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Deformations in Elasto-Plastic Media with Memory: the Inverse Problem

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Abstract: We consider anelastic media governed by constitutive equations with memory behavior, which depend on the physical properties of the medium itself. In this note we use a model of elasto-plastic media with two unspecified memory formalisms, which are determined by performing a single virtual experiment on a sample of the medium. As an application, using a mathematical memory formally mimicking the Caputo-Fabrizio fractional derivative, we show that, when the applied stress is asymptotically vanishing, then a shorter memory in the constitutive equation and/or a slower decay of the applied stress, generate larger asymptotic plastic residual strain. In the last part of the paper, we present a non-linear stress-strain constitutive equation, which is suitable for describing hysteresis loops with discontinuity in the first derivative of the cycle.

Keywords: Elasto-Plastic media, memory, inverse problem, constitutive equations.

1 Introduction

The fractional derivative is a memory formalism which takes into account how the internal structure of the medium considered changed during the experiments. The fractional derivative spread in the world of applications through a variety of constitutive equations also for generic speculations through thousands of articles in scientific journals and dozens of books, sometime as a geometric tool.

The fields considered for the applications of the fractional derivatives could be a market of the economy or an anelastic material or a generic mathematical tool as for instance the Hamilton Jacobi equations and the relativity theory in order to take into account the second law of thermodynamic, which had insufficient attention by contemporary research and considered mostly by engineers. When considering all these applications, then it comes naturally to doubt that the single memory formalism may be able to represent all the phenomena for instance concerning the anelastic properties of materials. It could be acceptable that it be the same memory formalism, when the Second Principle of Thermodynamics is involved, but then one asks [1] why should this formalism be a fractional derivative, and specifically the Caputo derivative, and not a generic similar mathematical memory formalism.

Elasto-plastic media have the property that, because of stress relaxation, part of the strains induced in these media will remain even when the stress will vanish. The mathematical study of these media goes back at least to Volterra and later to Graffi, who however did not succeed in producing a satisfactory mathematical model for them.

Satisfactory models were introduced later inserting in the constitutive equations the mathematical generic memory formalism [2] and fractional derivatives [3], [4]. But is clear that a single mathematical formalism cannot model all memory phenomena of nature. So that, there is the possibility to observe strains to which however we cannot associate a stress unless we know the history of the stress variations and the mathematical memory formalism of the medium.

This phenomenon is of great interest in the studies concerning earthquake predictions, an emblematic case is the monitoring of the soil deformations in the region of Pozzuoli [5] , where dramatic uplifts and horizontal deformations were observed in the recent decades [6], which generated great concern since similar uplifts preceded the formation of the volcano Monte Nuovo in 1538.

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Because of the supposed plasticity of the Earth crust a question arises concerning the deformations observed in the Phlegrean Fields: may we assume, without additional studies, the stresses associated to the subsequent deformations observed during the last decades without taking into account that the stresses generating them may have substantially decreased or even disappeared in the time interval during the observations, while the associated deformations have not disappeared.

The present note is structured in the following way: first are introduced, in the time and the Laplace Transform domains, the constitutive equations for plastic media which include generic memory formalisms, then the problem of retrieving the mathematical form of memory from appropriate laboratory experiments is considered and solved, follows an application is made for the theoretical case when the laboratory experiments give deformations constant or linearly changing. Finally the effects of a memory formalism mimicking the Caputo-Fabrizio fractional derivative are studied.

In the last section we present a non-linear model able to describe hysteresis cycles, which show a discontinuous first derivative, when the direction of the strain is reversed. For these phenomena is convenient to represent the problem through a non-linear model. Therefore, for this purpose, we study in this last section a non-linear stress-strain constitutive equation.

2 The Modeling of Plastic Media

Most literature on applied fractional calculus shows that the successful use of the presently available fractional derivatives is a proof of the presence of memory in many scientific phenomena; however this is a first order approximation in taking account the memory phenomenon which is needed mostly, but not only, for implying the second law of thermodynamics in the constitutive equations of elasticity.

The literature of the constitutive equations for the mathematical modeling of plastic phenomena is vast beginning with Volterra [7], who modeled the phenomenon using hereditary mathematical tools and whose dislocation theory is the base of a new branch of plasticity studies [8] where the phenomenon of plasticity is considered as due to the migration of dislocations.

It is to be noted also the rich book of Argon [9], which appeared before the quick diffusion of fractional calculus, with the presentation and discussion of a variety cases. Concerning hereditary phenomena are also of interest the volume of Graffi [10] and particularly the notes of Fichera [11], [12].

