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Some Symmetric Identities for the Generalized Hermite-Euler and Hermite-Genocchi Polynomials

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Abstract: In this paper, we introduce a new class of generalized Hermite-Euler and Hermite-Genocchi polynomials and derive some symmetric identities by applying the generating functions. Also, we obtain some potentially useful relations for the Bernoulli polynomials, Euler polynomials, power sum, alternating sum and Genocchi numbers. These results extend some known summations and identities of generalized Hermite-Euler and Hermite-Genocchi polynomials studied by Dattoli et al. [3], Pathan and Khan [9] and Khan [5].

Keywords: Hermite polynomials, Euler polynomials, Hermite-Euler polynomials, Genocchi polynomials, Hermite-Genocchi polynomials, power sum, alternating sum.

1 Introduction

The 2-variable Hermite Kampé de Fériet polynomials (2VHKdFP) $H_n(x,y)$ [1,3] are defined as:

$$H_n(x,y) = n! \sum_{r=0}^{[\frac{n}{2}]} \frac{y^r x^{n-2r}}{r!(n-2r)!}, \tag{1.1}$$

with the following generating function:

$$e^{xt+yt^2} = \sum_{n=0}^{\infty} H_n(x,y) \frac{t^n}{n!}. \tag{1.2}$$

On replacing x by $2x$ and setting $y=-1$, equation (1.2) reduces to the classical Hermite polynomials $H_n(x)$ (see [2]).

The generalized Bernoulli, Euler and Genocchi polynomials of (real or complex) order α are usually defined by means of the following generating functions (see [5]-[11]):

$$\left(\frac{t}{e^t-1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x) \frac{t^n}{n!}, (|\, t \,|< 2\pi; 1^{\alpha}=1), \tag{1.3}$$

$$\left(\frac{2}{e^t+1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x) \frac{t^n}{n!}, (|\, t \,|< \pi; 1^{\alpha}=1) \tag{1.4}$$

and

$$\left(\frac{2t}{e^t+1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} G_n^{(\alpha)}(x) \frac{t^n}{n!}, (|\, t \,|< \pi; 1^{\alpha}=1). \tag{1.5}$$

so that obviously

$B_n(x) = B_n^1(x), E_n(x) = E_n^1(x)$ and $G_n(x) = G_n^1(x), (n \in \mathbb{N})$,

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ $(\mathbb{N} = 1,2,3,\cdots)$.

For each integer $k \in \mathbb{N}_0$, $S_k(n)$ is defined by

$$S_k(n) = \sum_{i=0}^{n} i^k, \tag{1.6}$$

is called sum of integer powers or simply power sum.

The exponential generating function for $S_k(n)$ is given by [4]:

$$\sum_{k=0}^{\infty} S_k(n) \frac{t^k}{k!} = 1 + e^t + e^{2t} + \cdots + e^{nt} = \frac{e^{(n+1)t}-1}{e^t-1}. \tag{1.7}$$

For $k \in \mathbb{N}_0$ and $n \in \mathbb{N}$, $T_k(n)$ is defined by

$$T_k(n) = \sum_{i=0}^{n} (-1)^i i^k, \tag{1.8}$$

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is called the alternating sum of integer powers.

The exponential generating function $T_k(n)$ is defined by

$$\sum_{k=0}^{\infty} T_k(n)\frac{t^k}{k!} = \frac{1-(-1)^n e^{(n+1)t}}{1+e^t}. \tag{1.9}$$

Due to great importance and applications of Hermite-Euler and Hermite-Genocchi polynomials in several diverse fields (for example, number theory, combinatorics, classical and numerical analysis etc.), a number of authors have introduced and investigated several generalizations of these polynomials. In sequel of such type of works, in this paper, we introduce a new generalization of Hermite-Euler and Hermite-Genocchi polynomials. We also establish some elementary properties (for example, symmetric identities connection and summation formulae) for the polynomials introduced here by the approach given in the recent works of Yang et al. [11], Khan et al. [6] and Pathan and Khan [9].

