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Analytical Solution of Time-Fractional Navier-Stokes Equation by Natural Homotopy Perturbation Method

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Abstract: In this article, we study the analytical solution of time-fractional Navier-Stokes equation based on the combination of natural transform (NTM) and homotopy perturbation method (HPM). The analytical scheme gives a series solutions which converges rapidly within few iterations. The efficiency and simplicity of the scheme is clearly demonstrated, and the solutions obtained are compared with the solutions of the existing techniques.

Keywords: Natural homotopy perturbation method, time-fractional Navier-Stokes equations, Caputo fractional derivative, natural transform.

1 Introduction

In recent years, fractional calculus applications are widely applied in many areas of engineering and physical science processes [1,2,3,4,5,6,7,8,9,10]. Recently, partial differential equations with fractional order derivative are successfully applied to many mathematical models in mathematical biology, aerodynamics, control theory, fluid mechanic, analytical chemistry and so on. Fractional partial differential equations has been solved using many analytical and numerical methods such as homotopy analysis method [11], Adomian decomposition method [12,13,14,15,16,17,18], homotopy perturbation method [19,20,21,22,23,24,25,26], generalized differential transform method [27,1,2,3,28], Laplace decomposition method [29,30], natural homotopy perturbation method [31,32,33,34,35], natural decomposition method [36,37], fractional variational iteration method [38,39], to mentioned few. Navier-Stokes equations which describe the motion of viscous fluid was first named after George Gabriel Stokes and Claude-Louis Navier. Those equations are used to model flow in pipe, air flow around a wing, weather, oceans current, and so on. It is also applied in the design of cars, air craft, power stations, and in the study of magnetohydrodynamics if coupled with Max-wells equations. The standard Navier-Stokes equations with time fractional derivative written in operator form as [30,40]:

$$D_t^{\alpha} v(r,t) = Q + \eta\left(D_r^2 v + \frac{1}{r} D_r v\right), \quad 0 < \alpha \leq 1, \tag{1}$$

where the parameter α describes the order of the time fractional derivatives. The special case of Eq. (1) when $\alpha = 1$ reduces to the standard Navier-Stokes equation of the form:

$$D_t v(r,t) = Q + \eta\left(D_r^2 v + \frac{1}{r} D_r v\right), \quad 0 < \alpha \leq 1, \tag{2}$$

where $Q = \frac{\partial p*}{\rho \partial t}$, t is the time, η is the kinematics viscosity, $p*$ is the pressure, and ρ is the density.
The current development in fractional calculus have given impetus to research on fractional partial differential equations which deals with derivatives and integrals of arbitrary orders. We recall that J. H. He firstly proposed a semi-analytical

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technique called the homotopy perturbation technique, which is a coupling of classical perturbation technique and homotopy a concept in topology. Homotopy perturbation technique gained a considerable popularity due to its high accuracy and simplicity, and the crucial aspect of the homotopy perturbation technique is employment of the He's polynomials for computing the nonlinear terms [22]. Golmankhaneh et al. discussed the comparison of analytical solution of nonlinear Navier-Stokes equations, Sturm-Liouville and Burgers and using iterative methods [41]. Recently, Xu et al. studied the numerical solution of the space fractional Navier-Stokes equations by replacing Laplacian operator in Navier-Stokes equations by Riesz fractional derivatives [42]. Rashidi and Shahmohamadi discussed the analytical study of three-dimensional viscous flow near an infinite rotating disk using the variational iteration method (VIM), and the Padè approximant [43].
In this paper, we apply the natural transform method (NTM) and the homotopy perturbation method (HPM) called natural homotopy perturbation method (NHPM) to solved time-fractional Navier-Stokes equation with initial condition. The proposed analytical scheme is applied directly to time-fractional Navier-Stokes equation without taking any restrictive assumptions, discretization of variables, linearization, or transformation. It reduces the computational difficulties, avoids round off errors, and required a small computational size. The natural homotopy perturbation method is successfully applied to time-fractional Navier-Stokes equations, and the results obtained are in excellent agreement with the results of the existing methods.

