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A New Generalized Burr Family of Distributions Based on Quantile Function

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Abstract: We propose a new family from Burr XII distribution, called T-Burr family of distributions based on the T-$R\{\text{Y}\}$ framework. For this family, we consider the quantile functions of three well-known distributions, namely, Lomax, logistic and Weibull, and further developed three sub-families T-Burr$\{$Lomax$\}$, T-Burr$\{$Log-logistic$\}$ and T-Burr$\{$Weibull$\}$. Some mathematical properties such as quantile function, mode, Shannon entropy, moments, and mean deviations, of T-$R\{\text{Y}\}$ family are obtained. One special model, namely, Weibull-Burr$\{$Lomax$\}$ from T-Burr$\{$Lomax$\}$ family is considered and its properties are obtained. This model is flexible and can produce the shapes of the density such as left-skewed, right-skewed, symmetrical, J, and reversed-J, and can have constant, increasing and decreasing hazard rate shapes. The usefulness of this model is demonstrated through applications to censored and complete data sets.

Keywords: Burr XII distribution, generalization, quantile function, T-X family, T-$R\{Y\}$ family.

The Burr family of distributions (due to Burr, 1942) is a well-recognized family that contains twelve different functional forms. Among these forms, the Burr XII (or simply Burr) model is very popular and has wide applications in the fields of reliability, actuarial science, forestry, ecotoxiology, and survival analysis, among others.

The art of generalizing distributions is an old practice in which location, scale, shape, or inequality parameter(s) are inducted to the parent (or baseline) distributions. The induction of parameter(s) increases flexibility in terms of tail properties, and also improves goodness-of-fit of the proposed distribution. The modern parameter induction technique suggests inducting shape parameter(s) into the cumulative distribution function (CDF) or survival function (SF) of the baseline distribution. Azzalini (1985), Marshall and Olkin (1997), Gupta et al. (1998), and Zografos and Balakrishnan (2009) first started single shape parameter induction to the baseline distribution. Later, two- and three- parameters induction was proposed by Eugene et al. (2002), Cordeiro and de-Castro (2011), and Alexander et al. (2012) by introducing beta-G, Kumaraswamy-G, and McDonald-G classes, which have received wide recognition in statistical literature. In their approach, the properties of two distributions are mixed together for better exploration of the skewness and tail properties, and to enhance the goodness-of-fit of the distribution.

A rather more generalized approach of parameter induction was pioneered by Alzaatreh et al. (2013), by defining the transformed-transformer *(T-X)* technique. Let $r(t)$ be the probability density function (PDF) of a random variable $T \in [a,b]$ for $-\infty \le a < b < \infty$, and let $F(x)$ be the CDF of a random variable X such that the transformation $W(\cdot) : [0,1] \longrightarrow [a,b]$ satisfies the following conditions: (i) $W(\cdot)$ is differentiable and monotonically non-decreasing, and(ii) $W(0) \to a$ and $W(1) \to b$.

Alzaatreh et al. (2013) defined the CDF of the T-X family of distributions as

$$G(x) = \int_a^{W[F(x)]} r(t)\,dt. \tag{1}$$

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If $T \in (0,\infty)$, then, X is a continuous random variable and $W[F(x)] = -\log[1-F(x)]$. Then, the PDF corresponds to Eq. (1) is given by

$$g(x) = \frac{f(x)}{1-F(x)} r\Big(-\log\left[1-F(x)\right]\Big) = h_f(x)\, r\Big(H_f(x)\Big), \tag{2}$$

where $h_f(x) = \frac{f(x)}{1-F(x)}$ and $H_f(x) = -\log[1-F(x)]$ are the hazard and cumulative hazard rate functions corresponding to any baseline PDF $f(x)$, respectively.

Aljarrah et al. (2014) proposed the function $W(F(x))$ as the quantile function of a random variable Y and defined the T-$R\{Y\}$ family. Alzaatreh et al. (2014), Alzaatreh et al. (2015), and Almheidat et al. (2015) proposed and studied T-normal$\{Y\}$, T-Gamma$\{Y\}$, and T-Weibull$\{Y\}$ families of distributions, respectively. The beauty of this approach is that it allows us to study the simultaneous effect of the parameters (mostly shape parameters) of the three models at a time, and in this way, most of the data characteristics are captured. This method allow us to enhance the flexibility of the proposed model and provide better goodness-of-fits.

In this article, our objective is to propose the T-Burr family of distributions by using the T-$R\{Y\}$ approach pioneered by Aljarrah et al. (2014).

A random variable X is said to have a two-parameter Burr XII distribution if its CDF and PDF are, respectively, given by

$$\Pi_{c,k}(x) = 1-(1+x^c)^{-k} \tag{3}$$

and

$$\pi_{c,k}(x) = c\,k\,x^{c-1}\,(1+x^c)^{-(k+1)}, \quad x>0, \tag{4}$$

where $c>0$ and $k>0$ are both shape parameters. Henceforth, a random variable having PDF (4) is denoted by $X \sim$ Burr(c,k). The closed-form of the Burr CDF and SF ensure that the properties of the Burr distribution can be explored easily for censored and non-censored cases. In literature, some generalizations of the Burr distribution are reported viz. the beta-Burr XII distribution by Paranaiba et al. (2011), the Marshall-Olkin extended Burr XII distribution by Al-Saiari et al. (2014), the Kumaraswamy-Burr XII distribution by Paranaiba et al. (2013), the McDonald-Burr XII distribution by Gomes et al. (2015), odd Burr III by Jamal et al. (2017), and generalized Burr-G by Nasir et al. (2017).

