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The Approximate Solution of Nonlinear Fractional Optimal Control Problems by Measure Theory Approach

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Abstract: In this paper a novel strategy to find an approximate solution for nonlinear fractional optimal control problem is proposed. The measure theory is used to find the solution. There are different definitions for fractional derivatives and integrals. In this work, a new definition which is named conformable calculus is used. Analytically convergence of the proposed method is proved.

Keywords: Fractional optimal control problem, measure theory, conformable derivative, nonlinear optimal control, linear programming.

1 Introduction

The field of fractional calculus has a significant role in various disciplines such as engineering, biomechanics, electrochemical, etc [1,2,3,4,5,6]. Many real physical systems can be modeled more accurately by fractional order differential equations. Optimal control problem generally is defined as a function minimization over an admissible set of control and state functions subject to dynamic constraints on the state and control input. The Fractional Optimal Control Problem (FOCP) is an optimal control problem, in which the performance index or the differential equation governing the dynamic of the system or both contain at least one fractional order derivative term.

One of the efficient methods for solving the classic nonlinear optimal control problem is measure theory, see e.g.[7,8,9,10,11,12,13,14,15]. In the present research measure theory approach is extended for solving nonlinear FOCPs.

The most famous fractional derivatives that are applied by researchers are Riemann-Liouville, Caputo and Grünwald-Letnikov. These fractional derivatives do not satisfy most properties of classical calculus, such as product rule, chain rule, and Leibniz rule. Khalil et al. (2014) introduced a new well-behaved definition of derivative called conformable derivative to overcome these drawbacks. This new definition is theoretically easier and more adaptive with the conventional derivative properties.[16,17,18,19,20,21,22,23]. So this new concept motivated the authors to use conformable derivatives in solving FOCPs.

In order to solve such problems, first a fractional positive measure is defined, then measure theoretical method proposed in [10] is developed. In Section 2, we give some preliminaries. In Section 3, the classical FOCP is embedded into a new space (a space of measures). Then new form of the FOCP is an infinite dimensional linear programming problem (LPP) in Section 4. At the final stage, this LPP is approximated by a finite dimensional linear programming problem, where the optimal pair of state and control can be found by the solution of this finite dimensional LPP. Numerical examples are given in Section 5. In Section 6 we conclude our work.

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2 Preliminaries

Let $w = f(t)$ $(t \geq 0)$ be a real valued, continuous function and $\alpha > 0$ be a given real number. Consider the following definitions: (see [16]).

Definition 2.1 The Riemann-Liouville fractional integral of order α of the function f, is defined as

$$ {}_0I_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \qquad t > 0, $$

where $\Gamma(.)$ denotes the gamma function. As a property for the Riemann-Liouville fractional integration we have

$$ {}_0I_t^{\alpha} {}_0I_t^{\beta} f(t) = {}_0I_t^{\alpha+\beta} f(t), \qquad \alpha, \beta > 0. $$

Definition 2.2 The fractional derivative of $f(t)$ in the Riemann-Liouville sense is defined as follows

$$ {}_0^RD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{(dt)^n} [\int_0^t (t-\tau)^{n-\alpha-1} f(\tau) d\tau], \quad n-1 < \alpha < n, n \in N. $$

Definition 2.3 The fractional derivative of $f(t)$ in the Caputo sense is defined as follows

$$ {}_0^cD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \frac{d^n}{(d\tau)^n} f(\tau) d\tau, \quad n-1 < \alpha < n, n \in N. $$

Definition 2.4 Let $f : [0,\infty) \longrightarrow R$, then the conformable fractional derivative of $f(t)$ is defined as follows ([16]).

$$ T_{\alpha} f(t) = \lim_{\varepsilon \to 0} \frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon} \qquad 0 < \alpha < 1, t > 0. \tag{1} $$

We write sometimes $f^{\alpha}(t)$ for $T_{\alpha} f(t)$ to denote conformable fractional derivative of order α, also if $T_{\alpha} f(t)$ exists, then we say f is α-differentiable.

Let $\alpha \in (0,1]$ and f, g be α-differentiable for $t > 0$, then the following properties can be resulted from Definition 2.4. (see [**?**] for more details).

$$ T_{\alpha}(af + bg) = aT_{\alpha}(f) + bT_{\alpha}(g) \qquad a, b \in R, \tag{2} $$

$$ T_{\alpha}(t^p) = pt^{p-\alpha} \qquad p \in R, \tag{3} $$

$$ T_{\alpha}(\lambda) = 0, \qquad \lambda \text{ is a constant number}, \tag{4} $$

$$ T_{\alpha}(fg) = fT_{\alpha}(g) + gT_{\alpha}(f), \tag{5} $$

$$ T_{\alpha}(\frac{f}{g}) = \frac{gT_{\alpha}(f) - fT_{\alpha}(g)}{g^2}. \tag{6} $$

