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Some Characteristic Functions for the Eigen Solution of Nonlinear Spinor

Ying-Qiu Gu∗

School of Mathematical Science, Fudan University, Shanghai 200433, China

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Abstract: In this paper, we solve the eigen solutions to some nonlinear spinor equations, and compute several functions reflecting their characteristics. The numerical results show that, the nonlinear spinor equation has only finite meaningful eigen solutions, which have positive discrete mass spectra and anomalous magnetic moment. The weird properties of the nonlinear spinors might be closely related with the elementary particles and their interactions, so some deeper investigations on them are significant.

Keywords: nonlinear potential, anomalous magnetic moment, nonlinear spinor, Dirac equation

1 Introduction

Since Dirac established relativistic quantum mechanics, many scientists such as H. Weyl, W. Heisenberg, have attempted to associate the elementary particles with the eigenstates of the nonlinear spinor equation[1,2,3,4,5,6]. In 1951, R. Finkelsten solved some rigorous solutions of the nonlinear spinor equation by numerical simulation, and pointed out that the corresponding particles have quantized mass spectra[7,8]. The theoretical proof about the existence of solitons was investigated in [9,10,11,12, 13,14]. The symmetries and many conditional exact solutions of the nonlinear spinor equations are collected in [15].

In recent years, the nonlinear spinor models for dark energy and dark matter may give an explanation for the accelerating expansion of the universe. Some researches get a number of interesting results[16,17,18,19,20,21, 22]. In this paper, we define some functions which reflect the properties of eigen solutions to the nonlinear spinor equations, and compute the typical values, then extract some important information from the data. Some previous works are given in[23,24].

At first, we introduce some notations and conventions. Denote the Minkowski metric by $\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$, Pauli matrices by

$$\sigma = (\sigma^j) = \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}. \tag{1.1}$$

Define 4×4 Hermitian matrices as follows

$$\alpha^\mu = \left\{ \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix} \right\}, \quad \gamma = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix}, \tag{1.2}$$

where $\mu \in \{0,1,2,3\}$, $x^0 = ct$ and $\alpha^\mu = \gamma^0\gamma^\mu$. In this paper, we adopt the Hermitian matrices (1.2) instead of Dirac matrices γ^μ for the convenience of calculation. For Dirac's bispinor ϕ, the quadratic forms of ϕ are defined by

$$\check{\alpha}^\mu = \phi^+\alpha^\mu\phi, \quad \check{\gamma} = \phi^+\gamma\phi, \quad \check{\beta} = \phi^+\beta\phi, \tag{1.3}$$

where the superscript '+' stands for the transposed conjugation. By the definition (1.3) we have $\check{\alpha}^\mu = \phi^\dagger\gamma^\mu\phi$ etc., where $\phi^\dagger = \phi^+\gamma^0$ is the Dirac conjugation[25]. $\check{\alpha}^\mu$ is a contra-variant 4-vector, $\check{\gamma}$ a true scalar and $\check{\beta}$ a pseudo-scalar. they are not independent due to Pauli-Fierz identities[26,27], such as $\check{\alpha}_\mu\check{\alpha}^\mu = \check{\gamma}^2 + \check{\beta}^2$.

In general, the Lagrangian of the nonlinear bispinor ϕ with a vector potential A^μ and scalar G is given by[28]

$$\begin{aligned}\mathscr{L} = {} & \phi^+\alpha^\mu(\hbar i\partial_\mu - eA_\mu)\phi - \mu\check{\gamma} + V(\check{\gamma},\check{\beta}) - s\check{\gamma}G \\ & -\frac{1}{2}\partial_\mu A_\nu\partial^\mu A^\nu - \frac{1}{2}(\partial_\mu G\partial^\mu G - b^2G^2).\end{aligned} \tag{1.4}$$

In this paper, we only consider the case $V = V(\check{\gamma}) > 0$ is a concave function satisfying

$$V'(\check{\gamma})\check{\gamma} > V(\check{\gamma}), \quad (\text{for } \check{\gamma} > 0). \tag{1.5}$$