The outline of the paper is as follows. In Section 2, we review basic definitions of fractional calculus and natural transform. In Section 3, we provide a basic analysis of the natural homotopy perturbation method, to show its efficiency and high accuracy. The results of the application of the proposed method are given in Section 4. The work is concluded in Section 5.

2 Fractional Calculus and Natural Transform

Definition 1. The natural transform of the function $v(t) \geq 0$ is defined over the set of functions,
$A = \left\{ v(t) : \exists\, M,\, \tau_1,\, \tau_2 > 0,\, |v(t)| < Me^{\frac{|t|}{\tau_j}},\, if\ t \in (-1)^j \times [0,\infty) \right\}$,
by the following integral [44,45,46]:

$$\mathbb{N}^+[v(t)] = V(s,u) = \frac{1}{u}\int_0^\infty e^{\frac{-st}{u}} v(t)\,dt;\ \ s > 0,\ u > 0. \tag{3}$$

Definition 2. A function $f(t),\ t > 0$ is said to be in the space $C_\alpha^n,\ n \in N \cup \{0\}$, if $f^{(n)} \in C_\alpha$.

Definition 3. A real function $f(t),\ t > 0$ is said to be in the space $C_\alpha,\ \alpha \in \mathbb{R}$ if there exist a real number $p\ (> \alpha)$ such that $f(t) = t^p f_1(t)$ where $f_1(t) \in C[0,\infty)$. Clearly $C_\alpha \subset C_\beta$ if $\beta \leq \alpha$.

Definition 4. The left sided Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, of a function $f(t) \in C_\mu$, and $\mu \geq -1$ is defined as [30].

$$I^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)\,d\tau, & \alpha > 0,\ t > 0, \\ f(t), & \alpha = 0. \end{cases} \tag{4}$$

For the Riemann-Liouville fractional integral, we have:

$$I^\alpha t^y = \frac{\Gamma(y+1)}{\Gamma(y+\alpha+1)} t^{\alpha+y}. \tag{5}$$

Definition 5. The Natural transform $\mathbb{N}^+[I^\alpha v(t)]$ of the Riemann-Liouville fractional integral is defined as:

$$\mathbb{N}^+[I^\alpha v(t)] = \frac{s^\alpha}{u^\alpha} V(s,u). \tag{6}$$

Definition 6. The fractional derivative of the function $f(t)$ in Caputo sense fractional derivative $f,\ f \in C_{-1}^n,\ n \in \mathbb{N} \cup \{0\}$, is defined as [47,48].

$$D_t^\alpha f(t) = \begin{cases} I^{n-\alpha}\left[\frac{\partial^n f(t)}{\partial t^n}\right], & n-1 < \alpha < n,\ n\varepsilon\,\mathbb{N}, \\ \frac{\partial^n f(t)}{\partial t^n}, & \alpha = n. \end{cases} \tag{7}$$

Note that [14,15]:

$(i)\ I_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(t)}{(t-s)^{1-\alpha}} dt,\ \mu > 0\, t > 0,$

$(ii)\ D_t^{\alpha} f(x,t) = I_t^{n-\alpha} \left[\frac{\partial^n f(t)}{\partial t^n}\right], n-1 < \alpha \leq n.$

Definition 7. The Caputo fractional derivative of natural transform is defined as [31,32]:

$$\mathbb{N}^{+}\left[D_t^{\alpha} v(t)\right] = \frac{s^{\alpha}}{u^{\alpha}} V(s,u) - \sum_{k=0}^{n-1} \frac{s^{\alpha-(k+1)}}{u^{\alpha-k}} v^{(k)}(0+), \tag{8}$$

$n-1 \leq \alpha < 1.$

3 Mathematical Presentation of the Analytical Scheme

The basic analysis of the natural homotopy perturbation method is clearly illustrated, to show its efficiency and high accuracy by considering the general time-fractional Navier-Stokes equation of the form:

$$D_t^{\alpha} v(r,t) = Q + \eta \left(v_{rr} + \frac{1}{r} v_r\right), \quad 0 < \alpha < 1, \tag{9}$$

with the initial condition

$$v(r,0) = f(r), \tag{10}$$

where $Q = \frac{\partial p*}{\rho \partial t}$, t is the time, $p*$ is the pressure, ρ is the density, η is the kinematics viscosity, and $D_t^{\alpha} = \frac{\partial^{\alpha}}{\partial t^{\alpha}}$ is the Caputo fractional derivative.