The applications of fractional calculus to plasticity and hysteresis was considered also in the notes of [2] , [13], [14], [1] who studied the rheological properties of polycrystalline halite, appearing in nature in large thick deposits, which were considered for the disposal of radioactive waste and whose anelastic properties had been studied experimentally [15] and [16].

In order to find which could be the most appropriate model for the memory of an elasto-plastic medium let us consider the set of constitutive equation of rheological media already studied by [17], [2] and by [18] [14],[3], [4], which may be written in simplified form with the undefined memory kernels $h(t)$ and $m(t)$

2.1 Body Math

$$\begin{aligned} &h(t)\star\mathscr{D}^{(1)}\tau_{ij}+\mu(\tau_{ij}-\tfrac{1}{3}\delta_{ij}\tau_{rr}) \\ &=\mu(\delta_{ij}\lambda\varepsilon_{rr}+2m(t)\star\mathscr{D}^{(1)}\varepsilon_{ij}), \end{aligned} \tag{1}$$

where λ is Lame's coefficient, μ the shear modulus. While $\star$ denotes the convolution and τ_{ij} and ε_{ij} the stress and strain tensors, whose Laplace Transform (LT) with variable p is

$$pH(p)T_{ij}+\mu(T_{ij}-\frac{1}{3}\delta_{ij}T_{rr})=\mu\,[\delta_{ij}\lambda E_{rr}+2pM(p)E_{ij}] \tag{2}$$

provided $\varepsilon_{ij}(0)=\tau_{ij}(0)=0$. Capital letters indicate LT of the function with equal lower letter, $D^{(1)}$ means classic derivative of first order, while u is the order of the memory operators $h(u,t)$ and $m(u,t)$ are L^1 and have dimension of stress. Moreover, they are monotonically decreasing with

$$h(u,\infty)=0,\; h(0,t)=1,\; m(u,\infty)=0,\; m(0,t)=1.$$

The latter conditions, would imply that the application of the operators $h(0,t)*\mathscr{D}^{(1)}$, $m(0,t)*\mathscr{D}^{(1)}$ to a function reproduce the function itself as for the memory operators presently used many research. The ranges of u and v respectively depend on the problem considered. All the functions concerning the physical conditions of the medium modeled by equations (1) and (2) are assumed to be initially zero that is the medium to be initially at rest.

Examples of generic functions $h(u,t)$ and $m(u,t)$ are

$$h(t) = exp(-ut) \tag{3}$$

or

$$h(t) = \frac{1}{log(e+ut)}, \tag{4}$$

where, with $u = 0$, follows $h(0,t) = 1$, which imply that the operator of order zero reproduces the function, that are monotonically decreasing and satisfy the conditions $h(\infty,t) = 0$. The same is valid for $m(u,t)$.

When $u = 1$ the operators are not necessarily representing the first order derivative of the function $exp(-ut)$ and $\frac{1}{log(e+ut)}$ are not necessarily kernels of fractional derivatives of order u.

Obviously the functions $h(u,t)$, $m(u,t)$ would not define an operator with the all properties of the classical fractional derivative, but are simple, hopefully useful, memory formalisms which reproduce the function when the operator has order $u = 0$.

Classic examples of function $h(u,t) * \mathscr{D}^{(1)}$, $m(u,t) * \mathscr{D}^{(1)}$ are considered in Caputo and Caputo-Fabrizio fractional derivative without singularity [14].

3 The Experimental Retrieval of the Memory of a Medium

In order to find the memory of the medium we assume that an experiment is made on a cylindrical sample applying to it the constant stresses $\tau_{22} = \tau_{33}$ and τ_{11}, observing the corresponding strains and, in order to find the operators H and M obtain from equation (2)

$$\begin{aligned} pH(p)T_{11} + \mu(T_{11} - \frac{1}{3}T_{rr}) &= \mu\left[\lambda E_{rr} + 2pM(p)E_{11}\right], \\ pH(p)T_{22} + \mu(T_{22} - \frac{1}{3}T_{rr}) &= \mu\left[\lambda E_{rr} + 2pM(p)E_{22}\right], \\ pH(p)T_{33} + \mu(T_{33} - \frac{1}{3}T_{rr}) &= \mu\left[\lambda E_{rr} + 2pM(p)E_{33}\right]. \end{aligned} \tag{5}$$