The corresponding dynamical equation is given by

$$\alpha^\mu(\hbar i\partial_\mu - eA_\mu)\phi = (\mu c + sG - V')\gamma\phi, \tag{1.6}$$

$$\partial_\alpha\partial^\alpha A^\mu = e\check{\alpha}^\mu, \tag{1.7}$$

$$(\partial_\alpha\partial^\alpha + b^2)G = s\check{\gamma}. \tag{1.8}$$

∗ Corresponding author e-mail: yqgu@fudan.edu.cn

The Hamiltonian form of (1.6) is given by

$$\hbar i\partial_t\phi = \hat{H}\phi, \quad \hat{H} = c[eA_0+\alpha\cdot\hat{p}+(\mu c+sG-V')\gamma] \tag{1.9}$$

where $\hat{p} = -\hbar i\nabla - e\mathbf{A}$ is the momentum operator. For the angular momentum operator

$$\hat{J} = \mathbf{r}\times\hat{p}+\frac{1}{2}\hbar\gamma, \quad \gamma_k = \mathrm{diag}(\sigma_k,\sigma_k), \tag{1.10}$$

the eigenfunctions of $\hat{J}_3 = -\hbar i\partial_\varphi + \frac{1}{2}\hbar\gamma_3$ are given by

$$\hat{J}_3\phi_j = j_3\hbar\phi_j, \quad \phi_j = (u_1, u_2e^{\varphi i}, iv_1, iv_2e^{\varphi i})^T e^{j\varphi i}, \tag{1.11}$$

where the index 'T' stands for transpose, $j_3 = j+\frac{1}{2}, j\in\{0,\pm1,\pm2,\cdots\}$. For all the eigenfunctions, $\hat{J}_3$ is commutative with the nonlinear Hamiltonian operator like the linear case, so the solutions of (1.9) take the following form,

$$\phi_j = (u_1, u_2e^{\varphi i}, iv_1, iv_2e^{\varphi i})^T \exp(j\varphi i - \frac{mc^2}{\hbar}it), \tag{1.12}$$

where $u_k, v_k (k=1,2)$ are real functions of (r,θ). The normalizing condition becomes

$$2\pi\int_0^\infty r^2dr\int_0^\pi \sin\theta d\theta(u_1^2+u_2^2+v_1^2+v_2^2) = 1. \tag{1.13}$$

2 Properties of the dark nonlinear spinor

The simplest case of (1.9) is dark spinor described by the following dynamical equation,

$$\hbar i\partial_t\phi = \hat{H}\phi, \quad \hat{H} = c[\alpha\cdot\hat{p}+(\mu c - w\check{\gamma})\gamma]. \tag{2.1}$$

Different from the linear case, the nonlinear spinor equation generally has continuous spectra if the restriction of (1.13) is absent, so the normalizing condition becomes quantizing condition for nonlinear spinors, and the nonlinear coupling coefficient w is meaningful only if the solution satisfies the normalizing condition (1.13).

The eigen solutions to (2.1) with spin $j_3 = \pm\frac{1}{2}$ can be solved rigorously as follows

$$\begin{cases} \phi_{e\uparrow} = (g,0,if\cos\theta, if\sin\theta e^{\varphi i})^T\exp(-i\frac{mc^2}{\hbar}t), & \text{for } (\mathsf{P}=1,\ j_3=\frac{1}{2}) \\ \phi_{e\downarrow} = (0,g,if\sin\theta e^{-\varphi i}, -if\cos\theta)^T\exp(-i\frac{mc^2}{\hbar}t), & \text{for } (\mathsf{P}=1,\ j_3=-\frac{1}{2}) \\ \phi_{o\uparrow} = (f\cos\theta, f\sin\theta e^{\varphi i}, ig, 0)^T\exp(-i\frac{mc^2}{\hbar}t), & \text{for } (\mathsf{P}=-1,\ j_3=\frac{1}{2}) \\ \phi_{o\downarrow} = (f\sin\theta e^{-\varphi i}, -f\cos\theta, 0, ig)^T\exp(-i\frac{mc^2}{\hbar}t), & \text{for } (\mathsf{P}=-1,\ j_3=-\frac{1}{2}) \end{cases} \tag{2.2}$$

where $\mathsf{P}=1$ corresponds to even parity, and $\mathsf{P}=-1$ corresponds to odd parity. For the above eigenfunctions, we have

$$\check{\gamma} = \mathsf{P}(g^2-f^2), \quad 4\pi\int_0^\infty (g^2+f^2)r^2dr = 1. \tag{2.3}$$

The radial equation of even parity satisfies

$$\begin{cases} \frac{d}{dr}g = -\frac{1}{\hbar c}[(\mu+m)c^2 - wc(g^2-f^2)]f, \\ \frac{d}{dr}f = -\frac{1}{\hbar c}[(\mu-m)c^2 - wc(g^2-f^2)]g - \frac{2}{r}f. \end{cases} \tag{2.4}$$

For the odd parity, we have

$$\begin{cases} \frac{d}{dr}g = -\frac{1}{\hbar c}[(\mu-m)c^2 + wc(g^2-f^2)]f, \\ \frac{d}{dr}f = -\frac{1}{\hbar c}[(\mu+m)c^2 + wc(g^2-f^2)]g - \frac{2}{r}f. \end{cases} \tag{2.5}$$

The initial data of (2.4) and (2.5) satisfy $f(0)=0,\ g(0)>0$. For (2.4) and (2.5), we have positive mass $0<m<\mu$ if and only if $w>0$[23].