2 Some symmetry identities for the generalized Hermite-Euler polynomials

In this section, we introduce the generalized Hermite-Euler polynomials ${}_HE_n^{(\alpha)}(x,y)$ for a real or complex parameter α defined by means of the following generating function defined in a suitable neighborhood of $t=0$:

$$\left(\frac{2}{e^t+1}\right)^{\alpha} e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HE_n^{(\alpha)}(x,y)\frac{t^n}{n!}, \tag{2.1}$$

so that

$${}_HE_n^{(\alpha)}(x,y) = \sum_{s=0}^{n}\binom{n}{s} E_{n-s}^{(\alpha)} H_s(x,y).$$

Notice that (2.1) is the generalization of the following function defined by Dattoli et al. [3, p.386(1.6)]:

$$\left(\frac{2}{e^t+1}\right) e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HE_n(x,y)\frac{t^n}{n!}. \tag{2.2}$$

Theorem 2.1. Let a and b be positive integers with same parity, then

$$\sum_{i=0}^{a-1}(-1)^i a^n {}_HE_n^{(\alpha)}(bx+\frac{b}{a}i, b^2y)$$

$$= \sum_{i=0}^{b-1}(-1)^i b^n {}_HE_n^{(\alpha)}(ax+\frac{a}{b}i, a^2y). \tag{2.3}$$

Proof. Let us consider

$$f(t) = \left(\frac{2}{e^{at}+1}\right)^{\alpha} e^{abxt+a^2b^2yt^2}\frac{1+(-1)^{a+1}}{e^{bt}+1}e^{abt} \tag{2.4}$$

$$= \left(\frac{2}{e^{at}+1}\right)^{\alpha} e^{abxt+a^2b^2yt^2}\frac{1-(-e^{-bt})^a}{e^{bt}+1}$$

$$= \left(\frac{2}{e^{at}+1}\right)^{\alpha} e^{abxt+a^2b^2yt^2}\sum_{i=0}^{a-1}(-e^{bt})^i$$

$$= \left(\frac{2}{e^{at}+1}\right)^{\alpha} e^{a^2b^2yt^2}\sum_{i=0}^{a-1}(-1)^i e^{(bx+\frac{b}{a}i)at}$$

$$f(t) = \sum_{n=0}^{\infty}\sum_{i=0}^{a-1}(-1)^i a^n {}_HE_n^{(\alpha)}(bx+\frac{b}{a}i, b^2y)\frac{t^n}{n!}. \tag{2.5}$$

Since $(-1)^{a+1} = (-1)^{b+1}$, the expression for $f(t) = \left(\frac{2}{e^{at}+1}\right)^{\alpha} e^{abxt+a^2b^2yt^2}\frac{1+(-1)^{a+1}}{e^{bt}+1}e^{abt}$ Therefore, we obtain the following power series expansion for f(t) by symmetry

$$f(t) = \sum_{n=0}^{\infty}\sum_{i=0}^{b-1}(-1)^i b^n {}_HE_n^{\alpha}(ax+\frac{a}{b}i, a^2y)\frac{t^n}{n!}. \tag{2.6}$$

Equating the coefficients of $\frac{t^n}{n!}$ in (2.5) and (2.6), we get the desired result (2.3).

Remark 2.1. On setting $\alpha = 1$, $y = 0$, Theorem 2.1 reduces to the known result of Yang et al. [11, p.459(17)].

Corollary 2.1. For $\alpha = 1$ in Theorem 2.1, we obtain the following result:

$$\sum_{i=0}^{a-1}(-1)^i a^n {}_HE_n(bx+\frac{b}{a}i, b^2y) = \sum_{i=0}^{b-1}(-1)^i b^n {}_HE_n(ax+\frac{a}{b}i, a^2y). \tag{2.7}$$

Theorem 2.2. Let a and b be positive integers with same parity, then the following identity holds true:

$$\sum_{k=0}^{n}\binom{n}{k} a^k b^{n-k} {}_HE_k^{(\alpha)}(bx, b^2y)T_{n-k}(a)$$

$$= \sum_{k=0}^{n}\binom{n}{k} a^{n-k} b^k {}_HE_k^{(\alpha)}(ax, a^2y)T_{n-k}(b). \tag{2.8}$$

Proof. Let us consider

$$f(t) = \left(\frac{2}{e^{at}+1}\right)^{\alpha} e^{abxt+a^2b^2yt^2}\frac{1+(-1)^{a+1}}{e^{bt}+1}e^{abt} \tag{2.9}$$

$$= \left(\frac{2}{e^{at}+1}\right)^{\alpha} e^{abxt+a^2b^2yt^2}\frac{1-(-e^{-bt})^a}{e^t+1}e^{at}$$

$$= \sum_{k=0}^{\infty} {}_HE_k^{(\alpha)}(bx, b^2y)\frac{(at)^k}{k!}\sum_{n=0}^{\infty} T_n(a)\frac{(bt)^n}{n!}$$

Now replacing n by $n-k$ in the R.H.S. of above equation, we get

$$f(t) = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k} {}_HE_k^{(\alpha)}(bx, b^2y)a^k b^{n-k} T_{n-k}(a)\frac{(t)^n}{n!}. \tag{2.10}$$

Since $(-1)^{a+1} = (-1)^{b+1}$, the expression for $f(t) = \left(\frac{2}{e^{at}+1}\right)^{\alpha} e^{abxt+a^2b^2yt^2} \frac{1+(-1)^{a+1}}{e^{bt}+1} e^{abt}$ Therefore, we obtain the following power series expansion for f(t) by symmetry

$$f(t) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} {}_HE_k^{(\alpha)}(ax, a^2y) b^k a^{n-k} T_{n-k}(b) \frac{(t)^n}{n!}. \tag{2.11}$$

Now equating the coefficients of $\frac{t^n}{n!}$ in the last two expression for f(t), we get our desired result.