Summing equations (5) we find

$$H(p)T_{rr} = \mu(\frac{3\lambda}{p} + 2M)E_{rr}. \tag{6}$$

Substituting T_{rr} in equations (5) we obtain

$$\begin{aligned} (pH-\mu)T_{11} &= 2\mu pME_{11} + 2pME_{11} + \mu\left[\lambda + \frac{\mu}{3H}(\frac{3\lambda}{p} + 2M)\right]E_{ii}, \\ (pH-\mu)T_{22} &= 2\mu pME_{22} + 2pME_{22} + \mu\left[\lambda + \frac{\mu}{3H}(\frac{3\lambda}{p} + 2M)\right]E_{ii}, \\ (pH-\mu)33 &= 2\mu pME_{33} + 2pME_{33} + \mu\left[\lambda + \frac{\mu}{3H}(\frac{3\lambda}{p} + 2M)\right]E_{ii}. \end{aligned} \tag{7}$$

In order to simplify the formulae without losing in generality we assume $\tau_{22} = \tau_{33}$, which implies $\varepsilon_{22} = \varepsilon_{33}$. Then the third and second equations (5) are identical, therefore we use only the first and the second equation of the system (5), that is the system

$$\begin{aligned} T_{11} &= \left\{2\frac{\mu pM}{pH-\mu}\right\}E_{11} + \mu\left[\lambda + \frac{\mu}{3H}(\frac{3\lambda}{p} + 2M)\right]\frac{E_{ii}}{pH-\mu}, \\ T_{22} &= \left\{2\frac{\mu pM}{pH-\mu}\right\}E_{22} + \mu\left[\lambda + \frac{\mu}{3H}(\frac{3\lambda}{p} + 2M)\right]\frac{E_{ii}}{pH-\mu}, \end{aligned}$$

then subtracting we obtain

$$T_{11}-T_{22}=2\frac{\mu pM}{pH-\mu}(E_{11}-E_{22}) \tag{8}$$

which is a relation between H and M, where $T_{22}=T_{33}, E_{22}=E_{33},$ while T_{11} and E_{11} are known since we assume that the strain are measured experimentally.

We have then two equations, (8) and (6), identifying H and M, which we write as follows

$$(T_{11}-T_{22})pH-2\mu p(E_{11}-E_{22})=\mu(T_{11}-T_{22}) \tag{9}$$

which,assuming

$$b=(T_{11}-T_{22})\ ,\ \ a=T_{rr}$$

gives

$$M=-\frac{b}{2p}\frac{a-3\lambda E_{rr}}{a(E_{11}-E_{22})-bE_{rr}}, \tag{10}$$

$$H=-\frac{\mu}{p}\frac{3\lambda E_{rr}(E_{11}-E_{22})-E_{rr}b}{a(E_{11}-E_{22})-bE_{rr}}. \tag{11}$$

If the LT^{-1} of H and M exist the problem of identifying the memory formalism of the medium under examination is solved. However we prove that is satisfied the necessary condition for the existence of the LT^{-1} based on the assumed mathematical properties of the latter functions that are finite, continuous and monotonic. In fact since E_{22} and E_{11} are LT of the observed deformations, which we assume finite,we have

$$\lim_{p\to\infty}E_{ij}=0$$

and consequently from equations (10) and (11) we obtain

$$\lim_{p\to\infty}H=0\ ,\ \ \lim_{p\to\infty}M=0$$

which imply that the necessary condition for the of the existence of the LT^{-1} of M and H is satisfied.

We now verify the results obtained assuming that the strain resulting from the constant applied stress is constant. To this purpose let as consider equations (10) and set $E_{ij}=\frac{e_{ij}}{p}$; we find

$$M=-\frac{b}{2p}\frac{ap-3\lambda e_{rr}}{a(e_{11}-e_{22})-be_{rr}}, \tag{12}$$

$$H=-\frac{\mu}{p}\frac{3\lambda e_{rr}(e_{11}-e_{22})-e_{rr}b}{a(e_{11}-e_{22})-be_{rr}}, \tag{13}$$

which give

$$m(t)=-\frac{b}{2}\frac{a\delta(t)-3\lambda e_{rr}}{a(e_{11}-e_{22})-be_{rr}},$$

$$h(t)=\mu\frac{be_{rr}\delta(t)-3\lambda e_{rr}(e_{11}-e_{22})}{a(e_{11}-e_{22})-be_{rr}}.$$