Making transformation

$$a = \sqrt{\frac{\mu+m}{\mu-m}}, \quad r_0 = \frac{\hbar}{c\sqrt{\mu^2-m^2}} = \frac{(a^2+1)\hbar}{2a\mu c}, \quad \rho = \frac{r}{r_0}, \tag{2.6}$$

$$u = \sqrt{\frac{w(a^2+1)}{2a\mu c}}g, \quad v = -\sqrt{\frac{w(a^2+1)}{2a\mu c}}f. \tag{2.7}$$

where a is equivalent to the spectrum, r_0 takes the unit of length. (2.4) and (2.5) can be rewritten in a dimensionless form. For (2.4) we have

$$\begin{cases} u' = (a-u^2+v^2)v, & u(0)=u_0>0, \\ v' = (\frac{1}{a}-u^2+v^2)u - \frac{2}{\rho}v, & v(0)=0, \end{cases} \tag{2.8}$$

where prime stands for $\frac{d}{d\rho}$. For (2.5) we have

$$\begin{cases} u' = (\frac{1}{a}+u^2-v^2)v, & u(0)=u_0>0, \\ v' = (a+u^2-v^2)u - \frac{2}{\rho}v, & v(0)=0, \end{cases} \tag{2.9}$$

The normalizing condition (2.3) becomes

$$(a+a^{-1})^2\int_0^\infty (u^2+v^2)\rho^2d\rho = S^2 \equiv \frac{w\mu^2c^2}{\pi\hbar^3}, \tag{2.10}$$

where S is a dimensionless constant to be determined.

The computation shows that, for any given $a>1$, there exists a sequence of initial data $0<u(0)_1<u(0)_2<\cdots$, such that (2.4) and (2.5) have eigen solutions. The theoretical analysis proves that there are infinite eigen solutions for each a[11]. In [23] we have shown three families of eigen solutions with even parity and the first family of eigen functions with odd parity.

To describe the properties of the eigen solutions, we define the following dimensionless functions, which are continuous functions of spectrum a for the same family solutions.

1.The dimensionless norm $y(a)$

$$y \equiv \frac{1}{2}\lg\left((a+a^{-1})^2\int_0^\infty (u^2+v^2)\rho^2d\rho\right). \tag{2.11}$$

For the same family of eigen solution, y is a continuous function of a. By (2.10), the normalizing condition is equivalent to the equation $y = \lg S$.

2.The dimensionless energy $\mathscr{E}(a)$ in the Nöther's sense,

$$\begin{aligned} \mathscr{E} &\equiv \frac{1}{\mu c^2}\left(mc^2+\frac{1}{2}wc\int_0^\infty \check{\gamma}^2\cdot 4\pi r^2dr\right) \\ &= \frac{a^2-1}{a^2+1}+\frac{a}{a^2+1}\frac{\int_0^\infty(u^2-v^2)^2\rho^2d\rho}{\int_0^\infty(u^2+v^2)\rho^2d\rho}. \end{aligned} \tag{2.12}$$

3.The mean diameter of an eigen solution $d(a)$

$$d \equiv \frac{2}{\lambda}\frac{\int r|\phi|^2d^3x}{\int|\phi|^2d^3x} = \frac{a^2+1}{a}\frac{\int_0^\infty(u^2+v^2)\rho^3d\rho}{\int_0^\infty(u^2+v^2)\rho^2d\rho}, \tag{2.13}$$

where $\lambda = \frac{\hbar}{\mu c}$ is a universal Compton wave length for all solutions.

4.The total dimensionless inner pressure $P(a)$

$$
\begin{aligned}
P &\equiv \frac{1}{3\mu c^2}\left(mc^2 - \int_0^\infty (\mu c^2\check{\gamma} + \frac{1}{2} wc\check{\gamma}^2)\cdot 4\pi r^2 dr\right) \\
&= \frac{1}{3}\left(\frac{a^2-1}{a^2+1} - \frac{\int_0^\infty (u^2-v^2)\rho^2 d\rho}{\int_0^\infty (u^2+v^2)\rho^2 d\rho} - \frac{a}{a^2+1}\frac{\int_0^\infty (u^2-v^2)^2\rho^2 d\rho}{\int_0^\infty (u^2+v^2)\rho^2 d\rho}\right). \quad (2.14)
\end{aligned}
$$