Remark 2.2. For $\alpha = 1, y = 0$, Theorem 2.2 reduces to the known result of Yang et al. [[11],p.460(18)].

Corollary 2.2. Taking $\alpha = 1$ in Theorem 2.2, we deduce the following result:

$$\sum_{k=0}^{n} \binom{n}{k} {}_HE_k(bx, b^2y) a^k b^{n-k} T_{n-k}(a)$$

$$= \sum_{k=0}^{n} \binom{n}{k} {}_HE_k(ax, a^2y) b^k a^{n-k} T_{n-k}(b). \tag{2.12}$$

Theorem 2.3. Let a and b be positive integers and a be even, then the following identity holds true:

$$\sum_{i=0}^{b-1} {}_HE_n(ax + \frac{a}{b}i, a^2y) b^n$$

$$= \sum_{i=0}^{a-1} (-1)^{i+1} \frac{2}{n+1} {}_HB_{n+1}(bx + +\frac{b}{a}i, b^2y) a^n. \tag{2.13}$$

Proof. Let us consider

$$g(t) = \left(\frac{2}{e^{bt}+1}\right) e^{abxt+a^2b^2yt^2} \frac{1-e^{abt}}{1-e^{at}}$$

$$= \left(\frac{2}{e^{bt}+1}\right) e^{abxt+a^2b^2yt^2} \sum_{i=0}^{b-1} e^{ait}$$

$$= \left(\frac{2}{e^{bt}+1}\right) e^{a^2b^2yt^2} \sum_{i=0}^{b-1} e^{(ax+\frac{a}{b}i)bt}$$

$$g(t) = \sum_{n=0}^{\infty} \sum_{i=0}^{b-1} {}_HE_n(ax + \frac{a}{b}i, a^2y) \frac{b^n t^n}{n!}. \tag{2.14}$$

On the other hand, considering a is even, we have

$$g(t) = \left(\frac{2}{e^{bt}+1}\right) e^{abxt+a^2b^2yt^2} \frac{1-e^{abt}}{1-e^{at}}$$

$$= -\left(\frac{2}{e^{at}-1}\right) e^{abxt+a^2b^2yt^2} \frac{1-(-e^{bt})^a}{1-(-e^{bt})}$$

$$= -\frac{at}{at}\left(\frac{2}{e^{at}-1}\right) e^{abxt+a^2b^2yt^2} \sum_{i=0}^{a-1} (-e^{bit})$$

$$= -\frac{1}{at}\left(\frac{2}{e^{at}-1}\right) e^{a^2b^2yt^2} \sum_{i=0}^{a-1} (-1)^i e^{(bx+\frac{b}{a})at}$$

$$= -\frac{2}{at} \sum_{i=0}^{a-1} (-1)^i \sum_{n=0}^{\infty} {}_HB_n(bx + \frac{b}{a}i, b^2y) \frac{a^n t^n}{n!}$$

$$= -\frac{2}{at} \sum_{n=0}^{\infty} \sum_{i=0}^{a-1} (-1)^i {}_HB_n(bx + \frac{b}{a}i, b^2y) \frac{a^n t^n}{n!}$$

$$= -2 \sum_{n=0}^{\infty} \sum_{i=0}^{a-1} (-1)^i {}_HB_n(bx + \frac{b}{a}i, b^2y) \frac{a^{n-1} t^{n-1}}{n!}.$$

Replacing n by $n+1$ in the above equation, we get

$$g(t) = -2 \sum_{n=0}^{\infty} \sum_{i=0}^{a-1} (-1)^i \frac{1}{n+1} {}_HB_{n+1}(bx + \frac{b}{a}i, b^2y) \frac{a^n t^n}{n!}. \tag{2.15}$$

On equating the coefficients of $\frac{t^n}{n!}$ in (2.14) and (2.15), we get our required result (2.13).

Remark 2.3. For $y = 0$, Theorem 2.3 reduces to the known result of Yang et al. [[11],p.460(19)].