The paper is outlined as follows: In Section 2, we define the generalized family of Burr distribution and three associated generalized families from it viz. T-Burr$\{$Lomax$\}$, T-Burr$\{$Log-logistic$\}$, and T-Burr$\{$Weibull$\}$. In Section 3, we give some general properties of the T-Burr$\{Y\}$ family of distributions including the modes, moments, Shannon entropy, and mean deviations. In Section 4, three special sub-models, namely, Gamma-Burr$\{$log-logistic$\}$, Dagum-Burr$\{$Weibull$\}$, and Weibull-Burr$\{$Lomax$\}$ are considered. Some properties of Weibull-Burr$\{$Lomax$\}$ are discussed in detail. In Section 5, a simulation study is performed to assess the performance of the method of maximum likelihood estimation (MML) of Weibull-Burr$\{$Lomax$\}$ distribution. In Section 6, two applications of the Weibull-Burr$\{$Lomax$\}$ are presented for real-life data sets. In Section 7, we conclude the paper.

1 The proposed family

Let T, R, and Y be three random variables with their CDF $F_T(x) = \mathbb{P}(T \le x)$, $F_R(x) = \mathbb{P}(R \le x)$, and $F_Y(x) = P(Y \le x)$. The quantile functions of these three CDFs are $Q_T(u)$, $Q_R(u)$, and $Q_Y(u)$, where the quantile function is defined as $Q_Z(u) = \inf\{z : F_Z(z) \ge u\}$, $0<u<1$. The densities of T, R, and Y are denoted by $f_T(x)$, $f_R(x)$, and $f_Y(x)$, respectively. We assume the random variables $T \in (a,b)$ and $Y \in (c,d)$, for $-\infty \le a < b \le \infty$ and $-\infty \le c < d \le \infty$. Aljarrah et al. (2014) (see also Alzaatreh et al., 2014) presented the CDF of the T-$R\{Y\}$ family as follows:

$$F_X(x) = \int_a^{Q_Y(F_R(x))} f_T(t)\,dt = F_T\Big(Q_Y\big(F_R(x)\big)\Big). \tag{5}$$

The PDF and HRF that correspond to Eq. (5) are, respectively, given by

$$f_X(x) = f_R(x) \times \frac{f_T\Big(Q_Y\big(F_R(x)\big)\Big)}{f_Y\Big(Q_Y\big(F_R(x)\big)\Big)}$$

and

$$h_X(x) = h_R(x) \times \frac{h_T\Big(Q_Y\big(F_R(x)\big)\Big)}{h_Y\Big(Q_Y\big(F_R(x)\big)\Big)}.$$

Table 1: Quantile functions for different Y distributions.

S.No	Y	$Q_Y(u)$
1.	Lomax	$\beta\left[(1-u)^{-\frac{1}{\alpha}}-1\right]$
2.	Weibull	$\left[-\alpha^{-1}\ln(1-u)\right]^{\frac{1}{\beta}}$
3.	Log-Logistic	$\alpha\left[u^{-1}-1\right]^{-\frac{1}{\beta}}$

Let the random variable R follow the Burr distribution given in Eq. (3), then Eq. (5) gives the CDF of T-Burr$\{Y\}$ family as

$$F_X(x)=\int_a^{Q_Y(1-(1+x^c)^{-k})} f_T(t)dt = F_T(Q_Y(1-(1+x^c)^{-k})). \tag{6}$$

The PDF corresponding to Eq. (6) is given by

$$f_X(x)=ckx^{c-1}\,(1+x^c)^{-k-1}\,\frac{f_T\left(Q_Y(1-(1+x^c)^{-k})\right)}{f_Y\left(Q_Y(1-(1+x^c)^{-k})\right)},$$

which can be written as

$$f_X(x)=\text{burr}(c,k)\,\frac{f_T\left(Q_Y(\text{Burr}(c,k))\right)}{f_Y\left(Q_Y(\text{Burr}(c,k))\right)}, \tag{7}$$

where burr(c,k) and Burr(c,k) are the PDF and CDF of the Burr random variable, respectively. Henceforth, the family of distributions given in Eq. (7) is called the T-Burr$\{Y\}$ family and is denoted by T-Burr$\{Y\}$. The PDF in Eq. (7) is clearly a generalization of Burr distribution.

Many generalizations of the Burr distributions can be considered as members of T-Burr$\{Y\}$ family. When $T\sim$ Beta(a,b) and $Y\sim$ Uniform$(0,1)$, the T-Burr$\{Y\}$ reduces to the beta-Burr XII distribution (Paranaiba et al., 2011). When $T\sim$ Kumaraswamy(a,b) and $Y\sim$ uniform$(0,1)$, the T-Burr$\{Y\}$ reduces to the Kumaraswamy-Burr XII distribution (Paranaiba et al., 2013). When $T\sim$ McDonald(α,β,γ) and $Y\sim$ Uniform$(0,1)$, the T-Burr$\{Y\}$ reduces to the McDonald Burr XII distribution (Gomes et al., 2015). Table 1 gives three quantile functions of popular distributions, which will be used to generate T-Burr$\{Y\}$ sub-families in the following subsections.

Table 1 gives the quantile functions of well-known distributions. We can generate different generalized Burr families of T-Burr$\{Y\}$ by using these quantile functions to Eq. (7).

*Remark.*If X follows the T-Burr$\{Y\}$ family of distributions given by (6), then we have the following:

(i) $X\overset{d}{=}\left\{\left[1-F_Y(T)\right]^{-\frac{1}{k}}-1\right\}^{\frac{1}{c}}$.