Moreover, in the case that f is a differentiable function, one can prove that

$$ T_{\alpha} f(t) = t^{1-\alpha} \frac{df}{dt}. \tag{7} $$

As a special case for certain trigonometric functions, the following formulas can be easily obtained.

$$ T_{\alpha}(sinat) = at^{1-\alpha} cosat, a \in R, \tag{8} $$

$$ T_{\alpha}(cosat) = -at^{1-\alpha} sinat, a \in R. \tag{9} $$

Definition 2.5 Let $f : (0,t) \longrightarrow R$ be a continuous function and $\alpha \in (0,1)$, the conformable α-fractional integral of the function f is defined as follows:

$$ I_{\alpha}^{a} f(t) = I_1^a(t^{\alpha-1} f) = \int_a^t \tau^{\alpha-1} f(\tau) d\tau, \ a \geq 0. \tag{10} $$

Theorem 2.6 Let $f : [0,\infty] \longrightarrow R$ be a α-differentiable function. Let g be a function defined in the range of f and also differentiable, then we have the following rule:

$$T_\alpha(fog) = (T_\alpha f)(g).(T_\alpha g).g^{\alpha-1} \tag{11}$$

Proof: (see [17]).
Theorem 2.7 Let f be a α- differentiable function for $t > a$,$(a \geq 0)$ and $0 < \alpha \leq 1$, then

$$I_\alpha^a T_\alpha f(t) = f(t) - f(a). \tag{12}$$

Proof.: Since f is α- differentiable function, by (10) and (7) we have

$$I_\alpha^a T_\alpha f(t) = \int_a^t (\tau-a)^{\alpha-1} T_\alpha f(\tau) d\tau = \int_a^t (\tau-a)^{\alpha-1}(\tau-a)^{1-\alpha} \frac{df(\tau)}{d\tau} d\tau = f(t) - f(a).$$

Now, we recall that by the Riesz representation theorem [10,24], for a function $f \in C[0,1]$, there exists a Borel measure μ_α such that

$$\mu_\alpha(f) = \int_0^1 \tau^{\alpha-1} f(\tau) d\tau\ , \qquad 0 < \alpha < 1, \tag{13}$$

where μ_α is a positive and linear measure.

3 Problem Statement

Consider the following nonlinear FOCP:

$$\min \quad I(x(.),u(.)) = \textstyle\int_J f_0(t,x(t),u(t))dt \tag{14}$$

subject to

$$x^\alpha(t) = g(t,x(t),u(t)),\ (t,x(t),u(t)) \in \Omega, \tag{15}$$

$$x(t_0) = x^0, \qquad x(t_f) = x^1, \tag{16}$$

where $0 < \alpha < 1$. Let the trajectory (state) $x(t)$ and the control function $u(t)$ be vectors in R^n and R^m, respectively, and let t be a real non-negative number. Now consider:

$(i)$$J$=$[t_0,t_f]$ with $t_0 < t_f$. This is the time interval in which the FOCP will evolve.
(ii)A bounded, closed, pathwise-connected set A in R^n. The trajectory $x(t)$ is constrained to stay in A for $t \in J$.
(iii)A bounded, closed subset U in R^m. The control functions are taken values in U.
$(iv)$$\Omega = J \times A \times U$, and $g : \Omega \longrightarrow R^n$, is a continuous function.

without loss of generality, J is considered as $J = [0,1]$.
Definition 3.1 A pair $p = (x(.),u(.))$, is said to be admissible for the problem (15)- (16) if for all $t \in J$ the trajectory function $x(t) \in A$ is absolutely continuous and the control function $u(t) \in U$ is Lebesgue measurable, also the constrains of problem (15)- (16) are satisfied.

We denote by W the set of admissible pairs. If we find $p^* = (x^*(.),u^*(.)) \in W$ such that minimizes the performance criterion (14), then p^* is called as optimal solution.

Now, we use an embedding process into a space of measures. In fact, at this stage an admissible pair $p = (x(.),u(.))$ can be considered as something else, that is, a transformation can be established between the admissible pairs and some other entities, and show that this transformation is an injection; one-one mapping, but the new problem is a linear programming (LP) problem. So one can use all the benefits of solving LP problem. In the following paragraphs this embedding process is described precisely.