4 The Behavior of the Strain Resulting from Constitutive Equations which Simulate Caputo-Fabrizio Fractional Derivative

Now, we seek the values of e_{22} and e_{11} to be expected, when the stresses T_{11} and T_{22} are applied and using the following memory operators

$$h(t)=nu*exp(-ut),\quad m(t)=dv*exp(-vt) \tag{14}$$

whose LT are

$$H(p)=\frac{n}{p}(p+u),\quad M(p)=\frac{d}{p}(p+v). \tag{15}$$

that is the memory of the medium is represented by a fractional operators mimicking the fractional derivative in [14] with different order of differentiation. The memory formalisms (15) , although mimicking the Caputo-Fabrizio fractional derivative, have a different effect. In fact, they are nil at $t=0$ and, asymptotically they reach the values n and d respectively, while the Caputo and Caputo-Fabrizio derivatives are initially positive and asymptotically nil. Moreover while they act as high pass filters, the memory formalism represented by equations (15) acts as a low pass filter.

This type of memory may seem difficult to figure in solid or liquid substances or human phenomena however a simple example is the memory of elderly persons who remember better facts occurred in their remote past than the recent ones.

Using equations (10) and substituting the definitions equations (15) for $h(t)$ and $m(t)$ respectively, the result asymptotic strain comes from the solution of the following system

$$\frac{d}{p+v} = \frac{-\frac{b}{2p}(a-3\lambda(E_{11}+2E_{22})}{a(E_{11}-E_{22})-b(E_{11}+2E_{22})}, \tag{16}$$

$$\frac{n}{p+u} = \frac{-\frac{\mu}{p}3\lambda E_{rr}(E_{11}-E_{22})-b(E_{11}+2E_{22})}{a(E_{11}-E_{22})-b(E_{11}+2E_{22})}, \tag{17}$$

setting

$$\begin{gathered} E_{11}-E_{22}=R, \\ E_{11}+2E_{22}=S, \end{gathered} \tag{18}$$

the system (15) gives

$$\begin{gathered} daR = dbS - \tfrac{b}{2p}(p+v))(a-3\lambda S), \\ \tfrac{n(ar-bS)}{p+u} = \tfrac{\mu}{p}(3\lambda R-b), \end{gathered} \tag{19}$$

$$R = -\frac{b}{2pda}(p+v)(a-3\lambda dS)+\frac{dbS}{da}, \tag{20}$$

$$R = \frac{bS}{a}-\frac{\mu}{pan}S(3\lambda R-b)(p+u), \tag{21}$$

$$\begin{gathered} -\frac{b}{2pd}(p+v)+S\frac{b}{ad}\left\{\left[(\frac{3\lambda}{2p}(p+v)+d\right]-\frac{\mu}{pan}(p+u)b-\frac{b}{and}(1-\frac{3\lambda\mu}{2p^2}(p+v)(p+u)\right\}+ \\ +\frac{3\lambda\mu b}{2p^2a^2dn}(p+u)\left[\frac{3\lambda}{2p}(p+v)+d\right]S^2=0, \end{gathered} \tag{22}$$

multiplying now by $\frac{a}{b}$ we finally obtain the following second degree algebraic equation in the unknown S

$$\begin{gathered} -\tfrac{a(p+v)}{2pd}+\left[\tfrac{3\lambda S}{2pd}(p+v)+1\right]-\tfrac{\mu S}{pn}(p+u)-S+ \\ \tfrac{3\mu\lambda S(p+u)(p+v)}{2p^2nd}+\tfrac{3\lambda\mu S^2(p+u)}{padn}\left[\tfrac{3\lambda}{2\mu}(p+v)+d\right]=0, \end{gathered} \tag{23}$$

In order to find S we consider now the equation (23) and write it as follows

$$wS_2+gS+z=0, \tag{24}$$

$$z=-\frac{a}{2pd}(p+v), \tag{25}$$

$$g=\frac{3\lambda}{2pd}[(p+v)+1]-\frac{\mu}{pn}(p+u)-1+\frac{3\lambda\mu(p+u)(p+v)}{2p^2dn}, \tag{26}$$

$$w=\frac{3\lambda\mu(p+u)(p+v)+2pd}{2p^2adn}, \tag{27}$$

$$S=\frac{(-g\pm\,(g_2-4wz)0.5)}{2w}. \tag{28}$$

The limit with $p \to 0$ of the term $\frac{g}{2w}$ is

$$\lim_{p\to o} p\left\{\frac{3\lambda}{2pd}((p+v)+1)-\frac{\mu}{np}(p+u)-1+\frac{3\lambda\mu}{2p^2nd}(p+u)(p+v\right\},$$

$$\frac{2p^2adn}{6\lambda\mu(p+u)(3\lambda(p+v)+2pd/3\lambda}= \tag{29}$$

$$\lim_{p\to 0} p\frac{3\lambda p(p+u)(p+v)a}{3\lambda(p+u)(3\lambda(p+v)+2d}=0.$$

Also

$$\lim_{p\to 0}\frac{pz}{w}=0.$$

Follows then that the solutions (28) are asymptotically nil.

