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Existence of Smooth Solutions of Multi-Term Caputo-Type Fractional Differential Equations

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Abstract: This work deals with the initial value problem for the multi-term fractional differential equation. The fractional derivative is defined in the Caputo sense. Firstly the initial value problem is transformed into an equivalent Volterra-type integral equation under appropriate assumptions. Then, new existence results for smooth solutions are established by using the Schauder fixed point theorem.

Keywords: Caputo fractional derivative, initial value problem, multi-term fractional differential equation, existence of solution.

1 Introduction

Let's consider multi-term Caputo-type fractional differential equations of the form

$$D^{\alpha_n}u(t) = f(t, D^{\alpha_1}u(t), \dots, D^{\alpha_{n-1}}u(t)) \tag{1}$$

subject to initial conditions

$$u^{(i)}(0) = u_0^{(i)}, i = 0, 1, \dots, \lceil \alpha_n \rceil - 1, \tag{2}$$

where $\alpha_n > \alpha_{n-1} > \dots > \alpha_1 \geq 0$ and the symbol D^β denotes the Caputo-type fractional differential operator defined by ([1,2])

$$D^\beta u = J^{\lceil \beta \rceil - \beta} u^{(\lceil \beta \rceil)}.$$

Here J^γ is the Riemann-Liouville integral operator of order $\gamma \geq 0$ defined by J^0 being the identity operator and

$$J^\gamma u(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} u(s) ds$$

for $\gamma > 0$.

Fractional differential equations have boosted a great deal of interest in areas such as porous media, plasma dynamics, thermodynamics, cosmic rays, continuum mechanics, biological systems, electrodynamics, quantum mechanics [2,3,4,5,6,7,8]. In particular, the relaxation modulus and creep compliance of the multi-term fractional constitutive model which describe the linear viscoelastic behaviour are obtained from multi-term fractional differential equations [4,9].

In general, it is very difficult to obtain analytical solutions of fractional differential equations. Although an analytical expression of solutions of initial value problems of linear differential equations with constant coefficients and Caputo derivative was studied before [10], it is quite cumbersome to be handled. The predictor-corrector method is one of powerful tools for obtaining numerical solutions of fractional differential equations [11,12,13,14]. It was proved that the initial value problem (1)-(2) is equivalent to a fractional differential system when the solution u is in $C^{\lceil \alpha_n \rceil}[0, L]$ [12,14]. Based on the equivalence theorems, the studies used the predictor-corrector method to obtain numerical solutions of the initial

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value problem (1)-(2). Thus it is significant to study the existence of $\lceil\alpha_n\rceil$ times continuously differentiable solutions of the initial value problem (1)-(2).

Analytical properties of fractional differential equations can be investigated by considering equivalent Volterra-type integral equations [1,2,11,15,16,17]. The smoothness of solutions of single-term Caputo-type differential equations was already studied [15]. The existence of $\lceil\alpha_2\rceil$ times continuously differentiable solutions to two-term fractional differential equations of the form

$$D^{\alpha_2}u(t) = f(t, D^{\alpha_1}u(t)) \tag{3}$$

subject to initial conditions

$$u^{(i)}(0) = u_0^{(i)}, i = 0, 1, ..., \lceil\alpha_2\rceil - 1 \tag{4}$$

was also handled before [16]. The following lemmas are essential in [16].

Lemma 1([16]). *Let $\alpha_1, \alpha_2 \notin N$ and $\lceil\alpha_1\rceil < \lceil\alpha_2\rceil$. Suppose that $f(0,0) = 0$, $f(t,0) \neq 0$ on a compact subinterval of $(0,1]$ and $f : [0,1]\times R \to R$ is continuously differentiable. Then a function $u \in C^{\lceil\alpha_2\rceil}[0,1]$ is a solution of the initial value problem (3)-(4) if and only if*

$$u(t) = \sum_{i=0}^{\lceil\alpha_1\rceil-1} \frac{t^i}{i!}u_0^{(i)} + \int_0^t \frac{(t-s)^{\lceil\alpha_1\rceil-1}}{(\lceil\alpha_1\rceil-1)!}v(s)ds, t \in [0,1], \tag{5}$$

where $v \in C[0,1]$ is a solution of the integral equation

$$v(t) = \sum_{i=0}^{\lceil\alpha_2\rceil-\lceil\alpha_1\rceil-1} \frac{t^i}{i!}u_0^{(i+\lceil\alpha_1\rceil)} + \int_0^t \frac{(t-s)^{\alpha_2-\lceil\alpha_1\rceil-1}}{\Gamma(\alpha_2-\lceil\alpha_1\rceil)} f\left(s, \frac{1}{\Gamma(\lceil\alpha_1\rceil-\alpha_1)}\int_0^s (s-w)^{\lceil\alpha_1\rceil-\alpha_1-1}v(w)dw\right)ds. \tag{6}$$