Suppose that $p = (x(.),u(.))$ is an admissible pair for the problem (15)- (16) and B is an open ball in $\mathbb{R}^{n+1}$ containing $J \times A$. Let $\phi \in C^\alpha(B)$, where $C^\alpha(B)$ is the space of all real-valued continuously α- differentiable functions on B in the sense of conformable fractional derivative, such that the first derivative is also bounded. Since monomial functions are dense in the space of $C^{\alpha}(B)$,(see Theorem 2.1 from [16] and Weierstrass theorem in [24]), thus monomials depended on variables t and (or) x, can be considered as functions ϕ. Define ϕ^{g} by

$$\begin{aligned}\phi^{g}(t,x(t)) &= \phi^{\alpha}(t,x(t))\\ &= \frac{\partial^{\alpha}\phi}{(\partial t)^{\alpha}} + \frac{\partial^{\alpha}\phi}{(\partial x)^{\alpha}}\frac{\partial^{\alpha}x}{(\partial t)^{\alpha}}\\ &= \frac{\partial^{\alpha}\phi}{(\partial t)^{\alpha}} + \frac{\partial^{\alpha}\phi}{(\partial x)^{\alpha}}T_{\alpha}(x(t))x(t)^{\alpha-1}\\ &= \frac{\partial^{\alpha}\phi}{(\partial t)^{\alpha}} + [\frac{\partial^{\alpha}\phi}{(\partial x)^{\alpha}}x^{\alpha}(t)x(t)^{\alpha-1}]\\ &= \frac{\partial^{\alpha}\phi}{(\partial t)^{\alpha}} + [\frac{\partial^{\alpha}\phi}{(\partial x)^{\alpha}}g(t,x(t),u(t))]x(t)^{\alpha-1}, \qquad (17)\end{aligned}$$

where the right hand side of (17) can be obtained by applying (11) and replacing $x^{\alpha}(t)$ by $g(t,x(t),u(t))$. One needs to remember that $p=[x(.),u(.)]$ is an admissible pair. Now, by using respectively equations (10) and (7), we have

$$\begin{aligned}I_{\alpha}(\phi^{g}) = \int_0^1 \tau^{\alpha-1}\phi^{g}(\tau,x(\tau))d\tau &= \int_0^1 \tau^{\alpha-1}\phi^{\alpha}(\tau,x(\tau))\, d\tau\\ &= \int_0^1 \tau^{\alpha-1}(\tau^{1-\alpha}\frac{d\phi(\tau,x(\tau)}{d\tau})d\tau = \phi(1)-\phi(0) = \Delta\phi.\end{aligned} \qquad (18)$$

Let consider special space $D^{\alpha}(J^0)$, which is the space of all real-valued α-differentiable functions $\Psi(t)$ on J^0 in the sense of conformable fractional derivative with compact support in $J^0=(0,1)$, that is $\Psi(0)=\Psi(1)=0$. The functions $\Psi(t)\in D^{\alpha}(J^0)$ are formally considered as

$$\begin{aligned}\Psi(t) &= sin(2\pi rt)\ , \qquad r=1,2,...\\ \Psi(t) &= 1-cos(2\pi rt)\ , \qquad r=1,2,...\end{aligned}$$

Define

$$\begin{aligned}\Psi^{j}(t,x(t),u(t)) &= T_{\alpha}(x_j\Psi(t))\\ &= x_j(t)\ T_{\alpha}\Psi(t)+g_j(t,x,u)\Psi(t), \qquad j=1,2,...,n\end{aligned} \qquad (19)$$

where the right side of (19) can be obtained by applying equation (5) and replacing $T_{\alpha}(x_j)$ by $g_j(t,x(t),u(t))$. Since, Ψ has compact support in $J^0=(0,1)$, one can get:

$$\begin{aligned}I_{\alpha}(\Psi^{j}(t,x(t),u(t)) = \int_0^1 \tau^{\alpha-1}\Psi^{j}(\tau,x(\tau),u(\tau))d\tau &= \int_0^1 \tau^{\alpha-1}T_{\alpha}(x_j\Psi(t))\, d\tau\\ &= \int_0^1 \tau^{\alpha-1}(\tau^{1-\alpha}\tfrac{d(x_j\Psi(\tau))}{d\tau})\, d\tau\\ &= \int_0^1 d(x_j\Psi(t))dt = \Psi(1)-\Psi(0) = 0.\\ & j=1,2,...,n.\end{aligned} \qquad (20)$$

As a special case of choosing functionals on space Ω, Walsh functions are introduced as follows which are dependent only on the variable t:

$$\theta_s = \begin{cases}1\ , & t\in[(s-1)/L,s/L)\\ 0\ , & o.w\end{cases} \qquad (21)$$

where $L\in\mathbb{N}$ and $s=1,\cdots,L$. For these functions we have

$$I_{\alpha}(\theta_s) = \int_0^1 \tau^{\alpha-1}\theta_s(\tau)d\tau = \int_{(s-1)/L}^{s/L}\tau^{\alpha-1}\, d\tau = \tfrac{(s/L)^{\alpha}}{\alpha} - \tfrac{((s-1)/L)^{\alpha}}{\alpha} = \Delta\theta_s. \qquad (22)$$