(ii) $Q_X(u)=\left\{\left[1-F_Y\left(Q_T(u)\right)\right]^{-\frac{1}{k}}-1\right\}^{\frac{1}{c}}$.

(iii) if $Y\overset{d}{=}$ Burr(c,k), then $X\overset{d}{=}T$.

1.1 T-Burr$\{$Lomax$\}$ family of distributions

By using, Q_Y, the quantile function of the Lomax distribution in Table 1, the $F_X(x)$ in Eq. (7) can be written as

$$F_X(x)=F_T\left\{\beta\left\{(1+x^c)^{\frac{k}{\alpha}}-1\right\}\right\}. \tag{8}$$

When $\alpha=1$, the CDF in Eq. (8) becomes

$$F_X(x)=F_T\left\{\beta\left\{(1+x^c)^{k}-1\right\}\right\}. \tag{9}$$

The PDF corresponding to Eq. (9) is given by

$$f_X(x)=\beta\,\text{burr}(c,-k)\,f_T\left\{\beta\left\{(1+x^c)^{k}-1\right\}\right\},$$

where $burr(c,-k)=ckx^{c-1(1+x^c)^{k-1}}$.

1.2 T-Burr{Log-Logistic} family of distributions

By using, Q_Y, the quantile function of the Log-logistic distribution in Table 1, the $F_X(x)$ in Eq. (7) is given by

$$F_X(x) = F_T\left\{\alpha\left[(1+x^c)^k - 1\right]^{\frac{1}{\beta}}\right\}. \tag{10}$$

The PDF corresponding to Eq. (10) is given by

$$f_X(x) = \frac{\alpha}{\beta}\,\mathrm{burr}(c,k)\,\left[(1+x^c)^k - 1\right]^{\frac{1}{\beta}-1} f_T\left\{\alpha\left[(1+x^c)^k - 1\right]^{\frac{1}{\beta}}\right\}.$$

When $\beta = 1$, then the T-Burr{log-logistic} distribution reduces to T-Burr{Lomax} distribution.

1.3 T-Burr{Weibull} family of distributions

By using, Q_Y, the quantile function of the Weibull distribution in Table 1, the $F_X(x)$ in Eq. (7) is given by

$$F_X(x) = F_T\left\{\left[\frac{k}{\alpha}\ln(1+x^c)\right]^{\frac{1}{\beta}}\right\}. \tag{11}$$

When $\alpha = 1$ the CDF in (11) becomes

$$F_X(x) = F_T\left\{[k\ln(1+x^c)]^{\frac{1}{\beta}}\right\}. \tag{12}$$

The PDF corresponding to Eq. (12) is given by

$$f_X(x) = \frac{c\,k\,x^{c-1}}{\beta\,(1+x^c)}\,[k\ln(1+x^c)]^{\frac{1}{\beta}-1} f_T\left\{[k\ln(1+x^c)]^{\frac{1}{\beta}}\right\}. \tag{13}$$

2 Some properties of the T-Burr$\{Y\}$ family of distributions

In this section, some general properties of the T-Burr $\{Y\}$ family of distributions are provided including the modes, moments, Shannon entropy, and mean deviations.

2.1 Mode

The mode(s) of T-Burr$\{Y\}$ family can be obtained by finding the solution to the equation:

$$x = (c-1)\left[\frac{c(k+1)\,x^{c-1}}{1+x^c} - \Psi\left[Q'_Y\left[f_B(x)\right]\right] - \Psi\left\{f_T\left[Q_Y\left[f_B(x)\right]\right]\right\}\right], \tag{14}$$

where $\Psi(f) = f'/f$. The result in (14) can be proved by setting the derivative of the PDF in Eq. (7) equal to zero.

2.2 Moments

On the basis of Remark 1 (i), we have the rth moment of T-Burr$\{Y\}$ is

$$\mathbb{E}(X^r) = \mathbb{E}\left[\{1 - F_Y(T)\}^{-\frac{1}{k}} - 1\right]^{\frac{r}{c}}.$$

Using generalized binomial theorem, we have $(x+y)^r = \sum_{j=0}^{\infty}\binom{r}{j} x^{r-j} y^j \quad (|x| > |y|)$, we obtain

$$\mathbb{E}(X^r) = \sum_{j=0}^{\infty}\binom{\frac{r}{c}}{j}(-1)^j\,\mathbb{E}\{1 - F_Y(T)\}^{-\frac{(r-j)}{k}}. \tag{15}$$

The expression (15) leads to the rth moments of T-Burr{Lomax}, T-Burr{Log-logistic}, and T-Burr{Weibull} distributions as follows:

$$\mathbb{E}(X^r) = \sum_{j=0}^{\infty} \binom{\frac{r}{c}}{j} (-1)^j \, \mathbb{E}\left\{ \left(1+\frac{T}{\beta}\right)^{\frac{1}{k}(r-j)} \right\}, \tag{16}$$

$$\mathbb{E}(X^r) = \sum_{j=0}^{\infty} \binom{\frac{r}{c}}{j} (-1)^j \, E\left\{ \left(1+\left(\frac{T}{\beta}\right)^{\beta}\right)^{\frac{1}{k}(r-j)} \right\} \tag{17}$$

and

$$\mathbb{E}(X^r) = \sum_{j=0}^{\infty} \binom{\frac{r}{c}}{j} (-1)^j \, \mathbb{E}\left\{ \exp\left[\frac{1}{k}(r-j)\,T^{\beta}\right] \right\}. \tag{18}$$

2.3 Shannon entropy

The entropy of a random variable X is a measure of the variation of uncertainty. Entropy has many applications in the fields such as physics, chemistry, engineering, and economics among others. The Shannon entropy of a continuous random variable was introduced by Shannon in 1948.