The physical meanings of $y(a)$, $\mathscr{E}(a)$ and $d(a)$ are evident. The inner pressure $P(a)$ is defined from general relativity. For the nonlinear spinor (2.1) in curved space-time with diagonal metric, we have energy momentum tensor[16]

$$T^{\mu\nu} = \frac{1}{2}\Re\langle\phi^+(\rho^\mu i\partial^\nu + \rho^\nu i\partial^\mu)\phi\rangle + (V'\check{\gamma} - V)g^{\mu\nu}. \quad (2.15)$$

For static spinor, we have

$$P = \frac{1}{3}(T_0^0 - T_\mu^\mu) = \frac{1}{3}(m|\phi|^2 - \mu\check{\gamma} - \frac{1}{2}w\check{\gamma}^2). \quad (2.16)$$

The dimensionless form of the total inner pressure of the spinor becomes (2.14).

The curves of the dimensionless functions defined above are shown in Fig.1 and Fig.2. In Fig.1, the normalizing condition $y \equiv \lg S = 0.918$ and $y \equiv \lg S = 0.647$ are derived from the anomalous magnetic moment(AMM) of an electron according to different definition of mass, as computed in the next section. A rough computation was once given in [24].

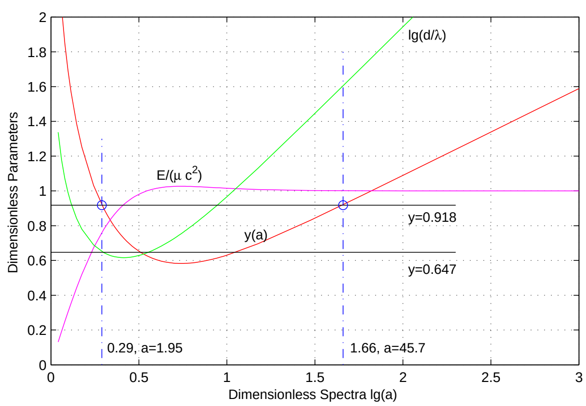

Fig. 1: The norm function $y(a)$, dimensionless energy $\mathscr{E} = \frac{E}{\mu c^2}$ and mean diameter $d(a)$ of a spinor. Only the solutions corresponding to the intersection $y(a) = \lg S$ are meaningful in physics

For an electron, we have $\mu \doteq m_e = 9.11\times 10^{-21}$kg, $\hbar = 1.055\times 10^{-34}$J.s, $c = 2.998\times 10^8$m/s. By (2.10) and $S = 8.277$, we can estimate the value

$$w = \frac{\pi\hbar^3 S^2}{\mu^2 c^2} \doteq 3.385\times 10^{-57}(\mathrm{Jsm}^2). \quad (2.17)$$

In this case, the nonlinear spinor equation has only two valid eigen solutions corresponding to $a = 1.95$ and $a = 45.7$. The norm function $y(a)$ of all other families of eigen solutions have no intersection points with $y = 0.918$.

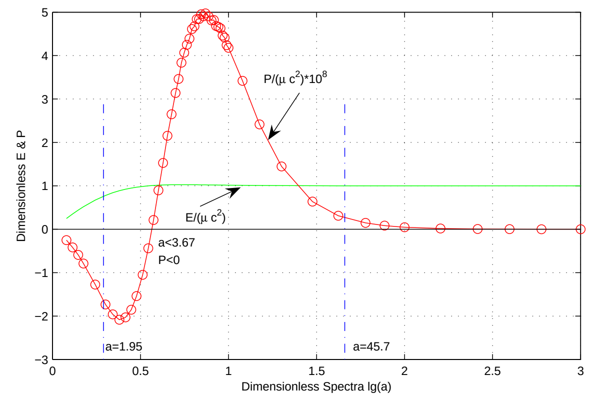

Fig. 2: The total energy $\mathscr{E}(a)$ and inner pressure $P(a)$ of a dark spinor

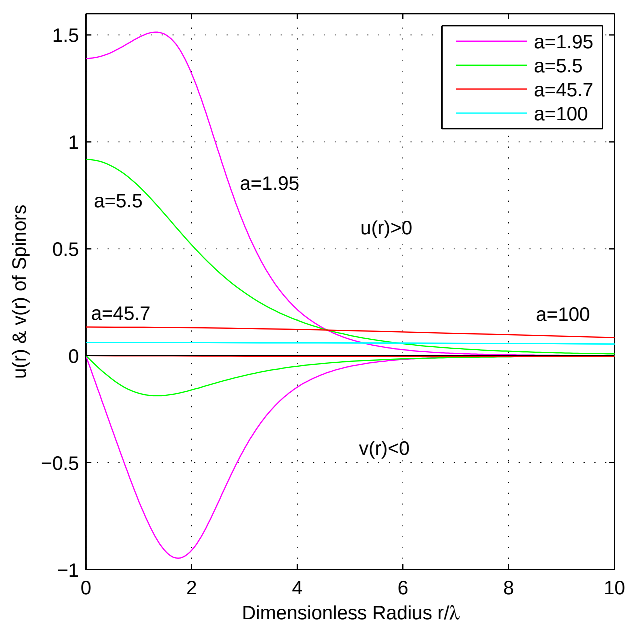

Fig. 3: The radial distribution of the nonlinear dark spinors

The radial functions (G,F) of solutions with even parity are shown in Fig.3, where