Now remember the classical fractional optimal control problem (FOCP) (14)-(16). Many difficulties may arise in solving this FOCP. The set of admissible pairs W may be empty; the functional measuring performance (14) may not achieve its minimum even if the set W is nonempty. The necessary conditions for optimality is not clear. To overcome these difficulties we attempt to change the problem and consider the admissible pair $p=(x(.),u(.))$ as some new objective. Let $f\in C^{\alpha}(\Omega)$, the following mapping is considered:

$$\Lambda_p: f\longrightarrow \int_0^1 \tau^{\alpha-1}f(\tau,x(\tau),u(\tau))d\tau, \qquad p\in W, \qquad (23)$$

where this mapping defines a positive linear functional on $C^{\alpha}(\Omega)$, that identify each admissible pair p by Λ_p. Now each linear positive functional Λ_p on $C^{\alpha}(\Omega)$, can be represented by a positive Borel measure ν_α, such that

$$\Lambda_p(f) = \int_0^1 \tau^{\alpha-1} f(\tau, x(\tau), u(\tau)) d\tau \equiv \nu_\alpha(f),$$

(see [9,11]). Using these concepts, we can put the FOCP (14)-(16) in its definite form. The positive linear functional Λ_p will be replaced by representing Borel measure ν_α, thus by considering the cost functional, we seek a minimizing measure $\nu_\alpha^* \in M^+(\Omega)$ (the space of all positive measures on Ω) which defined by the functional

$$I : \nu_\alpha \longrightarrow \nu_\alpha(f_0^*) \tag{24}$$

where

$$\nu_\alpha(f_0^*) = \nu_\alpha(t^{1-\alpha}\ f_0),$$

defined over the set of positive Borel measures on Ω, which satisfy

$$\begin{array}{ll} \nu_\alpha(\phi^g) = \Delta\phi,\ \phi \in C^{\alpha}(B), & \\ \nu_\alpha(\Psi^j) = 0, \quad \Psi \in D^{\alpha}(J^0), & \qquad j = 1, 2, ..., n \\ \nu_\alpha(\theta_s) = \Delta\theta_s,\ s = 1, \cdots, L. & \end{array} \tag{25}$$

So we choose the nonclassical problem (24)-(25) to replace the classical problem (14)-(16). One may consider (24)-(25) as a linear programming problem. We examine this problem in the next section.

4 Linear Programming

The set of all positive Borel measures of Ω satisfying (24)-(25) is defined as Q. If one consider the space $M^+(\Omega)$ with the *weak** $-$ *topology*, it can be seen from [12] that Q is compact. In the sense of this topology, the functional $I : Q \longrightarrow R$ defined by (24) is a linear and continuous functional on the compact set Q, thus attains its minimum on Q.

The linear programming (LP) problem (24)-(25) consisting in minimizing $\nu_\alpha(f_0^*)$ on the set of measures Q of $M^+(\Omega)$ described by (25) is an infinite-dimensional LP, the underlying space $M^+(\Omega)$ is not finite-dimensional and the number of constraints (25) is not finite. So the LP problem(24)-(25) is an infinite-dimensional linear programming problem. In this section we are going to approximate this infinite-dimensional LP by a finite-dimensional.

Let the set $\{\phi_i; i = 1, 2, \cdots\}$ be countable set of functions whose linear combinations are uniformly dense in $C^{\alpha}(B)$, remember that the functions $\Psi^j, j = 1, 2, ..., n$ and $\theta_s, s = 1, ..., L$ defined respectively in (19) and (21), are special cases of the first functions $\phi_i, i = 1, 2, \cdots$.

Theorem 4.1. Consider the LP problem of minimizing $\nu_\alpha(f_0^*)$ over the set $Q(M_1, M_2, L)$ of measures in $M^+(\Omega)$ satisfying

$$\begin{cases} \nu_\alpha(\phi_i^g) = \Delta\phi_i,\ i = 1, \cdots, M_1, \\ \nu_\alpha(\Psi_k^j) = 0, \quad k = 1, \cdots, M_2, j = 1, \cdots, n \\ \nu_\alpha(\theta_s) = \Delta\theta_s,\ s = 1, \cdots, L. \end{cases} \tag{26}$$

If $M_1 \to \infty$, $M_2 \to \infty$ and $L \to \infty$ then

$$\nu_{\alpha_{Q(M_1,M_2,L)}}(f_0^*) \longrightarrow \nu_{\alpha_Q}(f_0^*)$$

Proof: (see [13]).