Theorem 2.4. Let a and b be positive integers with same parity, then

$$\sum_{k=0}^{n} \binom{n}{k} E_{n-k}^{(\alpha-1)}(ay) a^k b^{n-k} \sum_{i=0}^{a-1} (-1)^i {}_HE_k^{(\alpha)}(bx + \frac{b}{a}i, b^2z)$$

$$= \sum_{k=0}^{n} \binom{n}{k} E_{n-k}^{(\alpha-1)}(by) b^k a^{n-k} \sum_{i=0}^{b-1} (-1)^i {}_HE_k^{(\alpha)}(ax + \frac{a}{b}i, a^2z). \tag{2.16}$$

Proof. Let us consider

$$h(t) = \frac{2^{2\alpha-1} e^{ab(x+y)t+a^2b^2zt^2(1+(-1)^{a+1})} e^{abt}}{(e^a t+1)^{\alpha}(e^b t+1)^{\alpha}} \tag{2.17}$$

$$= \left(\frac{2}{e^{at}+1}\right)^{\alpha} e^{abxt+a^2b^2zt^2} \frac{1-(-e^b t)^a}{e^{bt}+1} \left(\frac{2}{e^{bt}+1}\right)^{\alpha-1} e^{abyt}$$

$$= \left(\frac{2}{e^{at}+1}\right)^{\alpha} e^{abxt+a^2b^2zt^2} \sum_{i=0}^{a-1} (-e^{bt})^i \sum_{n=0}^{\infty} E_n^{(\alpha-1)}(ay) \frac{b^n t^n}{n!}$$

$$= \left(\frac{2}{e^{at}+1}\right)^{\alpha} e^{a^2b^2zt^2} \sum_{i=0}^{a-1} (-1)^i e^{(bx+\frac{b}{a}i)at} \sum_{n=0}^{\infty} E_n^{(\alpha-1)}(ay) \frac{b^n t^n}{n!}$$

$$= \sum_{i=0}^{a-1} (-1)^i \sum_{k=0}^{\infty} {}_HE_k^{(\alpha)}(bx + \frac{b}{a}i, b^2z) \frac{a^k t^k}{k!} \sum_{n=0}^{\infty} E_n^{(\alpha-1)}(ay) \frac{b^n t^n}{n!}$$

$$= \sum_{k=0}^{\infty} \sum_{i=0}^{a-1} (-1)^i {}_HE_k^{(\alpha)}(bx + \frac{b}{a}i, b^2z) \frac{a^k}{k!} \sum_{n=0}^{\infty} E_n^{(\alpha-1)}(ay) \frac{b^n t^{n+k}}{n!}.$$

On replacing n by $n-k$ in the above equation, we have

$$h(t) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} E_{n-k}^{(\alpha-1)}(ay) a^k b^{n-k} \sum_{i=0}^{a-1} (-1)^i {}_HE_k^{(\alpha)}(bx + \frac{b}{a}i, b^2z) \frac{t^n}{n!}. \tag{2.18}$$

We may also expand h(t) as:

$$h(t)=\left(\frac{2}{e^{bt}+1}\right)^{\alpha}e^{abxt+a^2b^2zt^2}\frac{1-(-e^{a}t)^{b}}{e^{at}+1}\left(\frac{2}{e^{at}+1}\right)^{\alpha-1}e^{abyt},$$

$$=\left(\frac{2}{e^{bt}+1}\right)^{\alpha}e^{abxt+a^2b^2zt^2}\sum_{i=0}^{b-1}(-e^{at})^{i}\sum_{n=0}^{\infty}E_n^{(\alpha-1)}(by)\frac{a^nt^n}{n!}$$

$$=\left(\frac{2}{e^{bt}+1}\right)^{\alpha}e^{a^2b^2zt^2}\sum_{i=0}^{b-1}(-1)^{i}e^{(ax+\frac{a}{b}i)bt}\sum_{n=0}^{\infty}E_n^{(\alpha-1)}(by)\frac{a^nt^n}{n!}$$

$$=\sum_{i=0}^{b-1}(-1)^{i}\sum_{k=0}^{\infty}{}_HE_k^{(\alpha)}(ax+\frac{a}{b}i,a^2z)\frac{b^kt^k}{k!}\sum_{n=0}^{\infty}E_n^{(\alpha-1)}(by)\frac{a^nt^n}{n!}$$

$$=\sum_{k=0}^{\infty}\sum_{i=0}^{b-1}(-1)^{i}{}_HE_k^{(\alpha)}(ax+\frac{a}{b}i,a^2z)\frac{b^k}{k!}\sum_{n=0}^{\infty}E_n^{(\alpha-1)}(by)\frac{a^nt^{n+k}}{n!}.$$

Further, on replacing n by $n-k$ in the above equation, we have

$$h(t)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}E_{n-k}^{(\alpha-1)}(by)b^ka^{n-k}$$

$$\times\sum_{i=0}^{b-1}(-1)^{i}{}_HE_k^{(\alpha)}(ax+\frac{a}{b}i,a^2z)\frac{t^n}{n!}. \tag{2.19}$$

After, equating the coefficients of $\frac{t^n}{n!}$ from (2.18) and (2.19), we get (2.16).