The extreme values theorem gives then

$$(e_{11}(\infty)+2e_{22}(\infty))=0.$$

Finally, from the first of equations (22) we have

$$\lim_{p\to 0} R(p)=-\frac{vb}{2d}$$

and

$$e_{11}-e_{22}=-\frac{bv}{2d}\ ,\ \ e_{11}+2e_{22}=0, \tag{30}$$

or $\frac{e_{11}}{e_{22}}=-1/2$ that is, in terms of Poisson ratio, the rigidity is zero.

As an example we may consider the case when applied stresses are asymptotically vanishing which is of interest for the studies of elasto-plastic media. For this case we assume that

$$\tau_{11}=2rexp(-qt),\quad \tau_{22}=\tau_{33}=r\ exp(-qt),\quad T_{rr}=\frac{3r}{(p+q)} \tag{31}$$

and the parameters have the following form

$$\lambda=\mu,\ a=4r/(p+q),\ b=r/(p+q),\ d=n, b/a=1/3,\ u=v, \tag{32}$$

then equation (22) is

$$\lim_{p\to 0} pR(p)=-\frac{b}{2vdq} \tag{33}$$

giving

$$e_{11}-e_{22}=-\frac{rv}{2dr}\ ;\quad e_{11}+2e_{22}=0, \tag{34}$$

$$e_{22}=\frac{rv}{6dq}\ ;\quad e_{11}=-\frac{rv}{3dq}, \tag{35}$$

the ratio $\frac{e_{22}}{e_{11}}=-1/2$, again in terms of Poisson ratio, implies zero rigidity.

It is important to note that shorter memory (larger value of v), slower decay of the applied stress (smaller value of q) and smaller effect of the memory (smaller amplitude of memory factor d) generate larger asymptotic residual strains.

Non-Linear Elasto-Plasticity by a Fractional Model

The linear model allows to describe plastic materials with clear viscous effects, whereby the hysteresis loops are quite smooth and stable. Other materials instead show a discontinuity in the first derivative of the cycle. In such a case it is necessary to represent the phenomenon through a non-linear model. Therefore, for this purpose, we study in this session a non-linear stress-strain constitutive equation.

So that, we leave from a mechanical system described by the differential system

$$\rho_0(x)v_{i/t}(x,t) = \tau_{ij/j}(x,t) + \rho_0(x)f_i(x,t), \tag{36}$$

where v_i is the velocity, f_i the external supply and τ_{ij} are the components of the stress tensor τ. Finally, ρ_0 is the density, which in the following we suppose constant and equal to 1.

The material is described by the constitutive equation in tensor form

$$\tau(x,t) = \boldsymbol{A}(x)\varepsilon(x,t) + \boldsymbol{P}(x,t), \tag{37}$$

where $\boldsymbol{A}(x)$ is a fourth order tensor and $\boldsymbol{P}(x,t)$ the plastic tensor, which satisfies a non-linear differential equation between ε, $\boldsymbol{P}$ given by

$$\lambda\varepsilon(t) = (\gamma\boldsymbol{P}^2(t)+1)\boldsymbol{P}(t) - \delta\boldsymbol{P}(t) - \mathbf{C}_0\mathscr{D}_t^{\alpha}\boldsymbol{P}(t), \tag{38}$$

while λ, γ, δ are positive scalars and $\mathbf{C}$ is a four order positive tensor, related with the constitutive properties of plastic material. In the following, we suppose $\lambda = 1$. Moreover, the fractional derivative ${}_0\mathscr{D}_t^{\alpha}\boldsymbol{P}(t)$ of order α, will be defined following the two view points considered in [14]. So, we have Caputo fractional model [19]

$$\mathscr{D}_t^{\alpha}\boldsymbol{P}(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^{\infty}\frac{\boldsymbol{P}'(\tau)}{(t-\tau)^{\alpha}}d\tau \tag{39}$$

or the fractional derivative defined in [14]

$$\hat{\mathscr{D}}_t^{\alpha}\boldsymbol{P}(t) = \frac{1}{1-\alpha}\int_0^{\infty}e^{-\frac{\bar{\alpha}}{1-\alpha}(t-\tau)}\boldsymbol{P}'(\tau)d\tau, \tag{40}$$

where we suppose $\boldsymbol{P}(t) = 0$ for $t \le 0$.