Operating the natural transform in Eq. (9), we get:

$$V(r,s,u) = \frac{f(r)}{s} + \frac{Q}{s^{\alpha+1}} + \frac{u^{\alpha}}{s^{\alpha}} \mathbb{N}^{+}\left[\eta v_{rr} + \eta \frac{1}{r} v_r\right]. \tag{11}$$

Computing the inverse natural transform of Eq. (11), we get:

$$v(r,t) = G(r,t) + \mathbb{N}^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}} \mathbb{N}^{+}\left[\eta v_{rr} + \eta \frac{1}{r} v_r\right]\right]. \tag{12}$$

Based on homotopy perturbation method, we get:

$$v(r,t) = \sum_{n=0}^{\infty} p^n v_n(r,t). \tag{13}$$

Substituting Eq.(13) into Eq.(12), we get:

$$\sum_{n=0}^{\infty} p^n v_n(r,t) = G(r,t) + p\left(\mathbb{N}^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}} \mathbb{N}^{+}\left[\eta \sum_{n=0}^{\infty} p^n v_{nrr} + \frac{\eta}{r} \sum_{n=0}^{\infty} p^n v_{nr}\right]\right]\right). \tag{14}$$

Equating the coefficients of like powers of p in Eq. (14), we get the following results:

$$p^0: v_0(r,t) = G(r,t),$$
$$p^1: v_1(r,t) = \mathbb{N}^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}} \mathbb{N}^{+}\left[\eta v_{0rr} + \eta \frac{1}{r} v_{0r}\right]\right],$$

$$p^2: v_2(r,t) = \mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbb{N}^+\left[\eta v_{1rr} + \eta\frac{1}{r}v_{1r}\right]\right],$$

$$p^3: v_3(r,t) = \mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbb{N}^+\left[\eta v_{2rr} + \eta\frac{1}{r}v_{2r}\right]\right]$$

$$\vdots,$$

and so on.
Thus, the analytical solutions of Eq. (9)-(10) is given by:

$$v(r,t) = \lim_{N\to\infty}\sum_{n=0}^{N} v_n(r,t). \tag{15}$$

4 Solved Applications

Applications of the analytical scheme to time-fractional Navier-Stokes equation is clearly presented in this section.

Example 1*Consider the following time-fractional Navier-Stokes equation of the form:*

$$D_t^\alpha v(r,t) = Q + v_{rr} + \frac{1}{r}v_r, \quad 0<\alpha<1, \tag{16}$$

with initial condition

$$v(r,0) = 1 - r^2. \tag{17}$$

Computing the natural transform in Eq. (16), we get:

$$V(r,s,u) = \frac{1-r^2}{s} + \frac{Q}{s^{\alpha+1}} + \frac{u^\alpha}{s^\alpha}\mathbb{N}^+\left[v_{rr} + \frac{1}{r}v_r\right]. \tag{18}$$

Operating the inverse natural transform on both sides of Eq. (18), we get:

$$v(r,t) = 1 - r^2 + \frac{Qt^\alpha}{\Gamma(\alpha+1)} + \mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbb{N}^+\left[v_{rr} + \frac{1}{r}v_r\right]\right]. \tag{19}$$

Based on homotopy perturbation scheme, we get:

$$v(r,t) = \sum_{n=0}^{\infty} p^n v_n(r,t). \tag{20}$$

Substituting Eq. (20) into Eq. (19), we get:

$$\sum_{n=0}^{\infty} p^n v_n(r,t) = 1 - r^2 + \frac{Qt^\alpha}{\Gamma(\alpha+1)} + p\left(\mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbb{N}^+\left[\sum_{n=0}^{\infty} p^n v_{nrr} + \frac{1}{r}\sum_{n=0}^{\infty} p^n v_{nr}\right]\right]\right). \tag{21}$$