Remark 2.4. On putting $z=0$, Theorem 2.4 reduces to a the known result of Yang et al. [[11],p.461(21)].

Corollary 2.4. On setting $\alpha=1$ and $y=0$ in Theorem 2.4, we get the following result:

$$\sum_{k=0}^{n}\binom{n}{k}a^kb^{n-k}\sum_{i=0}^{a-1}(-1)^{i}{}_HE_k(bx+\frac{b}{a}i,b^2z)$$

$$=\sum_{k=0}^{n}\binom{n}{k}b^ka^{n-k}\sum_{i=0}^{b-1}(-1)^{i}{}_HE_k(ax+\frac{a}{b}i,a^2z). \tag{2.20}$$

3 Some symmetry identities for the generalized Hermite-Genocchi polynomials

In this section, we introduce the generalized Hermite-Genocchi polynomials ${}_HG_n^{(\alpha)}(x,y)$ (for a real or complex parameter α) by means of the following generating function defined in a suitable neighborhood of $t=0$:

$$\left(\frac{2t}{e^{t}+1}\right)^{\alpha}e^{xt+yt^2}=\sum_{n=0}^{\infty}{}_HG_n^{(\alpha)}(x,y)\frac{t^n}{n!}, \tag{3.1}$$

so that

$${}_HG_n^{(\alpha)}(x,y)=\sum_{s=0}^{n}\binom{n}{s}G_{n-s}^{(\alpha)}H_s(x,y).$$

On setting $y=0$, equation (3.1) immediately reduces to (1.5).

For $\alpha=1$, equation (3.1) reduces to

$$\left(\frac{2t}{e^{t}+1}\right)e^{xt+yt^2}=\sum_{n=0}^{\infty}{}_HG_n(x,y)\frac{t^n}{n!}. \tag{3.2}$$

If we set $\alpha=0$ in (3.1) then we get the generating function given by (1.2).

Theorem 3.1. Let a and b be positive integers with same parity, then the following identity holds true:

$$\sum_{k=0}^{n}\binom{n}{k}G_{n-k}^{(\alpha-1)}(ay)a^kb^{n-k}\sum_{i=0}^{a-1}(-1)^{i}{}_HG_k^{(\alpha)}(bx+\frac{b}{a}i,b^2z)$$

$$=\sum_{k=0}^{n}\binom{n}{k}G_{n-k}^{(\alpha-1)}(by)b^ka^{n-k}\sum_{i=0}^{b-1}(-1)^{i}{}_HG_k^{(\alpha)}(ax+\frac{a}{b}i,a^2z). \tag{3.3}$$

Proof. Let us consider

$$f(t)=\frac{(2t)^{2\alpha-1}e^{ab(x+y)t+a^2b^2zt^2(1+(-1)^{a+1})}e^{abt}}{(e^at+1)^{\alpha}(e^bt+1)^{\alpha}}. \tag{3.4}$$

We can expand f(t) as:

$$f(t)=\left(\frac{2t}{e^{at}+1}\right)^{\alpha}e^{abxt+a^2b^2zt^2}\frac{1-(-e^{b}t)^{a}}{e^{bt}+1}\left(\frac{2t}{e^{bt}+1}\right)^{\alpha-1}e^{abyt}$$

$$=\left(\frac{2t}{e^{at}+1}\right)^{\alpha}e^{abxt+a^2b^2zt^2}\sum_{i=0}^{a-1}(-e^{bt})^{i}\sum_{n=0}^{\infty}G_n^{(\alpha-1)}(ay)\frac{b^nt^n}{n!}$$

$$=\left(\frac{2t}{e^{at}+1}\right)^{\alpha}e^{a^2b^2zt^2}\sum_{i=0}^{a-1}(-1)^{i}e^{(bx+\frac{b}{a}i)at}\sum_{n=0}^{\infty}G_n^{(\alpha-1)}(ay)\frac{b^nt^n}{n!}$$

$$=\sum_{i=0}^{a-1}(-1)^{i}\sum_{k=0}^{\infty}{}_HG_k^{(\alpha)}(bx+\frac{b}{a}i,b^2z)\frac{a^kt^k}{k!}\sum_{n=0}^{\infty}G_n^{(\alpha-1)}(ay)\frac{b^nt^n}{n!}$$

$$=\sum_{k=0}^{\infty}\sum_{i=0}^{a-1}(-1)^{i}{}_HG_k^{(\alpha)}(bx+\frac{b}{a}i,b^2z)\frac{a^k}{k!}\sum_{n=0}^{\infty}G_n^{(\alpha-1)}(ay)\frac{b^nt^{n+k}}{n!}.$$