For this non-linear problem (36), (37), (38) and (39) or (40), we study the coherence of this dynamic system with the thermodynamic laws. Hence, if we denote with $\mathscr{P}_m^i$ the mechanical power defined by

$$\mathscr{P}_m^i(x,t) = \tau(x,t)\cdot\dot{\varepsilon}(x,t). \tag{41}$$

Then, we have (see [20])

4.1 The Dissipation Principle

There exists a state function $\psi(x,t)$, called free energy, such that, for any thermodynamic process, we have

$$\dot{\psi}(x,t) \le \mathscr{P}_m^i(x,t). \tag{42}$$

Then, from the inequality (42) and the definition (41) of $\mathscr{P}_m^i$, we have

$$\begin{aligned}\mathscr{P}_m^i(x,t) &= \tau(x,t)\cdot\dot{\varepsilon}(x,t) = (\boldsymbol{A}(x)\varepsilon(x,t)+\boldsymbol{P}(x,t))\cdot\dot{\varepsilon}(x,t) = \\ &= (\frac{1}{2}\boldsymbol{A}(x)\varepsilon(x,t)\cdot\varepsilon(x,t)+\varepsilon(x,t)\cdot\boldsymbol{P}(x,t))^{\cdot} - \dot{\boldsymbol{P}}(x,t)\cdot((\gamma\boldsymbol{P}^2(t)+1)\boldsymbol{P}(x,t) - \delta\boldsymbol{P}(t) - \\ &-\mathbf{C}_0D_t^{\alpha}\boldsymbol{P}(x,t)) = (\frac{1}{2}\boldsymbol{A}(x)\varepsilon(x,t)\cdot\varepsilon(x,t)+\varepsilon(x,t)\cdot\boldsymbol{P}(x,t))^{\cdot} + ((\frac{1}{4}(\gamma\boldsymbol{P}^2(t)+1)^2)^{\cdot} - \\ &-\frac{\delta}{2}(\boldsymbol{P}^2(t))^{\cdot} - \mathbf{C}_0D_t^{\alpha}\boldsymbol{P}(x,t)\cdot\dot{\boldsymbol{P}}(x,t)).\end{aligned} \tag{43}$$

Now we work on the last term of (43), when the fractional derivative is defined by (39) , i. e.

$$\mathbf{C}\mathscr{D}_t^{\alpha}\boldsymbol{P}(x,t)\cdot\dot{\boldsymbol{P}}(x,t)=\frac{\mathbf{C}}{\Gamma(1-\alpha)}\int_0^{\infty}\frac{\boldsymbol{P}(x,t-s)}{s^{\alpha}}d\tau\cdot\dot{\boldsymbol{P}}(x,t)=$$

$$\frac{\alpha\mathbf{C}}{\Gamma(1-\alpha)}\int_0^{\infty}\frac{\boldsymbol{P}(x,t-s)-\boldsymbol{P}(x,t)}{(s)^{1+\alpha}}ds\cdot\dot{\boldsymbol{P}}(x,t).$$

So we denote with $e_1(x,t)$ the functional

$$e_1(x,t)=\frac{\alpha\mathbf{C}}{2\Gamma(1-\alpha)}(\int_0^{\infty}\frac{\boldsymbol{P}(x,t-s)-\boldsymbol{P}(x,t)}{(s)^{1+\alpha}}\cdot(\boldsymbol{P}(x,t-s)-\boldsymbol{P}(x,t))d\tau \tag{44}$$

then

$$\dot{e}_1(x,t)=\frac{\alpha\mathbf{C}}{\Gamma(1-\alpha)}\int_0^{\infty}\frac{\boldsymbol{P}(x,t-s)-\boldsymbol{P}(x,t)}{(s)^{1+\alpha}}ds\cdot\dot{\boldsymbol{P}}(x,t)=$$

$$=(\frac{\alpha\mathbf{C}}{\Gamma(1-\alpha)}(\int_0^{\infty}\frac{\boldsymbol{P}(x,t-s)-\boldsymbol{P}(x,t)}{(s)^{1+\alpha}}\cdot\frac{d}{dt}(\boldsymbol{P}(x,t-s)-\boldsymbol{P}(x,t))d\tau+ \tag{45}$$

$$\int_0^{\infty}\frac{\boldsymbol{P}(x,t-s)-\boldsymbol{P}(x,t)}{(s)^{1+\alpha}}\cdot\frac{d}{ds}(\boldsymbol{P}(x,t-s)-\boldsymbol{P}(x,t))d\tau).$$