Equating the coefficients of like powers of p in Eq. (21), we get the following results:

$$p^0: v_0(r,t) = 1 - r^2 + \frac{Qt^\alpha}{\Gamma(\alpha+1)},$$

$$p^1: v_1(r,t) = \mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbb{N}^+\left[v_{0rr} + \frac{1}{r}v_{0r}\right]\right] = \frac{-4t^\alpha}{\Gamma(\alpha+1)},$$

$$p^2: v_2(r,t) = \mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbb{N}^+\left[v_{1rr} + \frac{1}{r}v_{1r}\right]\right]$$
$$= 0,$$

$$p^3: v_3(r,t) = \mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbb{N}^+\left[v_{2rr} + \frac{1}{r}v_{2r}\right]\right]$$
$$= 0,$$
$$\vdots$$

and so on.
Thus, the analytical solutions of Eq.(16)-(17) is given by:

$$\begin{aligned} v(r,t) &= \lim_{N\to\infty}\sum_{n=0}^{N} v_n(r,t) \\ &= v_0(r,t) + v_1(r,t) + v_2(r,t) + v_3(r,t) + \cdots \\ &= 1 - r^2 + \frac{Qt^\alpha}{\Gamma(\alpha+1)} - \frac{4t^\alpha}{\Gamma(\alpha+1)} + 0 + 0 + \cdots \\ &= 1 - r^2 + \frac{(Q-4)t^\alpha}{\Gamma(\alpha+1)}. \end{aligned}$$

The solution of Eq.(16)-(17) in closed form is given by:

$$\begin{aligned} v(r,t) &= \lim_{N\to\infty}\sum_{n=0}^{N} v_n(r,t) \\ &= v_0(r,t) + v_1(r,t) + v_2(r,t) + v_3(r,t) + \cdots \\ &= 1 - r^2 + (Q-4)t, \end{aligned}$$

when $\alpha = 1$.

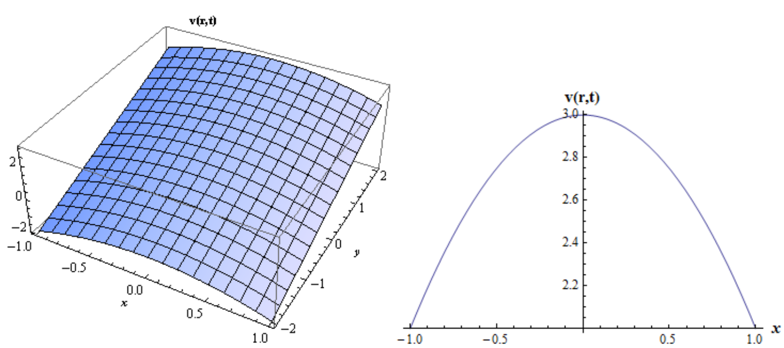

Fig. 1: 3D and 2D surfaces of the analytical solution of Eq (16)-(17) in the ranges $-1 < r < 1$, and $-2 < t < 2$, when $t = 2$, $\alpha = 1$, and $Q = 5$

The result is in complete agreement with [17,30,40].

Example 2*Consider the following time-fractional Navier-Stokes equation of the form:*

$$D_t^\alpha v(r,t) = v_{rr} + \frac{1}{r}v_r, \quad 0 < \alpha < 1, \tag{22}$$

with initial condition

$$v(r,0) = r. \tag{23}$$

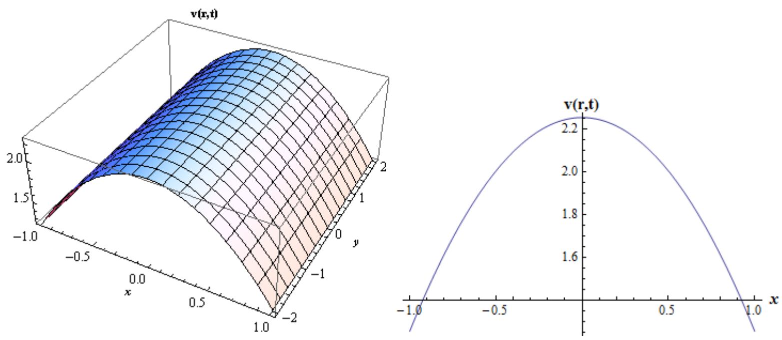

Fig. 2: 3D and 2D surfaces of the analytical solution of Eq (16)-(17) in the ranges $-1 < r < 1$, and $-2 < t < 2$, when $t = 2$, $\alpha = 3.5$, and $Q = 5$.