$$G(r) = \sqrt{\frac{w}{2\mu c}}g = \sqrt{\frac{a}{a^2+1}}u, \quad F(r) = -\sqrt{\frac{w}{2\mu c}}f = \sqrt{\frac{a}{a^2+1}}v. \quad (2.18)$$

The unit of the coordinate r is the universal Compton wave length $\frac{\hbar}{\mu c}$. So the images of different solutions are visually comparable in Fig.3.

3 The nonlinear spinor with electromagnetic interaction

The nonlinear spinor with self electromagnetic interaction was researched by a few authors. In 1966, M. Wakano has approximately analyzed the cases of A_0 dominance and $\mathbf{A}$ dominance when $w=0$, and reached the following conclusions[29]. In the case of $\mathbf{A}$ dominance, the eigen solutions or the solitons do not exist for the first order approximation. In the case of A_0 dominance, the eigen solutions exist but all with negative energy. In fact, the negative mass is equivalent to change the sign of A_0, which implies to transform the repulsive potential of A_0 into the absorbent one. M. Soler and A. F. Rañada calculated the eigen solutions of (1.9) by omitting $\mathbf{A}$. But they neglected the normalizing condition and did not use the true value of e[30,31]. Besides, the eigen solutions with Born-Infeld potential were studied in [32]. The detailed non-relativistic approximation of the many-spinors equations was given in [28]

In general, the coordinates r and θ can not be separable for nonlinear spinor with vector potential due to the term $\mathbf{A}$. However u_k and v_k can be conveniently expressed by spherical harmonics, and the equations of the radial functions can be derived via variation principle, because the eigen solutions are the critical points of the following energy functional

$$J=2\pi\int_0^\infty r^2dr\int_0^\pi \sin\theta d\theta(\phi^+\hat{H}\phi+\frac{1}{2}wc\check{\gamma}^2-\frac{1}{2}c\nabla A_\mu\cdot\nabla A^\mu-mc^2\check{\alpha}_0)+mc^2 \quad (3.1)$$

So the problem (1.9) can be changed into an ordinary differential equation system which can be solved by numerical computation.

In this paper, we only consider the eigen solutions with $\frac{1}{2}$-spin and even parity, which is the only valid case for a free electron. In the dimensionless form, we have the magnitude for the fields

$$|\mathbf{A}|\sim\frac{\alpha}{a}|g|,\quad |A_0|\sim\alpha|g|,\quad |f|\sim\frac{1}{a}|g|,\quad \alpha\doteq\frac{1}{137}, \quad (3.2)$$

where a is the dimensionless spectrum. Since the high order terms are caused by the vector potential $|\mathbf{A}|\sim\frac{\alpha}{a}|g|$, for adequately large a, we only keep the first order approximation for simplicity. Then we have

$$\phi\doteq(g,0,if\cos\theta,if\sin\theta e^{\varphi i})^T\exp(-i\frac{mc^2}{\hbar}t), \quad (3.3)$$

where g and f are real functions of r with $g(0)>0$. For large spectrum $a=49.12$, the relative error of the approximation is less than 10^{-4}, so the approximation is accurate enough to reveal the anomalous magnetic moment of a spinor with electromagnetic field. The less the value of a, the large the error of approximation.

The quadratic forms of ϕ are given by

$$\check{\alpha}_0=g^2+f^2,\quad \check{\gamma}=g^2-f^2,\quad \check{\alpha}=2gf\sin\theta(-\sin\varphi,\cos\varphi,0). \quad (3.4)$$

Correspondingly we have

$$A_0=A_0(r),\quad \mathbf{A}=A(r)\sin\theta(-\sin\varphi,\cos\varphi,0). \quad (3.5)$$

Substituting (3.4), (3.5) into (3.1) we get the energy functional

$$J\doteq 4\pi c\int_0^\infty r^2dr\Big\{\hbar[(f'+\frac{2}{r}f)g-g'f]+(\mu-m)cg^2-(\mu+m)cf^2-\frac{1}{2}w(g^2-f^2)^2+ e(g^2+f^2)A_0-\frac{4}{3}egfA+\frac{1}{2}A_0(\partial_r^2+\frac{2}{r}\partial_r)A_0-\frac{1}{3}A(\partial_r^2+\frac{2}{r}\partial_r-\frac{2}{r^2})A\Big\}+mc^2 \quad (3.6)$$