The first stage of the approximation scheme has been successfully completed. We have limited the number of constraints in the original linear program, but the underlying space $Q(M_1, M_2, L)$ is not finite-dimensional. In fact now we have a semi-infinite dimensional LP problem. It is possible, though, to develop a finite dimensional LP whose solution can be used to construct the optimal pair $p^* = (x^*(.), u^*(.))$.

Suppose $z = (t, x, u) \in \Omega$. A unitary atomic measure $\delta(z) \in M^+(\Omega)$, which is supported by the singleton set $\{z\}$, can be characterized by

$$\delta(z) f = f(z), \qquad f \in C(\Omega), \qquad z \in \Omega.$$

Now consider the following important theorem (see [10]).

Theorem 4.2. The optimal measure in the set $Q(M_1, M_2, L)$ at which the functional $\nu_\alpha \longrightarrow \nu_\alpha(f_0^*)$ attains its minimum has the form

$$\nu_\alpha^* = \sum_{k=1}^{M_1+M_2+L} \alpha_k^* \delta(z_k^*) \tag{27}$$

where the coefficients $\alpha_k^* \geq 0$ and $z_k^* \in \Omega$ are unknowns for $k = 1, \cdots, M_1 + M_2 + L$.

Thus by using(27), the LP problem(24)-(25) changes to the following nonlinear programming problem:

$$Min \ \sum_{k=1}^{M_1+M_2+L} \alpha_k^* f_0^*(z_k^*) \tag{28}$$

subject to

$$\begin{cases} \sum_{k=1}^{M_1+M_2+L} \alpha_k^* \phi_i^g(z_k^*) = \Delta \phi_i & i = 1,2,...,M_1 \\ \sum_{k=1}^{M_1+M_2+L} \alpha_k^* \Psi_h^j(z_k^*) = 0 & h = 1,2,...,M_2, j = 1,2,...,n \\ \sum_{k=1}^{M_1+M_2+L} \alpha_k^* \theta_s(z_k^*) = \Delta \theta_s & s = 1,2,...,L \\ \alpha_k^* \geq 0 & k = 1,2,...,M_1+M_2+L \end{cases} \tag{29}$$

where $z_k^* \in \Omega$.

Let $\omega = \{z_1, \cdots, z_N\}$ be a countable approximately dense subset of Ω. A measure $\nu^* \in M^+(\Omega)$ as a good approximation for ν_α^* can be found such that

$$\upsilon^* = \sum_{k=1}^{N} \alpha_k^* \delta(z_k), \tag{30}$$

where the coefficients α_k^* are the same as in the optimal measure ν_α^* in (27), and $z_k \in \omega, k = 1,2,...,N$. (see [9]).

By selecting $z_i; i = 1,...,N$ for sufficiently large N in ω and considering (30), then the nonlinear optimization problem (28)-(29) can be approximated by the following LP problem:

$$Min \ \sum_{j=1}^{N} \alpha_j^* f_0^*(z_j) \tag{31}$$

subject to

$$\begin{cases} \sum_{j=1}^{N} \alpha_j^* \phi_i^g(z_j) = \Delta \phi_i & i = 1,2,...,M_1 \\ \sum_{j=1}^{N} \alpha_j^* \Psi_k(z_j) = 0 & k = 1,2,...,M_2 \\ \sum_{j=1}^{N} \alpha_j^* \theta_s(z_j) = a_s & s = 1,2,...,L \\ \alpha_j^* \geq 0. \end{cases} \tag{32}$$

By the solution of the finite dimensional linear programming problem (31)-(32) one can find the coefficients α_j^* $(j = 1, \cdots, N)$, and using the method described in [9], the piecewise-constant optimal control function $u^*(.)$ can be obtained. Finally from the following fractional dynamical system

$$\begin{cases} x^\alpha(t) = g(t, x(t), u^*(t)), \ t \in [0,1] \\ x(0) = x^0, x(1) = x^1. \end{cases} \tag{33}$$

the optimal trajectory function $x^*(.)$ can be obtained.

5 Numerical Examples

In this section we solve some numerical examples by applying presented method in sections 3 and 4. After obtaining the optimal piecewise control function $u^*(.)$, we used Matlab Software to solve fractional order differential equation (33). In these examples the objective values I^* for $\alpha = 1$ and the optimal control functions $u^*(.)$ for some values of α are shown. Also by varying the values of α we obtain different trajectory functions $x^*(.)$. These examples demonstrate the efficiency of measure theory technique for solving linear and nonlinear FOCPs.

