdots\right)f_w(x,\theta)dx$$

$$\Rightarrow \qquad M_X(t)=\sum_{r=0}^{\infty}\frac{t^k}{k!}\int_0^{\infty}x^k f_w(x,\theta)dx$$

$$\Rightarrow \qquad M_X(t)=\sum_{k=0}^{\infty}\frac{t^k}{k!}E(X^k)$$

$$\Rightarrow \qquad M_X(t)=\sum_{k=0}^{\infty}\frac{\left(\frac{t}{2\theta}\right)^k}{k!}\frac{\Gamma(k+c+2)}{\Gamma(c+2)}. \tag{5.15}$$

Similarly, the characteristic function of weighted Ailamujia distribution is computed as:

$$\phi_X(t)=E\left(e^{itx}\right)=\int_0^{\infty}e^{itx}f_w(x,\theta)dx.$$

Using Taylor series

$$\phi_X(t)=\int_0^{\infty}\left(1+itx+\frac{(itx)^2}{2!}+\cdots\right)f_w(x,\theta)dx$$

$$\Rightarrow \qquad \phi_X(t)=\sum_{k=0}^{\infty}\frac{(it)^k}{k!}\int_0^{\infty}x^k f_w(x,\theta)dx$$

$$\Rightarrow \qquad \phi_X(t)=\sum_{k=0}^{\infty}\frac{(it)^k}{k!}E(X^k)$$

$$\Rightarrow \qquad \phi_X(t)=\sum_{k=0}^{\infty}\frac{\left(\frac{it}{2\theta}\right)^k}{k!}\frac{\Gamma(k+c+2)}{\Gamma(c+2)}. \tag{5.16}$$

## 6 Shannon's Entropy of Weighted Ailamujia Distribution

The Shannon entropy of a random variable X is a measure of the uncertainty and is given by $E[-\log(f(x))]$, where $f(x)$ is the probability function of the random variable X. The Shannon's entropy of the weighted Ailamujia model as follows:

$$H(x,\theta)=-E[\log\ f(x,\theta)]$$

$$=-E\left[\log\left(\frac{(2\theta)^{c+2}}{\Gamma(c+2)}x^{c+1}e^{-2\theta x}\right)\right]$$

$$=-\log\left(\frac{(2\theta)^{c+2}}{\Gamma(c+2)}\right)-E[(c+1)\log x-2\theta x]$$

$$H(x,\theta)=-(c+2)\log 2-(c+2)\log\theta+\log\Gamma(c+2)-(c+1)E[\log x]-2\theta E(x)$$

$$H(x,\theta)=-(c+2)\log 2-(c+2)\log\theta+\log\Gamma(c+2)-(c+1)I_1-2\theta E(x). \tag{6.1}$$

Now, $I_1=E(\log x)=\int_0^{\infty}\log x\, f(x)dx$

$$=\frac{(2\theta)^{c+2}}{\Gamma(c+2)}\int_0^{\infty}\log x\; x^{c+1}e^{-2\theta x}dx. \tag{6.2}$$

Put $2\theta x=t,\ as\ \ x\to 0, t\to 0; as\ \ x\to\infty, t\to\infty, dx=\frac{dt}{2\theta}$.

On solving (6.2) we have,

$$I_1=\frac{1}{\Gamma(c+2)}\left[\int_0^{\infty}e^{-t}t^{(c+2)-1}\log t dt-\log 2\theta\int_0^{\infty}t^{(c+2)-1}e^{-t}dt\right].$$

$$I_1=\frac{\Gamma'(c+2)}{\Gamma(c+2)}-\log 2\theta$$

$$I_1=\psi(c+2)-\log 2\theta\ . \tag{6.3}$$

Using the values of (6.3) and (5.1.2) in (6.1)

$$H(x,\theta)=-(c+2)\log 2-(c+2)\log\theta+\log\Gamma(c+2)-(c+1)[\psi(c+2)-\log 2\theta]-(c+2)$$

$$H(x,\theta)=\log\frac{\Gamma(c+2)}{2\theta}-(c+1)\psi(c+2)-(c+2) \tag{6.4}$$

## 7 Order Statistics

Let $X_{(1)}, X_{(2)}, X_{(3)}, \ldots, X_{(n)}$ be the ordered statistics of the random sample $X_1, X_2, X_3, \ldots X_n$ drawn from the continuous distribution with cumulative distribution function $F_X(x)$ and probability density function $f_X(x)$, then the probability density function of rth order statistics $X_{(r)}$ is given by:

$$f_{x(r)}(x,\theta)=\frac{n!}{(r-1)!(n-r)!}f(x)[F(x)]^{r-1}[1-F(x)]^{n-r}. \quad r=1, 2, 3,\ldots,n \tag{7.1}$$