From Theorem 2 of Aljarrah et al. (2014), the Shannon entropy of T-Burr $\{Y\}$ is given by

$$\eta_x = \eta_T + \mathbb{E}\left(\log f_Y(T)\right) + \mathbb{E}\left(\log Q'_{Burr}\left[F_Y(T)\right]\right), \tag{19}$$

where $Q'_{Burr} = \frac{1}{ck}\left[(1-\lambda)^{-\frac{1}{k}} - 1\right]^{\frac{1-c}{c}} (1-\lambda)^{-\frac{k+1}{k}}$ for all $\lambda \, \varepsilon \, (0,1)$ is the derivative of the quantile function of the Burr distribution.

From Eq. (19), the Shannon entropy for T-Burr{Lomax}, T-Burr{Log-logistic}, and T-Burr{Weibull distributions, respectively, is given by

$$\eta_x = \eta_T + \log\left(\frac{1}{\beta\, c\, k}\right) + (1-c)\,\mathbb{E}\left(\log X\right) + \left(1+\frac{1}{k}\right)\mathbb{E}\left[\log\left(1+\frac{T}{\beta}\right)\right], \tag{20}$$

$$\eta_x = (2-\beta)\eta_T + \log\left(\frac{\beta}{\alpha^{\beta}\, c\, k}\right) + (1-c)\mathbb{E}\left(\log X\right) + \left(\frac{1}{k}-1\right)\mathbb{E}\left[\log\left(1+\left(\frac{T}{\alpha}\right)^{\beta}\right)\right] \tag{21}$$

and

$$\eta_x = (2-\beta)\eta_T + \log\left(\frac{\beta}{c\, k}\right) + \frac{1}{k}\mathbb{E}\left(T^{\beta}\right) + (1-c)\mathbb{E}\left(\log X\right). \tag{22}$$

2.4 Mean deviation

The mean deviations from the mean and median are defined as

$$\delta_1 = 2\mu F(\mu) - 2I_c(\mu);\; \delta_2 = \mu - 2I_c(\mu), \tag{23}$$

where F_X is given by Eq. (6). The mean μ can be obtained from (15) with $r=1$. The median can be obtained from Remark 1(ii) after replacing u with 0.5. The first incomplete moment $I_c(s)$ is obtained as

$$I_c(s) = \int_0^s x\, f_X(x)\,dx = \int_0^{Q_Y(F_R(s))} Q_R(F_Y(w))\, f_T(w)\,dw. \tag{24}$$

On the basis of the result Eq. (24), the three first incomplete moments for T-Burr{Lomax}, T-Burr{Log-logistic}, and T-Burr{Weibull} families of distributions can be calculated as follows:

$$I_c(s) = \sum_{j=0}^{\infty} \binom{\frac{1}{c}}{j} (-1)^j \int_0^{\beta\left[(1+s^c)^k - 1\right]} \left(1+\frac{t}{\beta}\right)^{\frac{1}{k}(1-j)} f_T(t)\,dt,$$

$$I_c(s)=\sum_{j=0}^{\infty}\binom{\frac{1}{c}}{j}(-1)^j\int_0^{\alpha\left[(1+s^c)^k-1\right]^{\frac{1}{\beta}}}\left\{1+\left(\frac{t}{\beta}\right)^{\frac{1}{\beta}}\right\}^{\frac{1}{k}(1-j)} f_T(t)\,dt,$$

and

$$I_c(s)=\sum_{j=0}^{\infty}\binom{\frac{1}{c}}{j}(-1)^j\int_0^{[k\ln(1+s^c)]^{\frac{1}{\beta}}}\exp\left[\frac{T^{\beta}(1-j)}{k}\right] f_T(t)\,dt.$$

3 Special Sub-Models

In this section, we consider some different distributions for T random variable to generate some new models. We consider three special sub-models, namely, Gamma-Burr$\{$log-logistic$\}$, Dagum-Burr$\{$Weibull$\}$ and Weibull-Burr$\{$Lomax$\}$. We develop some properties of the Weibull-Burr$\{$Lomax$\}$ model as an illustration.

3.1 Gamma-Burr$\{$Log-logistic$\}$ distribution

If T follow the Gamma random variable with parameters a and b having CDF $F(t)=\gamma(a,t)/\Gamma(a) \quad ,t>0$, where $\gamma(a,x)=\int_0^x x^{a-1}\mathrm{e}^{-x}dx$ (the lower gamma function). Then from Eq. (11), the CDF of Gamma-Burr$\{$Log-Logistic$\}$ is given as

$$F_X(x)=\mathbb{P}\left(a,\alpha\left[\mathrm{Burr}(c,-k)\right]^{\frac{1}{\beta}}\right), \tag{25}$$

where $\mathrm{Burr}(c,-k)=(1+x^c)^k-1$ and $\mathbb{P}(a,x)=\frac{\gamma(a,x)}{\Gamma(a)}$.

Setting $\alpha=1$, the CDF in Eq. (25) becomes

$$F_X(x)=\mathbb{P}\left(a,\left[\mathrm{Burr}(c,-k)\right]^{\frac{1}{\beta}}\right). \tag{26}$$

The PDF corresponding to Eq. (26) is given by

$$f_X(x)=\frac{\mathrm{burr}(c,k)}{\Gamma(a)b^a\beta}\left(\mathrm{Burr}(c,-k)\right)^{\frac{a-\beta}{\beta}}\exp\left[\frac{1}{b}\left(\left(\mathrm{Burr}(c,-k)\right)^{\frac{1}{\beta}}\right)\right].$$

If a=1 then, the Gamma-Burr$\{$Log-Logistic$\}$ reduces to Exponential-Burr$\{$Log-Logistic$\}$.