Example 1

Consider the following fractional optimal control problem (see [25]):

$$Min \ I = 1/2 \int_0^1 (x^2(t) + u^2(t)) d(t), \tag{34}$$

subject to

$$x^{\alpha}(t) = -x(t) + u(t),$$
$$x(0) = 1,$$
$$0 < \alpha \leq 1.$$

Let $A = [0,1]$, $U = [-0.5, 0.5]$, we have chosen $M_1 = 4$, $M_2 = 0$, $L = 10$.
By using presented method, the objective value for $\alpha = 1$ is found as $I^* = 0.1943$. The obtained piecewise continuous control functions for $\alpha = 0.8, 0.9, 1$ are shown respectively in Figures 1-3. In Figures 4, the state functions $x(.)$ are shown for $\alpha = 0.8, 0.9, 1$.

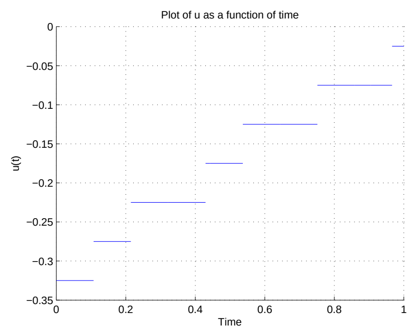

Fig. 1. Approximate solution of $u(.)$ for $\alpha = 0.8$ in Example 1 .

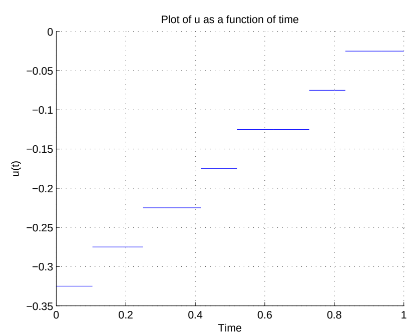

Fig. 2. Approximate solution of $u(.)$ for $\alpha = 0.9$ in Example 1 .

Example 2

Consider the following FOCP (see [10]):

$$Min\, I = \int_0^1 x^2(t)d(t), \tag{35}$$

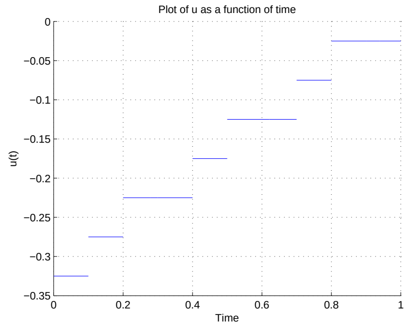

Fig 3. Approximate solution of $u(.)$ for $\alpha = 1$ in Example 1 .

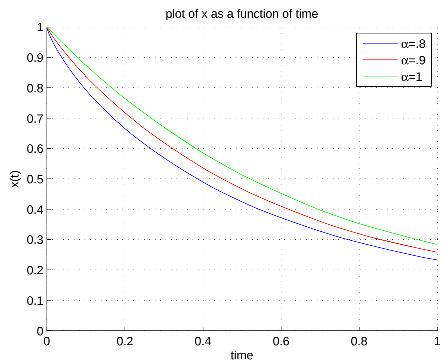

Fig. 4. Approximate solution of $x(.)$ for $\alpha = 0.8, 0.9, 1$ in Example 1.

subject to

$$
\begin{aligned}
&x^{(\alpha)}(t) = u(t),\\
&x(0) = 0, x(1) = 0.5,\\
&0 < \alpha \leq 1.
\end{aligned}
$$

Let $A = [0,1]$, $U = [0,1]$, we have chosen $M_1 = 2$, $M_2 = 8$, $L = 10$.
The value of the objective function for $\alpha = 1$ is found as $I^* = 0.0359$. In Figures 5 and 6 the optimal control function $u(.)$ for $\alpha = 1$ and trajectory functions $x(.)$ for $\alpha = 0.8, 0.9, 1$ are shown, respectively.

Example 3

Consider a two-dimensional nonlinear FOCP as follows (see [10]):

$$
Min\ I = \int_0^1 (x_1(t)^2 + x_2(t)^2) d(t), \tag{36}
$$

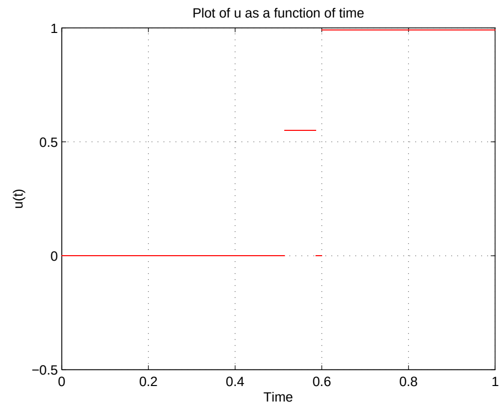

Fig. 5. Approximate solution of $u(.)$ for $\alpha = 1$ in Example 2 .