Using the equations (2.1) and (2.2), the probability density function of rth order statistics of weighted Ailamujia distribution is given by:

$$f_{w(r)}(x,\theta)=\frac{n!}{(r-1)!(n-r)!}\frac{(2\theta)^{c+2}}{\Gamma(c+2)}x^{c+1}e^{-2\theta x}\left[\frac{1}{\Gamma(c+2)}\gamma(c+2,2\theta x)\right]^{r-1}\left[1-\frac{1}{\Gamma(c+2)}\gamma(c+2,2\theta x)\right]^{n-r}. \tag{7.2}$$

Then, the pdf of first order $X_{(1)}$ weighted Ailamujia distribution is given by:

$$f_{1w}(x,\theta)=n\frac{(2\theta)^{c+2}}{\Gamma(c+2)}x^{c+1}e^{-2\theta x}\left[1-\frac{1}{\Gamma(c+2)}\gamma(c+2,2\theta x)\right]^{n-1}. \tag{7.3}$$

and the pdf of nth order $X_{(n)}$ weighted Ailamujia model is given as:

$$f_{nw}(x,\theta)=n\frac{(2\theta)^{c+2}}{\Gamma(c+2)}x^{c+1}e^{-2\theta x}\left[\frac{1}{\Gamma(c+2)}\gamma(c+2,2\theta x)\right]^{n-1}. \quad (7.4)$$

## 8 Estimation of Parameters

In this section, we estimate the parameters of the weighted Ailamujia distribution using different methods.

### *8.1 Method of Moments*

In order to obtain sample moments, we replace population moments with sample moments:

$$\frac{\sum_{i=1}^{n}x_i}{n}=\mu_1'$$

$$\Rightarrow \quad \bar{x}=\frac{c+2}{2\theta}$$

$$\Rightarrow \quad \hat{\theta}=\frac{c+2}{2\bar{x}} \quad (8.1)$$

Also, $$\frac{\sum_{i=1}^{n}x_i^2}{n}=\mu_2'$$

$$\Rightarrow \quad \frac{\sum_{i=1}^{n}x_i^2}{n}-\bar{x}^2=\mu_2$$

$$\Rightarrow \quad s^2=\frac{c+2}{4\theta^2}. \quad (8.2)$$

Using the value of $\theta$ given in (8.1) in eq. (8.2) we have

$$\Rightarrow \quad s^2=\frac{\bar{x}^2}{4(c+2)}$$

$$\Rightarrow \quad \hat{c}=\frac{\bar{x}^2}{4s^2}-2. \quad (8.3)$$

### *8.2 Method of Maximum Likelihood Estimation*

This is one of the most useful method for estimating the different parameters of the distribution. Let $X_1, X_2, X_3....X_n$ be the random sample of size n drawn from weighted Ailamujia distribution, then the likelihood function of weighted Ailamujia distribution is given as:

$$L(x\mid\theta)=\frac{(2\theta)^{n(c+2)}}{[\Gamma(c+2)]^n}\prod_{i=1}^{n}x_i^{c+1}e^{-2\theta\sum_{i=1}^{n}x_i} \quad (8.4)$$

The log likelihood function becomes:

$$\log L = n(c+2)\log 2 + n(c+2)\log\theta - n\log\Gamma(c+2) + (c+1)\sum_{i=1}^{n}\log x_i - 2\theta\sum_{i=1}^{n} x_i. \tag{8.5}$$

Differentiating the log-likelihood function with respect to $\theta$ and $c$ .This is done by partially differentiate (8.5) with respect to $\theta$ and $c$ and equating the result to zero, we obtain the following normal equations,

$$\frac{d\log L}{d\theta} = \frac{n(c+2)}{\theta} - 2\sum_{i=1}^{n} x_i = 0. \tag{8.6}$$

$$\frac{d\log L}{dc} = n\log 2 + n\log\theta - n\psi(c+2) + \sum_{i=1}^{n}\log x_i = 0, \tag{8.7}$$

where $\psi(c+2) = \frac{\Gamma'(c+2)}{\Gamma(c+2)}$ is the digamma function.