(a) (b)

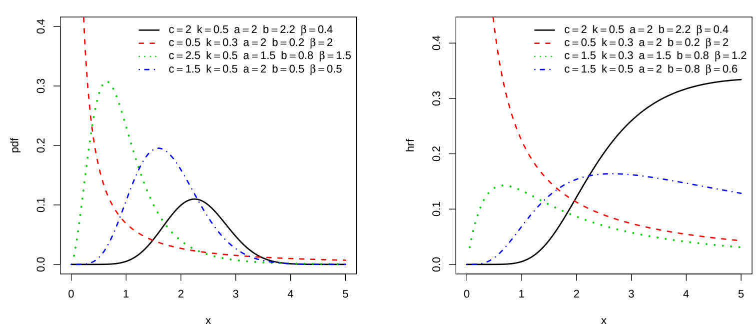

Fig. 1: Plots of PDF and HRF of Exponential-Burr$\{$Log-logistic$\}$ distribution.

The plots in Fig. (1) give (a) reversed J, symmetrical, and left-skewed density shapes, and (b) decreasing, increasing, and upside-down bathtub hazard rate shapes.

3.2 Dagum-Burr{Weibull} distribution

Let T follow the Dagum distribution with parameters a and b, $F_T(t) = [1+t^{-a}]^{-b}, \quad t > 0$. Then, the CDF of Dagum-Burr{Lomax} is as follows:

$$F_X(x) = \left[1 + [k \ln(1+x^c)]^{-\frac{a}{\beta}}\right]^{-b}. \tag{27}$$

Setting $a = 1$, the CDF in Eq. (27) becomes

$$F_X(x) = \left[1 + [k \ln(1+x^c)]^{-\frac{1}{\beta}}\right]^{-b}. \tag{28}$$

The PDF corresponding to Eq. (28) is given by

$$f_X(x) = \frac{c\,k\,b\,x^{c-1}}{\beta\,(1+x^c)} \frac{\left[1 + [k \ln(1+x^c)]^{-\frac{1}{\beta}}\right]^{-b-1}}{[k \ln(1+x^c)]^{\frac{1}{\beta}-1}}. \tag{29}$$

(a) (b)

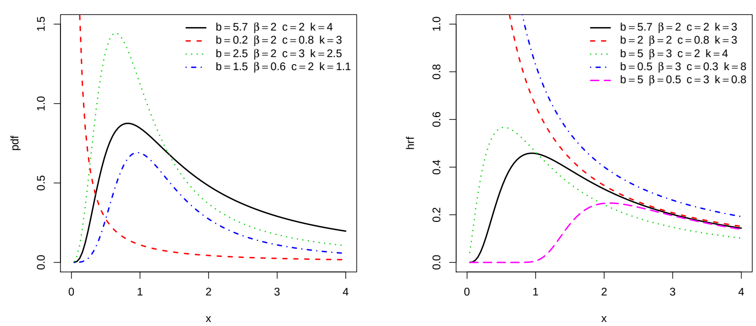

Fig. 2: Plots of PDF and HRF of Dagum-Burr{Weibull} distribution.

The plots in Fig. (2) give (a) reversed J and left-skewed shapes density shapes, and (b) decreasing and upside-down bathtub hazard rate shapes.

3.3 Weibull-Burr{Lomax} distribution

Let T follow the Weibull distribution with parameters a and b $F_T(t) = 1 - e^{-at^b}$. Then, the CDF of Weibull-Burr{Lomax} is as follows:

$$F_X(x) = 1 - \exp\left[-a\,\beta\left(\left\{(1+x^c)^{\frac{k}{\alpha}} - 1\right\}\right)^b\right]. \tag{30}$$

Setting $\beta = 1$ and $\alpha = 1$, the CDF in Eq. (30) becomes

$$F_X(x) = 1 - \exp\left[-a\left(\left\{(1+x^c)^k - 1\right\}\right)^b\right]. \tag{31}$$

The PDF corresponding to Eq. (31) is given by

$$f_X(x) = c\,k\,a\,b\,x^{c-1}(1+x^c)^{k-1}\left\{(1+x^c)^k - 1\right\}^{b-1} \times \exp\left[-a\left\{(1+x^c)^k - 1\right\}^b\right]. \tag{32}$$

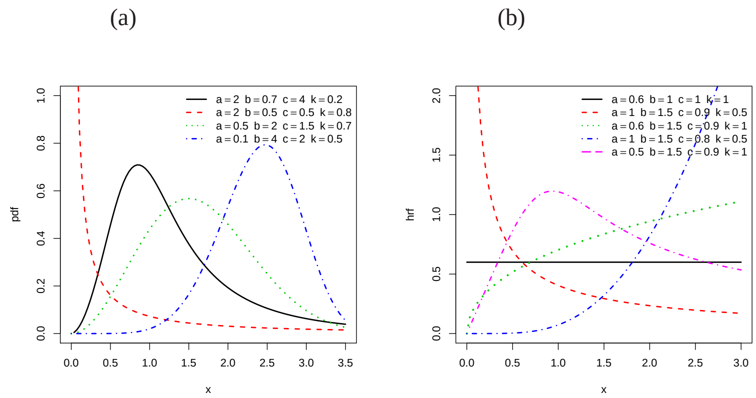

Fig. 3: Plots of PDF and HRF of Weibull-Burr{Lomax} distribution.