Lemma 2([16]). *Let $\alpha_1, \alpha_2 \notin N$ and $\lceil\alpha_1\rceil = \lceil\alpha_2\rceil$. Suppose that $f(0,0) = 0$, $f(t,0) \neq 0$ on a compact subinterval of $(0,1]$ and $f : [0,1]\times R \to R$ is continuously differentiable. Then a function $u \in C^{\lceil\alpha_2\rceil}[0,1]$ is a solution of the initial value problem (3)-(4) if and only if*

$$u(t) = \sum_{i=0}^{\lceil\alpha_1\rceil-1} \frac{t^i}{i!}u_0^{(i)} + \frac{1}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1-1}v(s)ds, t \in [0,1], \tag{7}$$

where $v \in C[0,1]$ is a solution of the integral equation

$$v(t) = \frac{1}{\Gamma(\alpha_2-\alpha_1)}\int_0^t (t-s)^{\alpha_2-\alpha_1-1}f(s,v(s)). \tag{8}$$

Based on lemma 1 and lemma 2, different studies [18,19] established the existence and uniqueness of solutions to the initial value problem (3)-(4) on the interval [0,1].

Our present paper is organized as follows. In Section 2, we transform the initial value problem (1)- (2) into an equivalent Volterra-type integral equation under proper assumptions. In particular, it is proved that lemma 1 and lemma 2 are incomplete. With the help of Section 2, the correct existence results for smooth solutions to the initial value problem (1)- (2) are established in Section 3. We conclude the paper on Section 4.

2 Equivalent Integral Equations

In this section it is proved that the solvability of the initial value problem (1)- (2) is equivalent to that of a Volterra-type integral equation.

Lemma 3([20]). *Let $I > 0$ and assume that $u \in C^m[0,I]$ and $m-1 < \beta < \rho < m$. Then, for all $k \in \{1,...,m-1\}$, $D^{\rho-m+k}u^{(m-k)}(t) = D^\rho u(t)$ and $D^{\rho-\beta}D^\beta u(t) = D^\rho u(t)$.*

The following lemma plays an important role in our consideration.

Lemma 4. *Let $I > 0$, $0 < \beta < 1$, $g \in C^1[0,I]$ and the function $F : [0,I] \to R$ is defined by*

$$F(t) = \int_0^t (t-s)^{-\beta}g(s)ds.$$

Then $F(t) \in C^1[0,I]$ if and only if $g(0) = 0$.

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Proof.

$$F(t)=\frac{t^{1-\beta}}{1-\beta}g(0)+\frac{1}{1-\beta}\int_0^t (t-s)^{1-\beta}g'(s)ds, t\in[0,I].$$

$$F'(t)=t^{-\beta}g(0)+\int_0^t (t-s)^{-\beta}g'(s)ds, t\in(0,I].$$

Thus, $F'(t)$ is continuous in $[0,I]$ if and only if $g(0)=0$.

Theorem 1.*Let $I>0$, $\alpha_n,\alpha_{n-1}\notin N$ and $\lceil\alpha_{n-1}\rceil+1<\alpha_n$. Suppose that $f(0,...,0)=0, u_0^{(\lceil\alpha_{n-1}\rceil)}=0$ and $f:[0,I]\times R^{n-1}\to R$ is continuously differentiable. Then a function $u\in C^{\lceil\alpha_n\rceil}[0,I]$ is a solution of the initial value problem (1)- (2) if and only if*

$$u(t)=\sum_{i=0}^{\lceil\alpha_{n-1}\rceil-1}\frac{t^i}{i!}u_0^{(i)}+\int_0^t\frac{(t-s)^{\lceil\alpha_{n-1}\rceil-1}}{(\lceil\alpha_{n-1}\rceil-1)!}v(s)ds, t\in[0,I], \tag{9}$$