On replacing n by $n-k$ in the above equation, we have

$$f(t)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}G_{n-k}^{(\alpha-1)}(ay)a^kb^{n-k}$$

$$\times\sum_{i=0}^{a-1}(-1)^{i}{}_HG_k^{(\alpha)}(bx+\frac{b}{a}i,b^2z)\frac{t^n}{n!}. \tag{3.5}$$

We may also expand $f(t)$ as:

$$f(t)=\left(\frac{2t}{e^{bt}+1}\right)^{\alpha}e^{abxt+a^2b^2zt^2}\frac{1-(-e^{a}t)^{b}}{e^{at}+1}\left(\frac{2t}{e^{at}+1}\right)^{\alpha-1}e^{abyt}$$

$$=\left(\frac{2t}{e^{bt}+1}\right)^{\alpha}e^{abxt+a^2b^2zt^2}\sum_{i=0}^{b-1}(-e^{at})^{i}\sum_{n=0}^{\infty}G_n^{(\alpha-1)}(by)\frac{a^nt^n}{n!}$$

$$=\left(\frac{2t}{e^{bt}+1}\right)^{\alpha}e^{a^2b^2zt^2}\sum_{i=0}^{b-1}(-1)^{i}e^{(ax+\frac{a}{b}i)bt}\sum_{n=0}^{\infty}G_n^{(\alpha-1)}(by)\frac{a^nt^n}{n!}$$

$$= \sum_{i=0}^{b-1}(-1)^i \sum_{k=0}^{\infty} {}_HG_k^{(\alpha)}(ax+\frac{a}{b}i,a^2z)\frac{b^kt^k}{k!}\sum_{n=0}^{\infty}G_n^{(\alpha-1)}(by)\frac{a^nt^n}{n!}$$
$$= \sum_{k=0}^{\infty}\sum_{i=0}^{b-1}(-1)^i {}_HG_k^{(\alpha)}(ax+\frac{a}{b}i,a^2z)\frac{b^k}{k!}\sum_{n=0}^{\infty}G_n^{(\alpha-1)}(by)\frac{a^nt^{n+k}}{n!}.$$

Replacing n by $n-k$ in the above equation, we have

$$f(t) = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}G_{n-k}^{(\alpha-1)}(by)b^ka^{n-k}$$
$$\times\sum_{i=0}^{b-1}(-1)^i {}_HG_k^{(\alpha)}(ax+\frac{a}{b}i,a^2z)\frac{t^n}{n!}. \quad (3.6)$$

By comparing the coefficients of $\frac{t^n}{n!}$ in (3.5) and (3.6), we obtain our desired result (3.3).

Corollary 3.1. If we put $\alpha = 1$ in Theorem 3.1, then we get the following result:

$$\sum_{k=0}^{n}\binom{n}{k}a^kb^{n-k}\sum_{i=0}^{a-1}(-1)^i {}_HG_k(bx+\frac{b}{a}i,b^2z)$$
$$=\sum_{k=0}^{n}\binom{n}{k}b^ka^{n-k}\sum_{i=0}^{b-1}(-1)^i {}_HG_k(ax+\frac{a}{b}i,a^2z). \quad (3.7)$$

Theorem 3.2. Let a and b be positive integers with same parity, then the following identity holds true:

$$\sum_{k=0}^{n}\binom{n}{k} {}_HG_k(bx,b^2y)a^kb^{n-k+1}T_{n-k}(a)$$
$$=\sum_{k=0}^{n}\binom{n}{k} {}_HG_k(ax,a^2y)b^ka^{n-k+1}T_{n-k}(b). \quad (3.8)$$