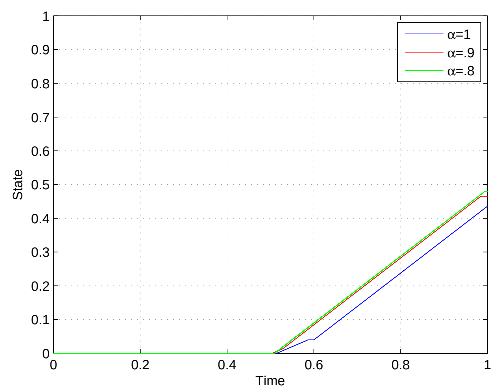

Fig. 6. Approximate solution of $x(.)$ for $\alpha = 0.8, 0.9, 1$ in Example 2 .

subject to

$$
\begin{aligned}
&x_1^{(\alpha)}(t) = x_2(t),\\
&x_2^{(\alpha)}(t) = (10x_1^3(t) + u(t)),\\
&x_1(0) = 0, x_1(1) = 0.1,\\
&x_2(0) = 0, x_2(1) = 0.3,\\
&0 < \alpha \le 1.
\end{aligned}
$$

Let $A = [0,1] \times [0,1]$, $U = [0,1]$, we have chosen $M_1 = 6$, $M_2 = 8$, $L = 10$. The objective value for $\alpha = 1$ is found as $I^* = 0.0311$. In Figures 7- 9, respectively we have shown the optimal control $u(.)$ for $\alpha = 1$ and trajectory functions $x_1(.)$ and $x_2(.)$ for $\alpha = 0.8, 0.9, 1$.

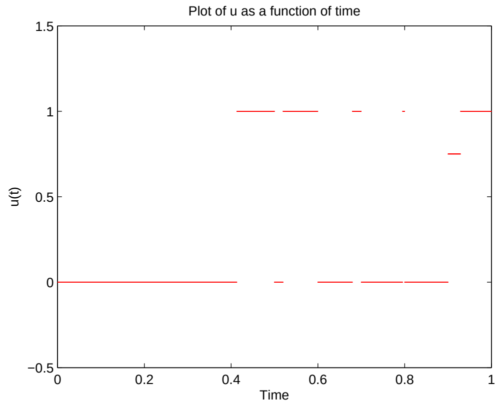

Fig. 7. Approximate solution of $u(.)$ for $\alpha = 1$ in Example 3 .

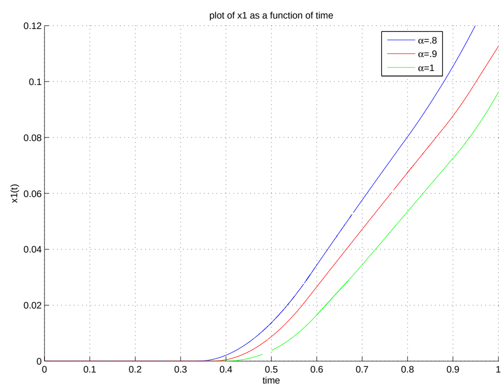

Fig. 8. Approximate solution of $x_1(.)$ for $\alpha = 0.8, 0.9, 1$ in Example 3.

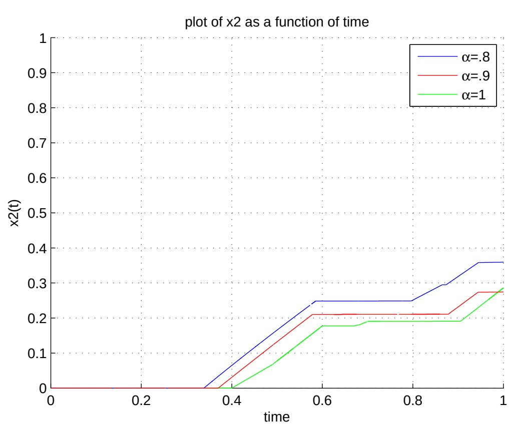

Fig. 9. Approximate solution of $x_2(.)$ for $\alpha = 0.8, 0.9, 1$ in Example 3.

Example 4

This example is linear time invariant fractional optimal control problem that can be found in [26] and stated as follows. We are going to find the optimal pair $p^* = (x^*(.), u^*(.))$, which minimizes the quadratic performance index

$$I = 1/2 \int_0^1 (x^2(t) + u^2(t))d(t), \tag{37}$$

and satisfies:

$$\begin{aligned} &x^{\alpha}(t) = -x(t) + u(t), \\ &x(0) = 1, \\ &x(1) = free, \\ &0 < \alpha \leq 1. \end{aligned} \tag{38}$$

We examined the solution of this example for different values of α. For this purpose, α is taken between 0.1 and 1. We need to mention that in this example, the final state is free, so the transversality conditions are totally different. To consider this situation in linear programming problem (31)-(32), in the right-hand side of the first M_1 equations in (32), where we have $\Delta\phi_i = \phi_i(1) - \phi_i(0)$; $i = 1, 2, ...M_1$, one needs to assume $\phi_i(1)$'s are also unknown variables. These variables must be found from solving linear programming problem (31)-(32). As Example 1, we assumed $A = [0, 1]$, $U = [-0.5, 0.5]$, and we have chosen $M_1 = 4$, $M_2 = 0$, $L = 10$. The state functions for different values of α are shown in Figure 10. The piecewise continuous control functions for some α ($\alpha = 0.1, \alpha = 0.3, \alpha = 0.5$) are shown respectively in Figures 11-13. The state functions shown in Figure 10, compared by Figure 1 in [26], show that the presented method, though simple and straight forward, achieves good results.