where $v\in C[0,I]$ is a solution of the integral equation

$$v(t)=\sum_{i=0}^{\lceil\alpha_n\rceil-\lceil\alpha_{n-1}\rceil-1}\frac{t^i}{i!}u_0^{(i+\lceil\alpha_{n-1}\rceil)}+\frac{1}{\Gamma(\alpha_n-\lceil\alpha_{n-1}\rceil)}\int_0^t (t-s)^{\alpha_n-\lceil\alpha_{n-1}\rceil-1}$$
$$f\left(s,\int_0^s\frac{(s-w)^{\lceil\alpha_{n-1}\rceil-\alpha_1-1}}{\Gamma(\lceil\alpha_{n-1}\rceil-\alpha_1)}v(w)dw,,\cdots,\int_0^s\frac{(s-w)^{\lceil\alpha_{n-1}\rceil-\alpha_{n-1}-1}}{\Gamma(\lceil\alpha_{n-1}\rceil-\alpha_{n-1})}v(w)dw\right)ds. \tag{10}$$

*Proof.*Using lemma 3, we have

$$D^{\alpha_n-\lceil\alpha_{n-1}\rceil}u^{(\lceil\alpha_{n-1}\rceil)}(t)=D^{\alpha_n}u(t)=f(t,u(t),D^{\alpha_1}u(t),\cdots,D^{\alpha_{n-1}}u(t)). \tag{11}$$

Applying the Riemann-Liouville integral $J^{\alpha_n-\lceil\alpha_{n-1}\rceil}$ for both sides of (11) and making the substitution $v(t)=u^{(\lceil\alpha_{n-1}\rceil)}(t)$, we obtain (9) and (10). In order to prove the converse, let $v\in C[0,I]$ be a solution of (10). By (9), it is clear to see that $u^{(\lceil\alpha_{n-1}\rceil)}(t)=v(t)$ and $u^{(j)}(0)=u_0^{(j)}$ for $j=0,1,...,\lceil\alpha_{n-1}\rceil$.
Since $\lceil\alpha_{n-1}\rceil+1<\alpha_n$, by (10), we can easily prove that $v(t)\in C^1[0,I]$. Differentiating (10), we have

$$D^jv(t)=\sum_{i=0}^{\lceil\alpha_n\rceil-\lceil\alpha_{n-1}\rceil-j-1}\frac{t^i}{i!}u_0^{(j+i+\lceil\alpha_{n-1}\rceil)}+\prod_{l=1}^{j}(\alpha_n-\lceil\alpha_{n-1}\rceil-l)\int_0^t\frac{(t-s)^{\alpha_n-\lceil\alpha_{n-1}\rceil-1-j}}{\Gamma(\alpha_n-\lceil\alpha_{n-1}\rceil)}$$
$$f\left(s,\int_0^s\frac{(s-w)^{\lceil\alpha_{n-1}\rceil-\alpha_1-1}}{\Gamma(\lceil\alpha_{n-1}\rceil-\alpha_1)}v(w)dw,\cdots,\int_0^s\frac{(s-w)^{\lceil\alpha_{n-1}\rceil-\alpha_{n-1}-1}}{\Gamma(\lceil\alpha_{n-1}\rceil-\alpha_{n-1})}v(w)dw\right)ds$$

and $D^jv(0)=D^{j+\lceil\alpha_{n-1}\rceil}u(0)=u_0^{(j+\lceil\alpha_{n-1}\rceil)}$ for $j=0,1,...,\lceil\alpha_n\rceil-\lceil\alpha_{n-1}\rceil-1$.

$$D^{\lceil\alpha_n\rceil-\lceil\alpha_{n-1}\rceil-1}v(t)=u_0^{(\lceil\alpha_n\rceil-1)}+\prod_{l=1}^{\lceil\alpha_n\rceil-\lceil\alpha_{n-1}\rceil-1}(\alpha_n-\lceil\alpha_{n-1}\rceil-l)\int_0^t\frac{(t-s)^{\alpha_n-\lceil\alpha_n\rceil}}{\Gamma(\alpha_n-\lceil\alpha_{n-1}\rceil)}$$

$$f\left(s,\int_0^s\frac{(s-w)^{\lceil\alpha_{n-1}\rceil-\alpha_1-1}}{\Gamma(\lceil\alpha_{n-1}\rceil-\alpha_1)}v(w)dw,,\cdots,\int_0^s\frac{(s-w)^{\lceil\alpha_{n-1}\rceil-\alpha_{n-1}-1}}{\Gamma(\lceil\alpha_{n-1}\rceil-\alpha_{n-1})}v(w)dw\right)ds.$$