The approximation is only caused by the vector potential $\mathbf{A}$. By variation, we get a closed system of ordinary differential equations

$$\begin{cases} g'=-\frac{1}{\hbar}[(\mu+m)c-eA_0-w(g^2-f^2)]f-\frac{2}{3\hbar}eAg,\\ f'=-\frac{1}{\hbar}[(\mu-m)c+eA_0-w(g^2-f^2)]g+(\frac{2}{3\hbar}eA-\frac{2}{r})f,\\ A_0''+\frac{2}{r}A_0'=-e(g^2+f^2),\quad A''+\frac{2}{r}A'-\frac{2}{r^2}A=-2egf.\end{cases} \quad (3.7)$$

Make transformation

$$r_0=\frac{\hbar}{\sqrt{\mu^2-m^2}c},\quad a=\sqrt{\frac{\mu+m}{\mu-m}},\quad \alpha=\frac{e^2}{4\pi\hbar}=\frac{1}{137.035999}, \quad (3.8)$$

$$\rho=\frac{r}{r_0},\quad u=\sqrt{\frac{wr_0}{\hbar}}g,\quad v=-\sqrt{\frac{wr_0}{\hbar}}f,\quad P=\frac{er_0}{\hbar}A_0,\quad Q=\frac{2er_0}{3\hbar}A, \quad (3.9)$$

where P is dimensionless potential, which can not be confused with the pressure defined in (2.14). Substituting them into (3.7), we get the dimensionless form

$$\begin{cases} u'=(a-P-u^2+v^2)v-Qu,\\ v'=(\frac{1}{a}+P-u^2+v^2)u+(Q-\frac{2}{\rho})v,\\ P''+\frac{2}{\rho}P'=-\alpha\frac{u^2+v^2}{\int_0^\infty(u^2+v^2)\rho^2d\rho},\\ Q''+\frac{2}{\rho}Q'-\frac{2}{\rho^2}Q=\frac{4\alpha}{3}\frac{uv}{\int_0^\infty(u^2+v^2)\rho^2d\rho},\end{cases} \quad (3.10)$$

In (3.10) only a is a free parameter, which acts as the spectrum similar to the dark case of $e=0$. The normalizing condition is still (2.10). (3.10) is independent on the undetermined coefficient w, but it becomes a global problem. The natural boundary conditions are given by

$$\begin{cases} u(0)>0,\ \ v(0)=P'(0)=Q(0)=Q'(0)=0,\\ u\to u_\infty e^{-\rho},\ \ v\to\frac{u_\infty}{a}e^{-\rho},\ \ P\to\frac{\alpha}{4\pi\rho},\ \ Q\to\frac{Q_\infty}{\rho^2},\ \ (\rho\to\infty).\end{cases} \quad (3.11)$$

The solutions of (P,Q) can be expressed as

$$P=\frac{\alpha}{\int_0^\infty(u^2+v^2)\rho^2d\rho}\int_\rho^\infty\frac{1}{\rho^2}\int_0^\rho\left[u^2(\tau)+v^2(\tau)\right]\tau^2d\tau d\rho, \quad (3.12)$$

$$Q=\frac{-4\alpha}{3\rho^2\int_0^\infty(u^2+v^2)\rho^2d\rho}\int_0^\rho\rho^2\int_\rho^\infty u(\tau)v(\tau)d\tau d\rho. \quad (3.13)$$

We have $P>0,Q>0$ for the meaningful solutions. The solution of (3.12) and (3.13) can be soundly solved by iterative algorithm.

The total energy of the system in Nöther's sense is given by

$$E=2\pi\int_0^\infty r^2dr\int_0^\pi\sin\theta d\theta(\phi^+\hat{H}\phi+\frac{1}{2}wc\check{\gamma}^2-\frac{1}{2}c\nabla A_\mu\cdot\nabla A^\mu), \quad (3.14)$$

Substituting (3.8), (3.9) into it, we get the dimensionless form

$$\mathscr{E}=\frac{E}{\mu c^2}=\frac{a^2-1}{a^2+1}+\frac{a}{a^2+1}\frac{\int_0^\infty[(u^2-v^2)^2-P(u^2+v^2)-2Quv]\rho^2d\rho}{\int_0^\infty(u^2+v^2)\rho^2d\rho}. \quad (3.15)$$