By solving equations (8.6) and (8.7), the maximum likelihood estimators of the parameters of the weighted Ailamujia distribution are obtained.

## 9 Likelihood Ratio Test

Let $x_1, x_2, \ldots x_n$ be a random sample drawn from Ailamujia distribution or weighted Ailamujia model. We test the hypothesis

$$H_0 : f(x) = f(x,\theta) \;\; v/s \; H_1 : f(x) = f_w(x,\theta)$$

In order to test whether the random sample comes from Ailamujia distribution or weighted Ailamujia distribution, we use the following test statistic

$$\Delta = \frac{L_1}{L_0} = \prod_{i=1}^{n}\frac{f_w(x,\theta)}{f(x,\theta)}$$

$$\Delta = \frac{L_1}{L_0} = \prod_{i=1}^{n}\frac{\dfrac{(2\theta)^{c+2}}{\Gamma(c+2)} x_i^{\,c+1} e^{-2\theta\, x_i}}{4x\theta^2 e^{-2\theta x_i}}$$

$$\Delta = \frac{(2\theta)^{cn}}{\left[4\theta^2\Gamma(c+2)\right]^n}\prod_{i=1}^{n} x_i^{\,c}.$$

We reject the null hypothesis if

$$\Delta = \left(\frac{1}{2\theta\Gamma(c+2)}\right)^{nc}\prod_{i=1}^{n} x_i^{\,c} > k$$

$$\Delta^* = \prod_{i=1}^{n} x_i^{\,c} > k^*, \text{ where } k^* = k\left(2\theta\,\Gamma(c+2)\right)^{nc}. \tag{9.1}$$

For a large sample of size n, 2log $\Delta$ is distributed as chi-square distribution with one degree of freedom. Thus p-value is obtained from the chi-square distribution. Also, we reject the null hypothesis, when probability value is given by

$P\left(\Delta^* > \beta^*\right)$, where $\beta^* = \prod_{i=1}^{n} x_i$ is less than a specified level of significance, where $\prod_{i=1}^{n} x_i$ is the observed value of the statistic $\Delta^*$.

## 10 Applications

To compare the flexibility of the Weighted Ailamujia distribution over the existing sub models, three real data sets are used. The analysis involved in this study has been performed with the help of R software.

*Data set I:* The first data set, strength data, which were originally reported by Badar and Priest (1982) and it represents the strength measured in GPA for single carbon fibers and impregnated 1000-carbon fiber tows. Single fibers were tested under tension at gauge lengths of 10 mm with sample size (n = 63).

This data set consists of observations: 1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.937, 2.977, 2.996, 3.030, 3.125, 3.139, 3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346, 3.377, 3.408, 3.435, 3.493, 3.501, 3.537, 3.554, 3.562, 3.628, 3.852, 3.871, 3.886, 3.971, 4.024, 4.027, 4.225, 4.395, 5.020. This data set is previously used by M. E. Mead et al (2017)

For the comparison of the weighted models and its sub models we have taken into consideration the criteria like AIC (Akaike information criterion), AICC (corrected Akaike information criterion) and BIC (Bayesian information criterion). The distribution which has lesser values of AIC, AICC and BIC is considered as better.

$$\text{AIC=2k-2logL}, \quad \text{AICC=AIC+}\frac{2k(k+1)}{n-k-1} \text{ and BIC=klogn-2logL},$$

where k is the number of parameters in the statistical model, n is the sample size and -2log*L* is the maximized value of the log-likelihood function under the considered model.

The summary of the data is given in Table 1.The estimates of the parameters, log-likelihood and Akaike Information Criteria (AIC) for the data one are generated and the result is as presented in Table 2.

**Table 1:** Data summary for the Data set one

| Minimum | 1st Qu. | Median | Mean | 3rd Qu. | Max. |
|---|---|---|---|---|---|
| 1.901 | 2.554 | 2.996 | 3.059 | 3.422 | 5.020 |

**Table 2:** Performance of distributions (S.E in parenthesis) for the Data set one

| Distribution | Parameter Estimates | | -2Log L | AIC | AICC | BIC |
|---|---|---|---|---|---|---|
| | $c$ | $\theta$ | | | | |
| Weighted Ailamujia Dist. | 13.792676 (2.784627) | 2.581264 (0.462437) | 120.121 | 124.121 | 124.321 | 128.4073 |
| Ailamujia Dist. | _ | 0.326872 (0.0291198) | 220.6962 | 222.6962 | 222.7618 | 224.8393 |
| Length Biased Ailamujia Dist. | _ | 0.490308 (0.03566451) | 195.9081 | 197.9081 | 197.9737 | 200.0512 |
| Area Biased Ailamujia Dist. | _ | 0.653744 (0.0411819) | 179.3941 | 181.3941 | 181.4597 | 183.5372 |