Then, by (44) we obtain the identity

$$\dot{e}_1(x,t)=\frac{\alpha}{\Gamma(1-\alpha)}(\int_0^{\infty}\frac{d}{dt}(\frac{(\boldsymbol{P}(x,t-s)-\boldsymbol{P}(x,t))}{(s)^{1+\alpha}}\cdot\mathbf{C}(\boldsymbol{P}(x,t-s)-\boldsymbol{P}(x,t)))d\tau- \tag{46}$$

$$\frac{\alpha(1+\alpha)}{2\Gamma(1-\alpha)}(\int_{-\infty}^{t}\mathbf{C}\frac{\boldsymbol{P}(x,t)-\boldsymbol{P}(x,\tau)}{(t-\tau)^{2+\alpha}}\cdot(\boldsymbol{P}(x,t)-\boldsymbol{P}(x,\tau))d\tau)$$

because the tensor $\mathbf{C}$ is positive defined, then we conclude that the dissipation $D(x,t)\geq 0$ is defined by

$$D(x,t)=\frac{\alpha(1+\alpha)}{2\Gamma(1-\alpha)}(\int_{-\infty}^{t}\mathbf{C}\frac{\boldsymbol{P}(x,t)-\boldsymbol{P}(x,\tau)}{(t-\tau)^{2+\alpha}}\cdot(\boldsymbol{P}(x,t)-\boldsymbol{P}(x,\tau))d\tau). \tag{47}$$

Hence, from (45)-(47) we have

$$\mathscr{P}_m^i(x,t)=(\frac{1}{2}\boldsymbol{A}(x)\varepsilon(x,t)\cdot\varepsilon(x,t)+\varepsilon(x,t)\cdot\boldsymbol{P}(x,t))^{\cdot}+((\frac{1}{4}(\gamma\boldsymbol{P}^2(t)+1)^2)^{\cdot}-$$

$$-\frac{\delta}{2}(\boldsymbol{P}^2(t))^{\cdot}++\dot{\boldsymbol{P}}(x,t))\cdot(\frac{\alpha\mathbf{C}}{\Gamma(1-\alpha)}\int_{-\infty}^{t}\frac{\boldsymbol{P}(x,t)-\boldsymbol{P}(x,\tau)}{(t-\tau)^{1+\alpha}}d\tau-D(x,t). \tag{48}$$

Then, from (42)

$$\dot{\psi}(x,t)\leq(\frac{1}{2}\boldsymbol{A}(x)\varepsilon(x,t)\cdot\varepsilon(x,t)+\varepsilon(x,t)\cdot\boldsymbol{P}(x,t))^{\cdot}+((\frac{1}{4}(\gamma\boldsymbol{P}^2(t)+1)^2)^{\cdot}-\frac{\delta}{2}(\boldsymbol{P}^2(t))^{\cdot}+$$

$$+\frac{\alpha}{\Gamma(1-\alpha)}(\int_0^{\infty}\frac{d}{dt}(\frac{(\boldsymbol{P}(x,t-s)-\boldsymbol{P}(x,t))}{(s)^{1+\alpha}}\cdot\mathbf{C}(\boldsymbol{P}(x,t-s)-\boldsymbol{P}(x,t)))d\tau-D(x,t)$$

so that

$$\psi(t)=(\frac{1}{2}\boldsymbol{A}(x)\varepsilon(x,t)\cdot\varepsilon(x,t)+\varepsilon(x,t)\cdot\boldsymbol{P}(x,t))+((\frac{1}{4}(\gamma\boldsymbol{P}^2(t)+1)^2)-\frac{\delta}{2}(\mathbf{P}^2(t))+$$

$$+\frac{\alpha}{\Gamma(1-\alpha)}(\int_0^{\infty}\frac{d}{dt}(\frac{(\boldsymbol{P}(x,t-s)-\boldsymbol{P}(x,t))}{(s)^{1+\alpha}}\cdot\mathbf{C}(\boldsymbol{P}(x,t-s)-\boldsymbol{P}(x,t)))d\tau.$$