The mass of a particle is a complex classical concept, which depends on the method of measurement and the context of theory. Using different definition of mass, we will get different spectrum a and constant S. In what follows, we take m_e and μ as the classical mass for computation. To get the anomalous magnetic moment, we introduce an infinitesimal external magnetic field $\mathbf{B}_{ext} = (0,0,B)$ with

$$\mathbf{A}_{ext} = \frac{1}{2}B(-y,x,0) = \frac{1}{2}Br\sin\theta(-\sin\varphi,\cos\varphi,0). \quad (3.16)$$

Adding $\mathbf{A}_{ext}$ to (3.5) and substituting it into (3.6), we get the increment of the energy

$$\Delta\mathscr{E} = |\frac{8\pi}{3}ec\int_0^\infty gfr^3dr|B \equiv \mu_z B, \quad (3.17)$$

where μ_z is the magnetic moment of the spinor. The dimensionless form is given by

$$\mu_z = \frac{2(a^2+1)}{3a}\frac{k|\int_0^\infty uv\rho^3 d\rho|}{\int_0^\infty(u^2+v^2)\rho^2 d\rho}\cdot\mu_B, \quad \mu_B \equiv \frac{e\hbar}{2m_k}, \quad (3.18)$$

where the constant μ_B is the Bohr magneton,

$$k = \begin{cases} 1 & \text{if } m_k = \mu, \\ \mathscr{E} & \text{if } m_k = m_e. \end{cases} \quad (3.19)$$

By (3.18), we get the anomalous magnetic moment of a particle

$$\Delta g \equiv \frac{\mu_z - \mu_B}{\mu_B} = \frac{2(a^2+1)k|\int_0^\infty uv\rho^3 d\rho|}{3a\int_0^\infty(u^2+v^2)\rho^2 d\rho} - 1. \quad (3.20)$$

The empirical value of the AMM of an electron is $\Delta g = 0.001159652$. The computational result shows nonlinear potential can provide an explanation for AMM.

To compare with the dark spinor, we also define the dimensionless norm by (2.11). The normalizing condition (2.10) is equivalent to $y = \lg S$. The dimensionless functions $(\mathscr{E},\Delta g,y)$ are all continuous functions of a for the same family of solutions. Fig.4 shows how to determine the spectrum a by the empirical AMM. Different definition of mass leads to different value of a.

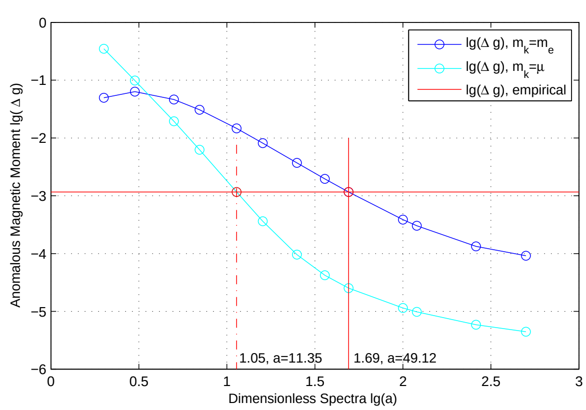

Fig. 4: The anomalous magnetic moment of the system (3.10) vs. the spectra a, the true value for an electron is $\Delta g = 0.001159652$ or $\lg(\Delta g) = -2.936$

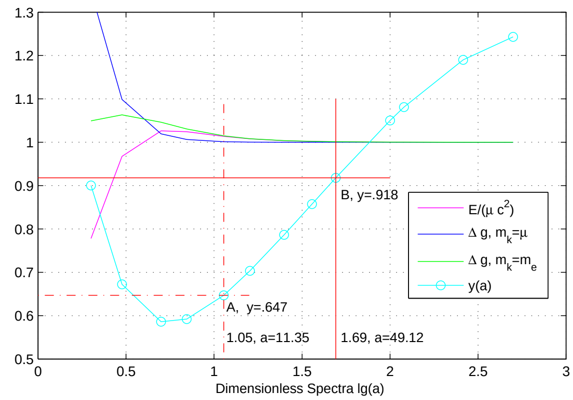

Fig. 5: The dimensionless functions $(\mathscr{E}(a),\Delta g(a),y(a))$. The constant S is determined by the intersection points A or B, which correspond to the empirical anomalous magnetic moment

In Fig.4, the trends of Δg shows that Δg is a decreasing function of a, and $\Delta g \to 0$ as $a \to \infty$. By the empirical data of Δg, we can compute the following undetermined parameters, If taking $m_k = \mathscr{E}\mu$, we have

$$a = 49.12,\ S = 8.277,\ w = 3.385\times10^{-57}\text{Jsm}^2,\ E_V = 1.088\text{keV},\ E_A = 85\text{eV} \quad (3.21)$$

If taking $m_k = \mu$, we have

$$a = 11.35,\ S = 4.434,\ w = 9.723\times10^{-58}\text{Jsm}^2,\ E_V = 15.08\text{keV},\ E_A = 330\text{eV} \quad (3.22)$$

Fig.5 shows the realistic values of some parameters such as the total energy $\mathscr{E}$, the norm function $y(a)$. The constants S or w is determined by normalizing condition $y = \lg S$, and then all other parameters can be computed. By Fig.5, we learn that, the value of a is larger than that of dark spinor, namely, the electromagnetic interaction increases the rest mass m of a spinor.