When $k = 1$, the distribution in (32) reduces to Weibull distribution with parameters a, b and c. The plots in Fig. (3) give (a) reversed J, left-skewed, right-skewed, and symmetrical density shapes and (b) increasing, decreasing, upside-down bathtub, and constant hazard rate shapes. The quantile function of Weibull-Burr{Lomax} can be obtained form Remark 1(ii)

$$Q_X(u) = \left[\left(1 + \left[-\frac{1}{a}\ln(1-u)\right]^{\frac{1}{b}}\right)^{\frac{1}{k}} - 1\right]^{\frac{1}{c}}.$$

The Weibull-Burr{Lomax} mode can be obtained form Equation (14).

$$\frac{d}{dx} f_T(x) = \frac{c-1}{x} + (k-1)\frac{cx^{c-1}}{1+x^c} + (b-1)ck\frac{x^{c-1}(1+x^c)^{k-1}}{\{(1+x^c)^k - 1\}} - abck\left(\left\{(1+x^c)^k - 1\right\}\right)^{b-1}(1+x^c)^{k-1}x^{c-1}.$$

The mode(s) will be the solution of the above equation.

Moments of Weibull-Burr{Lomax} can be obtained form Eq. (16) as

$$\mathbb{E}(X^r) = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}(-1)^j \binom{\frac{r}{c}}{j}\binom{\frac{r-j}{k}}{i}\left[\gamma\left(1+\frac{i}{b}, a\right) + \Gamma\left(1+\frac{\frac{r-j}{k}-i}{b}, a\right)\right], \tag{33}$$

where $\gamma(a,x) = \int_0^x t^{a-1}e^{-t}\,dt$ and $\Gamma(a,x) = \int_x^\infty t^{a-1}e^{-t}\,dt$ are the lower and upper incomplete gamma functions, respectively.

From Eq. (20), the Shannon entropy for X that follows the Weibull-Burr{Lomax} is given by

$$\eta_X = \eta_T - \log(ck) + \left(\frac{1+k}{k}\right)\mathbb{E}(\log(1+T)) + (1-c)\mathbb{E}(\log X),$$

where $\eta_T = \log(ab) + \left(1+\frac{1}{b}\right)\xi - a$, ξ is the Euler gamma constant, and

$$\mathbb{E}(\log(1+T)) = \frac{b-1}{b}\log a.\exp(-a) - \mathbb{E}I(-a) + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}a^{\frac{n}{b}}}{n}\Gamma\left(-\frac{n}{b}+1, a\right) + \sum_{n=1}^{\infty}\frac{(-1)^n}{na^{\frac{n}{b}}}\gamma\left(\frac{n}{b}+1, a\right)$$

(Aljarrah et al., 2015), where $\mathbb{E}I(x) = \int_{-\infty}^{x} t^{-1}e^t dt$ is the exponential integral (Abramowitz and Stegun 1972), $\mathbb{E}(\log X) = \lim_{r\to 0}\frac{d}{dr}\mathbb{E}(X^r)$ and $\mathbb{E}(X^r)$ is given in (33).

Estimation of the parameters: Let $X_1, X_2, \ldots, X_n$ be a random sample from the Weibull-Burr{Lomax} distribution defined in Eq. (32). Then the log-likelihood function is given by

$$\ell(\Theta) = n\log(a\,b\,c\,k) + (c-1)\sum_{i=1}^{n}\log x_i + (k-1)\sum_{i=1}^{n}\log(1+x_i^c) + (b-1)\sum_{i=1}^{n}\log\left\{(1+x_i^c)^k - 1\right\} - a\sum_{i=1}^{n}\left\{(1+x_i^c)^k - 1\right\}^b.$$

The score vector are as follows:

$$U_a = \frac{n}{a} - \sum_{i=1}^{n}\left\{(1+x_i^c)^k - 1\right\}^b,$$

$$U_b = \frac{n}{b} + \sum_{i=1}^{n}\log\left\{(1+x_i^c)^k - 1\right\} - a\sum_{i=1}^{n}\left\{(1+x_i^c)^k - 1\right\}^b \log\left\{(1+x_i^c)^k - 1\right\},$$

$$U_c = \frac{n}{c} + \sum_{i=1}^{n}\log x_i + (k-1)\sum_{i=1}^{n}\left[\frac{x_i^c\log x_i}{1+x_i^c}\right] + (b-1)\sum_{i=1}^{n}\left[\frac{k\,(1+x_i^c)^{k-1}\,x_i^c\log x_i}{(1+x_i^c)^k - 1}\right] - a\,b\sum_{i=1}^{n}\left\{(1+x_i^c)^k - 1\right\}^{b-1} k\,(1+x_i^c)^{k-1}x_i^c\log x_i,$$

$$U_k = \frac{n}{k} + \sum_{i=1}^{n}\log(1+x_i^c) + (b-1)\sum_{i=1}^{n}\left[\frac{(1+x_i^c)^k\log(1+x_i^c)}{(1+x_i^c)^k - 1}\right] - a\,b\sum_{i=1}^{n}\left\{(1+x_i^c)^k - 1\right\}^{b-1}(1+x_i^c)^k\log(1+x_i^c).$$

Setting U_b, U_a, U_k, and U_c equal to zero and solving these equations simultaneously obtains the maximum likelihood estimates (MLEs).

4 Simulation of Weibull-Burr{Lomax}

In this section, we perform a simulation study to assess the performance of maximum likelihood method used to estimate parameters of the Weibull-Burr{Lomax} distribution. We consider simulations for sample sizes (n=100, 200, 500) by using R-language. We simulate 1,000 samples for the true parameter values I: c=2 k=0.5 a=1 b=1 and II: c=3 k=1.5 a=1.5 b=0.5 to obtain average estimates (AEs), biases, and mean square errors (MSEs) of the parameters. These values are listed in Table 2. The values of the biases and MSEs decrease as the sample size increases. The results of the Table 2 indicate that the method of MLE performs well in estimating the model parameters of the proposed distribution.