Operating the natural transform in Eq. (22), we get:

$$V(r,s,u) = \frac{r}{s} + \frac{u^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\left[v_{rr} + \frac{1}{r}v_{r}\right]. \tag{24}$$

Computing the inverse natural transform on both sides of Eq. (24), we get:

$$v(r,t) = r + \mathbb{N}^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\left[v_{rr} + \frac{1}{r}v_{r}\right]\right]. \tag{25}$$

Based on the homotopy perturbation method, we get:

$$v(r,t) = \sum_{n=0}^{\infty} p^{n} v_{n}(r,t). \tag{26}$$

Substituting Eq.(26) into Eq.(25), we get:

$$\sum_{n=0}^{\infty} p^{n} v_{n}(r,t) = r + p\left(\mathbb{N}^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\left[\sum_{n=0}^{\infty} p^{n} v_{nrr} + \frac{1}{r}\sum_{n=0}^{\infty} p^{n} v_{nr}\right]\right]\right). \tag{27}$$

Equating the coefficients of like powers of p in Eq. (27), we get the following results:

$$\begin{aligned} p^{0}: v_{0}(r,t) &= r, \\ p^{1}: v_{1}(r,t) &= \mathbb{N}^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\left[v_{0rr} + \frac{1}{r}v_{0r}\right]\right] \\ &= \frac{1}{r}\frac{t^{\alpha}}{\Gamma(\alpha+1)}, \end{aligned}$$

$$\begin{aligned} p^{2}: v_{2}(r,t) &= \mathbb{N}^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\left[v_{1rr} + \frac{1}{r}v_{1r}\right]\right] \\ &= \frac{1^{2}}{r^{3}}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \end{aligned}$$

$$\begin{aligned} p^{3}: v_{3}(r,t) &= \mathbb{N}^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\left[v_{2rr} + \frac{1}{r}v_{2r}\right]\right] \\ &= \frac{1^{2}\times 3^{2}}{r^{5}}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}, \end{aligned}$$

$$
\begin{aligned}
p^4: v_4(r,t) &= \mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbb{N}^+\left[v_{3rr}+\frac{1}{r}v_{3r}\right]\right] \\
&= \frac{1^2\times 3^2\times 5^2}{r^7}\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}, \\
&\vdots
\end{aligned}
$$

and so on.
Thus, the analytical solutions of Eq.(22)-(23) is given by:

$$
\begin{aligned}
v(r,t) &= \lim_{N\to\infty}\sum_{n=0}^{N} v_n(r,t) \\
&= v_0(r,t)+v_1(r,t)+v_2(r,t)+v_3(r,t)+\cdots \\
&= r+\frac{1}{r}\frac{t^\alpha}{\Gamma(\alpha+1)}+\frac{1^2}{r^3}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{1^2\times 3^2}{r^5}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\cdots \\
&= r+\sum_{n=1}^{\infty}\frac{1^2\times 3^2\times 5^2\cdots\times(2n-3)^2}{r^{2n-1}}\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}.
\end{aligned}
$$

The solution of Eq.(22)-(23) in closed form is given by:

$$
\begin{aligned}
v(r,t) &= \lim_{N\to\infty}\sum_{n=0}^{N} v_n(r,t) \\
&= v_0(r,t)+v_1(r,t)+v_2(r,t)+v_3(r,t)+\cdots \\
&= r+\sum_{n=1}^{\infty}\frac{1^2\times 3^2\times 5^2\cdots\times(2n-3)^2}{r^{2n-1}}\frac{t^n}{n!},
\end{aligned}
$$

when $\alpha = 1$.

The result is in complete agreement with [17,30,40].

5 Conclusion

In this article, an analytical scheme called natural homotopy perturbation method (NHPM) is propose to solve time-fractional Navier-Stokes equation. Based on the scheme, solution in closed form of the time-fractional Navier-Stokes equation was successfully obtained within few iterations.The NHPM series solutions converges rapidly with high accuracy. The fractional order derivatives are computed in Caputo sense. Obviously, the computational simplicity of the analytical method shows that it can be use to study many problems in physical science and engineering.

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