Table 2, displays the Maximum Likelihood estimates of the model parameters. It is obvious that Weighted Ailamujia provides a better fit as compared to other sub-models since it has lowest value of -2logL, Akaike Information Criterion

(AIC), AICC (corrected Akaike information criterion) and Bayesian Information Criterion (BIC). Hence, the Weighted Ailamujia distribution performs better than other generalizations of Ailamujia distribution.

*Data set II:* The second data set represent the tensile strength, measured in GPa, of 69 carbon fibers tested under tension at gauge lengths of 20mm which were originally reported by Bader and Priest (1982), the data is given as: 1.312, 1.314, 1.479, 1.552, 1.700, 1.803, 1.861, 1.865, 1.944, 1.958, 1.966, 1.997, 2.006, 2.021, 2.027, 2.055, 2.063, 2.098, 2.14, 2.179, 2.224, 2.240, 2.253,2.270, 2.272, 2.274, 2.301, 2.301, 2.359, 2.382, 2.382, 2.426, 2.434, 2.435, 2.478, 2.490, 2.511,2.514, 2.535, 2.554, 2.566, 2.57, 2.586, 2.629, 2.633, 2.642, 2.648, 2.684, 2.697, 2.726, 2.770,2.773, 2.800, 2.809, 2.818, 2.821, 2.848, 2.88, 2.954, 3.012, 3.067, 3.084, 3.090, 3.096, 3.128,3.233, 3.433, 3.585, 3.585. This data set is previously studied by Shanker (2016).

The summary of the data is given in Table 3.The estimates of the parameters, log-likelihood and Akaike Information Criteria (AIC) for the data one are generated and the result is as presented in Table 4.

**Table 3:** Data summary for the Data set second

| Minimum | 1st Qu. | Median | Mean | 3rd Qu. | Max. |
|---|---|---|---|---|---|
| 1.312 | 2.098 | 2.478 | 2.451 | 2.773 | 3.585 |

**Table 4:** Performance of distributions (S.E in parenthesis) for the Data set second

| Distribution | Parameter Estimates | | -2Log L | AIC | AICC | BIC |
|---|---|---|---|---|---|---|
| | $c$ | $\theta$ | | | | |
| Weighted Ailamujia Dist. | 9.985018 (2.0127004) | 2.443931 (0.4191273) | 112.7086 | 116.7086 | 116.8904 | 121.1768 |
| Ailamujia Dist. | _ | 0.4079413 (0.03472606) | 211.3986 | 213.3986 | 213.4583 | 215.6327 |
| Length Biased Ailamujia Dist. | _ | 0.6119119 (0.0425307) | 184.5081 | 186.5081 | 186.5678 | 188.7422 |
| Area Biased Ailamujia Dist. | _ | 0.8158827 (0.04911029) | 166.6796 | 168.6796 | 168.7393 | 170.9137 |

Table 4, displays the Maximum Likelihood estimates of the model parameters. It is obvious that Weighted Ailamujia provides a better fit as compared to other sub-models since it has lowest value of -2logL, Akaike Information Criterion (AIC), AICC (corrected Akaike information criterion) and Bayesian Information Criterion (BIC). Hence, the Weighted Ailamujia distribution performs better than other generalizations of Ailamujia distribution.

*Data set III:* The third data set was originally reported by Bader and Priest (1982), on failure stresses (in GPa) of 65 single carbon fibers of lengths 50 mm, respectively. The data set is given as follows 1.339, 1.434, 1.549, 1.574, 1.589, 1.613, 1.746, 1.753, 1.764,1.807, 1.812, 1.84, 1.852, 1.852, 1.862, 1.864, 1.931, 1.952, 1.974, 2.019, 2.051, 2.055, 2.058,2.088, 2.125, 2.162, 2.171, 2.172, 2.18, 2.194, 2.211, 2.27, 2.272, 2.28, 2.299, 2.308, 2.335,2.349, 2.356, 2.386, 2.39, 2.41, 2.43, 2.431, 2.458, 2.471, 2.497, 2.514, 2.558, 2.577, 2.593,2.601, 2.604, 2.62, 2.633, 2.67, 2.682, 2.699, 2.705, 2.735, 2.785, 3.02, 3.042, 3.116, 3.174. This data set has been used by Al-Mutairi et al. (2013). The summary of the data is given in Table 5.The estimates of the parameters, log-likelihood and Akaike Information Criteria (AIC) for the data one are generated and the result is as presented in Table 6.