Table 2: Estimated AEs, biases, and MSEs of the MLEs of parameters of Weibull-Burr{Lomax} distribution based on 1000 simulations of with n=100, 200, and 500.

		I			II		
n	parameters	A.E	Bias	MSE	A.E	Bias	MSE
100	c	2.752	0.752	4.622	4.571	1.571	11.839
	k	0.554	0.054	0.059	1.844	0.344	0.955
	a	1.385	0.385	1.710	1.663	0.163	1.432
	b	1.074	0.074	0.439	0.557	0.057	0.202
200	c	2.298	0.298	1.185	4.021	1.021	6.407
	k	0.538	0.038	0.033	1.618	0.118	0.311
	a	1.380	0.380	1.588	1.503	0.043	0.356
	b	1.041	0.041	0.244	0.546	0.046	0.122
500	c	2.046	0.246	1.128	3.680	0.680	4.757
	k	0.501	0.001	0.017	1.610	0.110	0.146
	a	1.020	0.300	0.895	1.418	0.003	0.246
	b	1.038	0.038	0.153	0.527	0.037	0.118

5 Application

This section provides two applications, one for complete (uncensored) data sets and the other for censored data sets to show how the Weibull-Burr{Lomax} (for short, W-Bu{Lx}) distribution can be applied in practice. In these applications, the distribution parameters are estimated by using the maximum likelihood method. The Akaike information criterion (AIC), Anderson-Darling (A^*), Cramer-von Mises (W^*), and Kolmogrov-Smirnov (K-S) statistics are obtained to compare the fitted models. In general, a smaller value of the statistics corresponds to a better fit to the data. The plots of the fitted PDFs and CDFs of some distributions are displayed for visual comparison. The required computations are performed in R-language.

5.1 Uncensored (or complete) data sets

In this subsection, we show that how W-Bu$\{$Lx$\}$ distribution can be applied in practice for two complete (uncensored) real data sets. We fit the W-Bu$\{$Lx$\}$, Kumaraswamy Burr (Kw-Bu), Beta Burr (B-Bu), Beta exponential (B-Exp), Burr, and Weibull to this data set.

The data set of 50 observations, with a hole diameter and sheet thickness of 9 and 2 mm, respectively, is given in Table 3. Hole diameter readings are taken on jobs with respect to one hole, selected, and fixed as per a predetermined orientation. The data set is given by Dasgupta (2011).

Table 3: Data set 1

0.06	0.12	0.14	0.04	0.14	0.16	0.08	0.26	0.32	0.22
0.16	0.12	0.24	0.06	0.02	0.18	0.22	0.14	0.22	0.16
0.12	0.24	0.06	0.02	0.18	0.22	0.14	0.02	0.18	0.22
0.14	0.06	0.04	0.14	0.22	0.14	0.06	0.04	0.16	0.24
0.16	0.32	0.18	0.24	0.22	0.04	0.14	0.26	0.18	0.16

The summary statistics from the first data set are as follows: $\bar{x} = 0.152$, $s = 0.0061$, $\gamma_1 = 0.0061$, and $\gamma_2 = 2.301226$, where γ_1 and γ_2 are the sample skewness and kurtosis, respectively.

Table 4: MLEs and their standard errors (in parentheses) for data set 1.

Distribution	a	b	c	k	α	β
W-Bu$\{$Lx$\}$	0.565	0.807	1.663	19.342	-	-
	(0.82)	(0.41)	(1.11)	(22.99)		
Kw-Bu	0.227	11.522	8.340	-	39.720	
	(0.028)	(3.658)	(0.007)	-	(0.999)	-
B-Bu	27.607	9.738	5.070	-	0.029	-
	(87.432)	(1.951)	(10.925)	-	(0.032)	
B-Exp	2.667	18.006	-	-	-	0.9321
	0.5042	99.87	-	-	-	4.96
Burr	-	-	2.043		37.66	-
	-	-	(0.231)	-	(14.540)	-
Weibull	34.45	2.002	-	-	-	-
	(13.755)	(0.235)	-	-	-	-

Table 5: The value ℓ, $W*$, $A*$, KS, P-value for data set 1.

Dist	ℓ	$W*$	$A*$	KS	P-Value
W-Bu$\{$Lx$\}$	59.62026	0.1103664	0.6764127	0.1269	0.3969
Kw-Bu	57.88482	0.1976216	1.119699	0.1597	0.1558
B-Bu	54.90359	0.3194159	1.75434	0.2073	0.02716
B-Exp	54.62055	0.3224291	1.777851	0.2098	0.02455
Burr	57.10991	0.2166066	1.227761	0.1689	0.1153
Weibull	57.30266	0.212311	1.203196	0.1691	0.1144

From Table 5, we see that the W-Bu$\{$Lx$\}$ model has the smallest values of the $A*$, $W*$, and K-S statistics among the fitted B-Bu, Kw-Bu, B-Exp, Burr and Weibull distributions, thereby suggesting that the W-Bu$\{$Lx$\}$ model provides the best fit. Thus, the W-Bu$\{$Lx$\}$ model could be chosen as the most adequate model to explore this data set. The histogram of the first data set and the estimated PDFs and CDFs of the W-Bu$\{$Lx$\}$ model and the competitive models are displayed in Figure 4.