Proof. Let us consider

$$f(t) = \left(\frac{2abt}{e^{at}+1}\right)e^{abxt+a^2b^2yt^2}\frac{1+(-1)^{a+1}e^{abt}}{e^{bt}+1} \quad (3.9)$$
$$= \left(\frac{2abt}{e^{at}+1}\right)e^{abxt+a^2b^2yt^2}\frac{1-(-e^{bt})^a}{e^{bt}+1}$$
$$= \left(\frac{2abt}{e^{at}+1}\right)e^{abxt+a^2b^2yt^2}\sum_{i=0}^{a-1}(-e^{bt})^i$$
$$= \sum_{i=0}^{a-1}(-1)^ib\left(\frac{2at}{e^{at}+1}\right)e^{abxt+a^2b^2yt^2}e^{bit}$$
$$= \sum_{i=0}^{a-1}(-1)^ib\sum_{k=0}^{\infty} {}_HG_k(bx,b^2y)\frac{a^kt^k}{k!}\sum_{n=0}^{\infty}(bi)^n\frac{t^n}{n!}.$$

Replacing n by $n-k$ in the above equation, we have

$$f(t) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k} {}_HG_k(bx,b^2y)a^kb^{n-k+1}\sum_{i=0}^{a-1}(-1)^ii^{n-k}\right)\frac{t^n}{n!}$$
$$f(t) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k} {}_HG_k(bx,b^2y)a^kb^{n-k+1}T_{n-k}(a)\right)\frac{t^n}{n!}. \quad (3.10)$$

Since $(-1)^{a+1} = (-1)^{b+1}$, the expression for $f(t) = \left(\frac{2abt}{e^{at}+1}\right)^{\alpha}e^{abxt+a^2b^2yt^2}\frac{1+(-1)^{a+1}}{e^{bt}+1}e^{abt}$ Therefore, we obtain the following power series expansion for f(t) by symmetry

$$f(t) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k} {}_HG_k(ax,a^2y)b^ka^{n-k+1}T_{n-k}(b)\right)\frac{(t)^n}{n!}. \quad (3.11)$$

By equating the coefficients of $\frac{t^n}{n!}$ in the last two expression for $f(t)$, we arrive at our desired result (3.8).

Remark 3.1. For $x,y = 0$, Theorem 3.2 reduces to the known result of Yang et al. [[11],p.462(22)].

Theorem 3.3. Let m and n be positive integers, then the following identity holds true:

$$\sum_{k=0}^{m}\binom{m}{k}B_k(n,x)T_{m-k}(n) = 2^{m-1}\left(B_m[\frac{n}{2},\frac{x}{2}]+(-1)^{n+1}B_m(n,\frac{x}{2})\right); \quad (3.12)$$
$$\sum_{k=0}^{m}\binom{m}{k}E_k(n,x)S_{m-k}(n) = \frac{2^{m+2}}{m+1}\left(B_{m+1}(n,\frac{x}{2})-B_{m+1}(\frac{n}{2},\frac{x}{2})\right). \quad (3.13)$$

Proof. Let us consider

$$f(t) = \left(\frac{t}{e^t-1}\right)e^{nt+xt}\frac{1+(-1)^{n+1}}{e^t+1}e^{nt} \quad (3.14)$$
$$= \sum_{k=0}^{\infty}B_k(n,x)\frac{t^k}{k!}\sum_{m=0}^{\infty}T_m(n)\frac{t^m}{m!}.$$

Replacing m by $m-k$ in the above equation, we have

$$f(t) = \sum_{m=0}^{\infty}\sum_{k=0}^{m}\binom{m}{k}B_k(n,x)T_{m-k}(n)\frac{t^m}{m!}. \quad (3.15)$$

On the other hand

$$f(t) = \frac{te^{nt+xt}+(-1)^{n+1}te^{2nt+xt}}{e^{2t}-1}$$
$$= \left(\frac{t}{e^{2t}-1}e^{nt+xt}\right)+\frac{(-1)^{n+1}te^{2nt+xt}}{e^{2t}-1}$$
$$= \frac{1}{2}\sum_{m=0}^{\infty}B_m[\frac{n}{2},\frac{x}{2}]\frac{(2t)^m}{m!}+\frac{(-1)^{n+1}}{2}\sum_{m=0}^{\infty}B_m(n,\frac{x}{2})\frac{(2t)^m}{m!}$$
$$f(t) = \left(\sum_{m=0}^{\infty}2^{m-1}\left(B_m[\frac{n}{2},\frac{x}{2}]+(-1)^{n+1}B_m(n,\frac{x}{2})\right)\right)\frac{t^m}{m!}. \quad (3.16)$$

On comparing the coefficients of $\frac{t^m}{m!}$ in (3.15) and (3.16), we arrive at the desired result (3.12).