**Table 5:** Data summary for the Data set third

| Minimum | 1st Qu. | Median | Mean | 3rd Qu. | Max. |
|---|---|---|---|---|---|
| 1.339 | 1.931 | 2.272 | 2.244 | 2.558 | 3.174 |

**Table 6:** Performance of distributions (S.E in parenthesis) for the Data set third

| Distribution | Parameter Estimates | | -2Log L | AIC | AICC | BIC |
|---|---|---|---|---|---|---|
| | $c$ | $\theta$ | | | | |
| Weighted Ailamujia Dist. | 8.288217 (1.776177) | 2.288005 (0.404793) | 95.19269 | 99.19269 | 99.38624 | 103.5415 |
| Ailamujia Dist. | _ | 0.4456238 (0.0390836) | 187.1479 | 189.1479 | 189.2114 | 191.3223 |
| Length Biased Ailamujia Dist. | _ | 0.6684355 (0.04786755) | 161.3061 | 163.3061 | 163.3696 | 165.4805 |
| Area Biased Ailamujia Dist. | _ | 0.8912474 (0.05527275) | 144.0011 | 146.0011 | 146.0646 | 148.1755 |

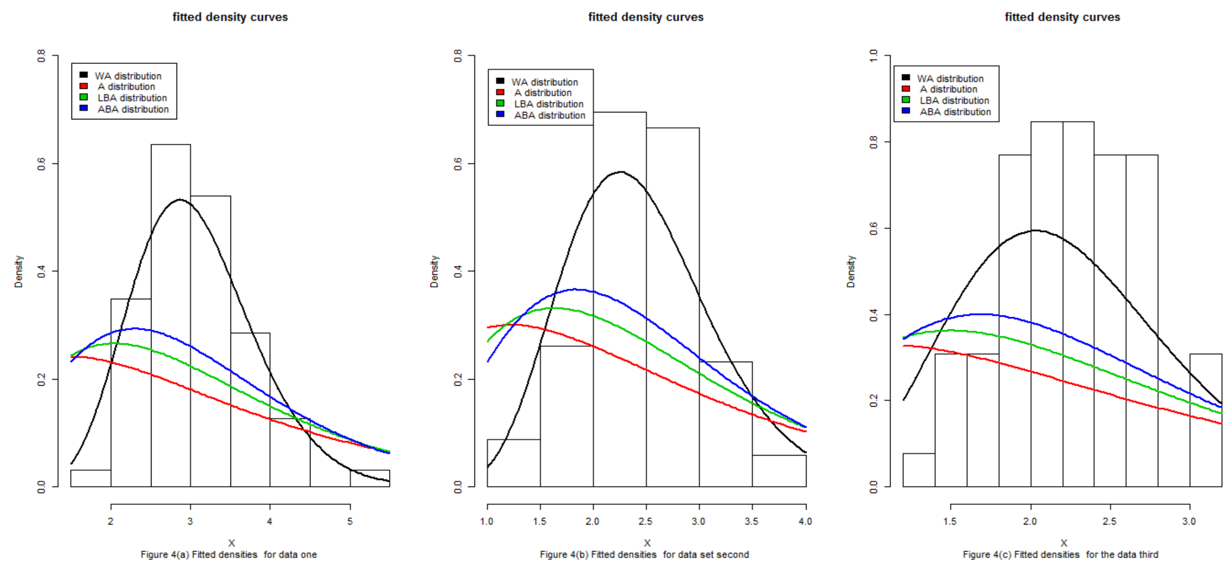


**Figure 4:** Plots of the fitted Weighted Ailamujia, Ailamujia, Length Biased Ailamujia, and Area Biased Ailamujia distributions for data sets 1, 2 and 3.

This manuscript deals with the weighted Ailamujia distribution and studies its different statistical properties. In this paper, moments, mode, harmonic mean, survival function and hazard rate, method of moments and maximum likelihood estimates of the parameters have been obtained. The newly proposed model has been applied to the different real life data sets and the results obtained prove that the proposed weighted Ailamujia model is better fit than its sub models.

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