5.2 Censored data set

In this subsection, we fit the W-Bu$\{$Lx$\}$ model to a censored data set. We use AIC and BIC statistics to compare the fits of the W-Bu$\{$Lx$\}$ with Kw-Bu and B-Bu distribution. The data below are remission times, in weeks, for a group of 30 patients with leukemia who received similar treatment, as quoted in Lawless (2003). Asterisks denote censoring times.

Consider a data set $D=(x,r)$, where $x=(x_1,x_2,...,x_n)^T$ is the observed failure times, and $r_i=(r_1,r_2,...,r_n)^T$ are the censored failure times. The r_i is equal to 1 if a failure is observed and 0 otherwise. Assume that the data are independently and identically distributed and come from a distribution with PDF given by Eq. (32). Let $\Theta=(c,k,a,b)^T$ denote the vector of parameters. The likelihood of Θ can be written as

$$\ell(D;\Theta)=\prod_{i=1}^{n}\left[f(x_i;\Theta)\right]^{r_i}\left[1-F(x_i;\Theta)\right]^{1-r_i}. \tag{34}$$

Then, the log-likelihood is reduced as follows:

$$\ell=\sum_{i=1}^{n} r_i\left[\log(c\,k\,a\,b)+(c-1)\log x_i+(k-1)\log(1+x_i^c)+(b-1)\log\left\{(1+x^c)^k-1\right\}-a\left\{(1+x^c)^k-1\right\}^b\right]$$
$$+\sum_{i=1}^{n}(1-r_i)\left[-a\left\{(1+x^c)^k-1\right\}^b\right]. \tag{35}$$

The log-likelihood function can be maximized numerically to obtain the MLEs. Various routines are available for numerical maximization of ℓ. We use the routine `optim` in the R software.

Table 6: Data set 2

1	1	2	4	4	6	6	6	7
8	9	9	10	12	13	14	18	19
24	26	29	31*	42	45*	50*	57	
60	71*	85*	91.					

Table 7: Data set 2

Model	Parameters	MLE	Standard error	Log-Likelihood	*AIC*	*BIC*
W-Bu$\{$*Lx*$\}$	c	1.2902	0.7573	-108.2892	224.5785	230.1832
	k	0.0675	0.0600			
	a	9.2729	19.2363			
	b	1.9982	0.4593			
Kw-Bu	c	1.6530	0.3119	-111.7468	231.4935	237.0983
	k	15.7654	14.8965			
	a	12.3872	9.3006			
	b	0.0051	0.0023			
B-Bu	c	0.2236	0.2691	-108.3125	224.6249	230.2297
	k	2.9653	7.0795			
	a	0.6207	0.2738			
	b	26.4381	30.1542			

We observed that *AIC* and *BIC* statistics W-Bu$\{$Lx$\}$ are lower than the Kw-Bu and B-Bu distributions.

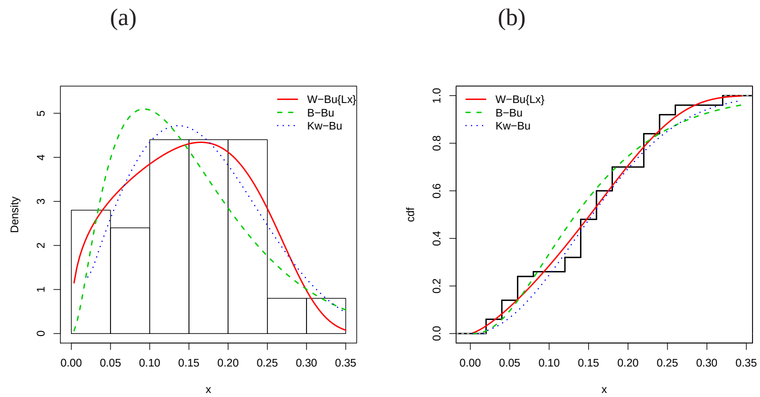

Fig. 4: Estimated PDFs and CDfs for data set 1.

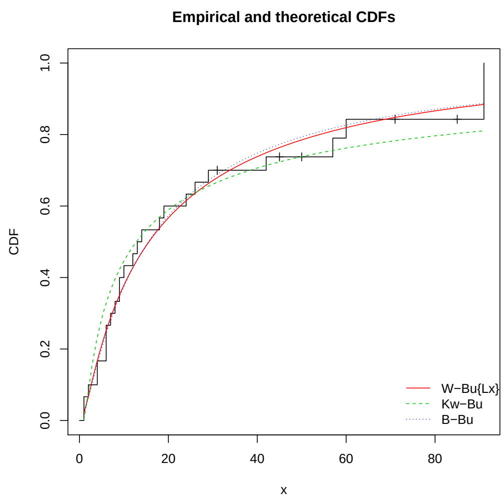

Fig. 5: Plots of estimated CDF for data set 2.

6 Conclusions and Results

Recently, statisticians and researchers have focused on developing flexible distributions to facilitate better modeling of lifetime data. Consequently, a significant progress has been made toward the generalization of some well-known lifetime models. In this context, we define the T-Burr$\{Y\}$ class of distributions, and three new distributions Gamma-Burr{Log-logistic}, Dagum-Burr{Weibull}, and Weibull-Burr{Lomax}, are introduced. We obtain explicit expressions for their quantile functions, ordinary and central moments, mean deviations, and Shannon entropy. We also presented two applications of the proposed family to real-life data sets (censored and complete) to illustrate the usefulness of the new family.

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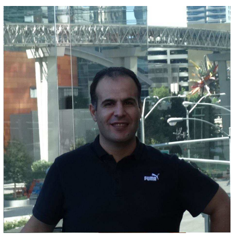

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