Similarly by considering,

$$g(t) = \left(\frac{2}{e^t+1}\right)e^{nt+xt}\frac{e^{nt}-1}{e^t-1} \quad (3.17)$$

$$= \sum_{k=0}^{\infty} E_k(n,x)\frac{t^k}{k!}\sum_{m=0}^{\infty} S_m(n)\frac{t^m}{m!}.$$

Replacing m by $m-k$ in the above equation, we have

$$g(t) = \sum_{m=0}^{\infty}\sum_{k=0}^{m}\binom{m}{k} E_k(n,x)S_{m-k}(n)\frac{t^m}{m!}. \tag{3.18}$$

On the other hand

$$g(t) = \left(\frac{2}{e^t+1}\right)e^{nt+xt}\frac{e^{nt}-1}{e^t-1}$$

$$= \frac{2e^{2nt+xt}-2e^{nt+xt}}{e^{2t}-1}$$

$$= \frac{2e^{2nt+xt}}{e^{2t}-1} - \frac{2e^{nt+xt}}{e^{2t}-1}$$

$$= \frac{2}{t}\sum_{m=0}^{\infty} B_m(n,\frac{x}{2})\frac{(2t)^m}{m!} - \frac{2}{t}\sum_{m=0}^{\infty} B_m(\frac{n}{2},\frac{x}{2})\frac{(2t)^m}{m!}$$

$$= \left(\sum_{m=0}^{\infty} 2^{m+1}\left(B_m(n,\frac{x}{2}) - B_m(\frac{n}{2},\frac{x}{2})\right)\right)\frac{t^{m-1}}{m!}.$$

Replacing m by $m+1$ in the above equation, we get

$$g(t) = \sum_{m=0}^{\infty}\left(\frac{2^{m+2}}{m+1}\left(B_{m+1}(n,\frac{x}{2}) - B_{m+1}(\frac{n}{2},\frac{x}{2})\right)\right)\frac{t^m}{m!}. \tag{3.19}$$

On comparing the coefficients of $\frac{t^m}{m!}$ in (3.18) and (3.19), we get the desired result (3.13).

Remark 3.2. For $x = 0$, Theorem 3.3 reduces to the known result of Yang et al. [[11],p.463(26)].

Theorem 3.4. Let m and n be positive integers, then the following identity holds true:

$$\sum_{k=0}^{m} G_k(n,x)S_{m-k}(n) = 2^{m+1}\left(B_m(n,\frac{x}{2}) - B_m(\frac{n}{2},\frac{x}{2})\right). \tag{3.20}$$

Proof. Let us consider

$$h(t) = \left(\frac{2t}{e^t+1}\right)e^{nt+xt}\frac{e^{nt}-1}{e^t-1}$$

$$= \sum_{k=0}^{\infty} G_k(n,x)\frac{t^k}{k!}\sum_{m=0}^{\infty} S_m(n)\frac{t^m}{m!}$$

$$h(t) = \sum_{m=0}^{\infty}\sum_{k=0}^{m}\binom{m}{k} G_k(n,x)S_{m-k}(n)\frac{t^m}{m!}. \tag{3.21}$$

On the other hand

$$h(t) = \left(\frac{2t}{e^t+1}\right)e^{nt+xt}\frac{e^{nt}-1}{e^t-1}$$

$$= \frac{2te^{2nt+xt}-2te^{nt+xt}}{e^{2t}-1}$$

$$= \frac{2te^{2nt+xt}}{e^{2t}-1} - \frac{2te^{nt+xt}}{e^{2t}-1}$$

$$= 2\sum_{m=0}^{\infty} B_m(n,\frac{x}{2})\frac{(2t)^m}{m!} - 2\sum_{m=0}^{\infty} B_m(\frac{n}{2},\frac{x}{2})\frac{(2t)^m}{m!}$$

$$h(t) = \sum_{m=0}^{\infty}\left(2^{m+1}\left(B_m(n,\frac{x}{2}) - B_m(\frac{n}{2},\frac{x}{2})\right)\right)\frac{t^m}{m!}. \tag{3.22}$$

On comparing the coefficients of $\frac{t^m}{m!}$ in (3.21) and (3.22), we get the desired result (3.20).

Remark 3.3. For $x = 0$, Theorem 3.4 reduces to the known result of Yang et al. [[11],p.463(27)].

4 Conclusion

Recently, many authors namely, Khan [5], Yang et al. [11], Khan et al. [6] and Pathan and Khan [9], have introduced and investigated several generalizations of Hermite-Euler and Hermite-Genocchi polynomials. In a sequel of such type of works, in this paper, we have introduced a new generalization of Hermite-Euler and Hermite-Genocchi polynomials. We have also established some elementary properties (for example, symmetric identities, summation formulae, power sum and alternating sum) for the polynomials introduced here. The results presented in this paper are more general in nature. Therefore, by the results established here, we may derive some other interesting special cases.

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