Since $\alpha \doteq \frac{1}{137}$ is quite small, by (3.2) we learn that, if $a > 10$, the electromagnetic field only have a little influence on the eigen solution. Fig.6 shows the comparison of the dimensionless fields when $a = 49.12$.

4 Discussion and conclusion

We have solved the particle-like eigen solutions to some nonlinear spinor equations, and computed several functions which reflect their characteristics. The numerical results show that, the nonlinear spinor equations have positive discrete mass spectra and anomalous magnetic moment. These unusual properties of spinor may have close relationship with the nature of the elementary particles.

1.By $P \to 0$ and (2.14), for $V = \frac{1}{2}w\check{\gamma}^2$ we find

$$mc^2 \to \int_0^\infty (\mu c^2\check{\gamma} + \frac{1}{2}wc\check{\gamma}^2)\cdot 4\pi r^2 dr. \quad (4.1)$$

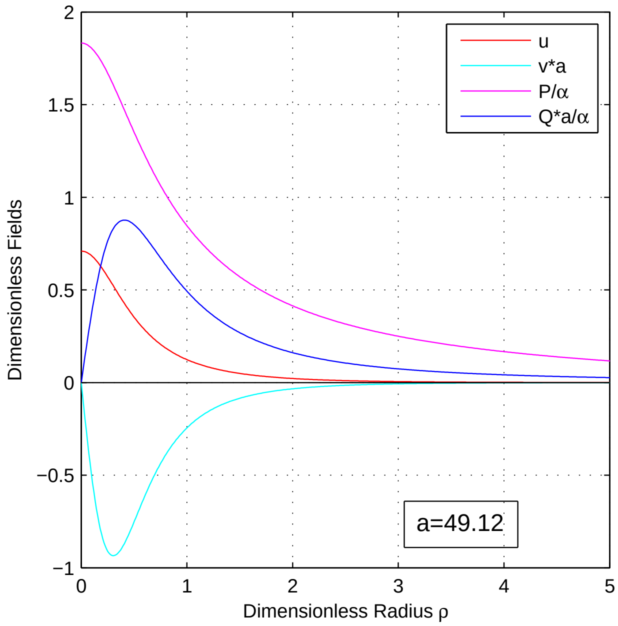

Fig. 6: The dimensionless radial functions, (u, v) correspond to spinor fields. (P, Q) correspond to the dimensionless potentials.

More calculations show that such relation also holds for other kind nonlinear potential $V(\check{\gamma})$ satisfying $V'\check{\gamma}-V>0$, namely we always have $|P|\ll E$. An interesting problem is whether the error is just caused by numerical approximation and $P=0$ is a rigorous relation generally valid for nonlinear spinors.

2.All dimensionless energy $\mathscr{E}(a)$ have a similar trend $\mathscr{E}\to 1(a\to\infty)$. For large enough a, we always have $E\to\mu c^2$.

3.For the nonlinear spinor equation with a scalar interactive potential

$$\alpha^\mu \hbar i\partial_\mu\phi=(\mu c+sG-V')\gamma\phi,\quad (\partial_\alpha\partial^\alpha+b^2)G=\lambda s\check{\gamma}, \tag{4.2}$$

similar to (3.12) and (3.13), G can be expressed as

$$G(r)=\frac{\lambda s}{r}\int_0^r e^{-b(r-\tau)}d\tau\int_\tau^\infty \check{\gamma}(\xi)\xi e^{-b(\xi-\tau)}d\xi, \tag{4.3}$$

so the solution to (4.2) can be soundly solved by iteration. For the AMM Δg defined by (3.20), computations show that we always have $\Delta g\sim 0$ similar to the above cases with electromagnetic interaction.

4.The energy functional of the nonlinear spinor system (3.1) is indefinite, so the stability of the solutions is different from that of the positive definite system. There are some works on this problem[33,34,35].

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Ying-Qiu Gu Senior Engineer, received the PhD degree in Institute of Mathematics at Fudan University. His research interests are in the areas of fundamental physics including the mathematical methods and models for elementary systems, general relativity, foundation of physics.