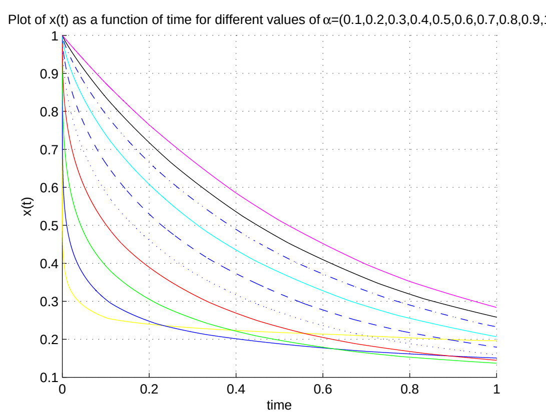

Fig. 10. Approximate solution of $x(.)$ in Example 4 for different values of α (yellow: α = 0.1, blue: α = 0.2, green:α= 0.3, red:α = 0.4, dotted-blue: α = 0.5, dashed-blue: α = 0.6, solid-cyan: α = 0.7, dashed-dotted-blue:α = 0.8, solid-black:α = 0.9, solid-magenta:α = 1).

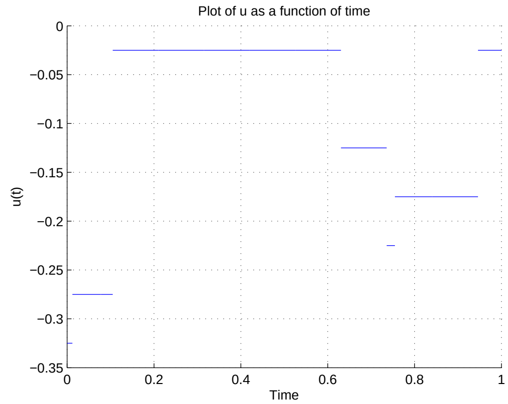

Fig. 11. Approximate solution of $u(.)$ for $\alpha = 0.1$ in Example 4.

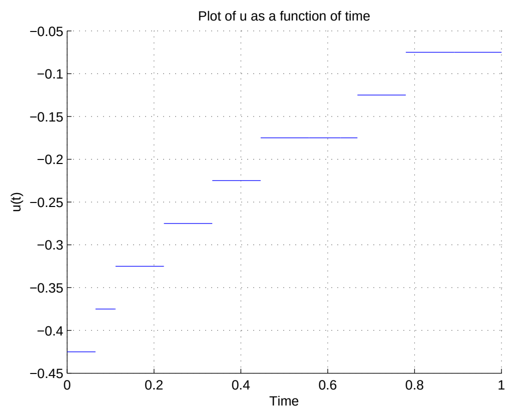

Fig. 12. Approximate solution of $u(.)$ for $\alpha = 0.3$ in Example 4.

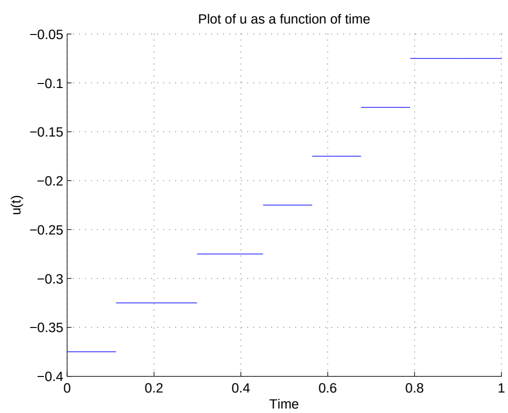

Fig. 13. Approximate solution of $u(.)$ for $\alpha = 0.5$ in Example 4.

6 Conclusion

In this paper, a direct extension of measure theory approach to solve nonlinear fractional optimal control problems is illustrated. By applying an embedding process and using the properties of positive Borel measure, functional analysis and linear programming, we present a new and useful technique for solving FOCPs. The most important characteristic of the proposed measure theory approach is its simplicity in dealing with nonlinear FOCPs. Computer simulations for different examples show that the proposed method is easy, linear and less time-consuming.

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