Now, we study the system related with the constitutive equation (38), when we use the fractional derivatives (40) and we compare the internal mechanical power $\mathscr{P}_m^i(x,t)$ with the new power $\hat{\mathscr{P}}_m^i(x,t)$ through the derivative (40). Then, we have

$$\hat{\mathscr{P}}_m^i(x,t)=(\frac{1}{2}\boldsymbol{A}(x)\varepsilon(x,t)\cdot\varepsilon(x,t)+\varepsilon(x,t)\cdot\boldsymbol{P}(x,t))^{\cdot}- \tag{49}$$

$$-\dot{\mathbf{P}}(x,t)\cdot((\gamma \boldsymbol{P}^2(t)+1)\boldsymbol{P}(x,t)-\delta \boldsymbol{P}(t)-\mathbf{C}\hat{\mathscr{D}}_{\mathbf{t}}^{\alpha}\boldsymbol{P}(x,t))\cdot\dot{\boldsymbol{P}}(x,t).$$

Hence, if we define the functional e_2 by

$$e_2(x,t) = \tag{50}$$
$$= \frac{\alpha}{2(1-\alpha)}(\int_0^\infty e^{-\frac{\bar{\alpha}}{1-\alpha}(t-\tau)}\mathbf{C}_2(x)(\boldsymbol{P}(x,t-s)-\boldsymbol{P}(x,t))\cdot(\boldsymbol{P}(x,t-s)-\boldsymbol{P}(x,t))d\tau$$

then we obtain

$$\dot{e}_2(x,t) = \frac{\alpha}{\Gamma(1-\alpha)}(\int_0^\infty \frac{d}{dt}(\frac{(\boldsymbol{P}(x,t-s)-\boldsymbol{P}(x,t))}{(s)^{1+\alpha}}\cdot\mathbf{C}_2(\boldsymbol{P}(x,t-s)-\boldsymbol{P}(x,t)))d\tau- \tag{51}$$
$$-\frac{(1-\alpha)}{2\alpha(1-\alpha)}(\int_0^\infty e^{-\frac{\bar{\alpha}}{1-\alpha}(t-\tau)}\mathbf{C}_2(x)(\boldsymbol{P}(x,t-s)-\boldsymbol{P}(x,t))\cdot(\boldsymbol{P}(x,t-s)-\boldsymbol{P}(x,t))d\tau,$$

finally the dissipation $\hat{D}(x,t)$ is given by

$$\hat{D}(x,t) = \frac{(1-\alpha)}{2\alpha(1-\alpha)}(\int_0^\infty e^{-\frac{\bar{\alpha}}{1-\alpha}(t-\tau)}\mathbf{C}_2(x)(\boldsymbol{P}(x,t-s)-\boldsymbol{P}(x,t))\cdot(\boldsymbol{P}(x,t-s)-\boldsymbol{P}(x,t))d\tau. \tag{52}$$

Hence, from [5] we have

$$\hat{\mathscr{P}}_m^i(x,t) = (\frac{1}{2}\boldsymbol{A}(x)\varepsilon(x,t)\cdot\varepsilon(x,t)+\varepsilon(x,t)\cdot\boldsymbol{P}(x,t))+((\frac{1}{4}(\gamma\boldsymbol{P}^2(t)+1)^2)-$$
$$-\frac{\delta}{2}(\boldsymbol{P}^2(t)))^{\cdot}+\frac{\alpha}{\Gamma(1-\alpha)}\frac{d}{dt}(\int_0^\infty(\frac{(\boldsymbol{P}(x,t-s)-\boldsymbol{P}(x,t))}{(s)^{1+\alpha}}\cdot\mathbf{C}_2(\boldsymbol{P}(x,t-s)- \tag{53}$$
$$-\boldsymbol{P}(x,t)))d\tau-\hat{D}(x,t).$$

5 Conclusion

Concerning the modeling of phenomena using constitutive equations with fractional derivatives the literature proves the presence of memory in many phenomena but this is often only a first order approximation in taking account of the memory of the phenomenon since we may not rule out the exist other memory formalisms giving a better modeling of the phenomenon and the fractional derivative is often needed, but not only, for accounting for the second law of thermodynamics.

In order to distinguish the different types of memory, we note that when e_{ij} and τ_{ij} are measured with the appropriate experiment and the elastic parameters are known then the analytical expression of the memory operators of the medium $H(p)$ and $M(p)$ are obtained from equations (3), (11) and (12). In other words one may infer the memory properties of the medium directly from the experimental data even if the memory operators is not a fractional derivatives.

Finally, In the last section, for describing hysteresis loops with discontinuity in the first derivative of the processes, we propose a non-linear stress-strain model, which is able to provide convenient hysteresis cycles.

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