Since $v(t)\in C^1[0,I]$ and $v(0)=u_0^{(\lceil\alpha_{n-1}\rceil)}=0$, and by using lemma 4, we have

$$\int_0^s\frac{(s-w)^{\lceil\alpha_{n-1}\rceil-\alpha_{n-1}-1}}{\Gamma(\lceil\alpha_{n-1}\rceil-\alpha_{n-1})}v(w)dw\in C^1[0,I].$$

and thus

$$f\left(s,\int_0^s\frac{(s-w)^{\lceil\alpha_{n-1}\rceil-\alpha_1-1}}{\Gamma(\lceil\alpha_{n-1}\rceil-\alpha_1)}v(w)dw,,\cdots,\int_0^s\frac{(s-w)^{\lceil\alpha_{n-1}\rceil-\alpha_{n-1}-1}}{\Gamma(\lceil\alpha_{n-1}\rceil-\alpha_{n-1})}v(w)dw\right)\in C^1[0,I].$$

Since $f(0,...,0)=0$, and using by Lemma 4, $D^{\lceil\alpha_n\rceil-\lceil\alpha_{n-1}\rceil-1}v(t)\in C^1[0,I]$. Therefore $v\in C^{\lceil\alpha_n\rceil-\lceil\alpha_{n-1}\rceil}[0,I]$ and $u\in C^{\lceil\alpha_n\rceil}[0,I]$.

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Theorem 2.*Let $I>0,\alpha_{n-1}\in N,\alpha_n\notin N$ and $\alpha_{n-1}+1<\alpha_n$. Suppose that $f(0,...,0,u_0^{(\alpha_{n-1})})=0$, $f:[0,I]\times R^{n-1}\to R$ is continuously differentiable. If $n>2$ and $\alpha_{n-1}-\alpha_{n-2}<1$, then we suppose that $u_0^{(\alpha_{n-1})}=0$. Then a function $u\in C^{\lceil\alpha_n\rceil}[0,I]$ is a solution of the initial value problem (1)-(2) if and only if*

$$u(t)=\sum_{i=0}^{\alpha_{n-1}-1}\frac{t^i}{i!}u_0^{(i)}+\int_0^t\frac{(t-s)^{\alpha_{n-1}-1}}{(\alpha_{n-1}-1)!}v(s)ds,t\in[0,I], \tag{12}$$

where $v\in C[0,I]$ is a solution of the integral equation

$$v(t)=\sum_{i=0}^{\lceil\alpha_n\rceil-\alpha_{n-1}-1}\frac{t^i}{i!}u_0^{(i+\alpha_{n-1})}+\int_0^t\frac{(t-s)^{\alpha_n-\alpha_{n-1}-1}}{\Gamma(\alpha_n-\alpha_{n-1})} f\left(s,\int_0^s\frac{(s-w)^{\alpha_{n-1}-\alpha_1-1}}{\Gamma(\alpha_{n-1}-\alpha_1)}v(w)dw,\cdots,\int_0^s\frac{(s-w)^{\alpha_{n-1}-\alpha_{n-2}-1}}{\Gamma(\alpha_{n-1}-\alpha_{n-2})}v(w)dw,v(s)\right)ds. \tag{13}$$

*Proof.*Using lemma 3, we have

$$D^{\alpha_n-\alpha_{n-1}}u^{(\alpha_{n-1})}(t)=D^{\alpha_n}u(t)=f(t,u(t),D^{\alpha_1}u(t),\cdots,D^{\alpha_{n-1}}u(t)). \tag{14}$$

Applying the Riemann-Liouville integral $J^{\alpha_n-\alpha_{n-1}}$ for both sides of (14) and making the substitution $v(t)=u^{(\alpha_{n-1})}(t)$, we obtain (12) and (13). In order to prove the converse, let $v\in C[0,I]$ be a solution of (13). By (12), it is easy to see that $u^{(\alpha_{n-1})}(t)=v(t)$ and $u^{(j)}(0)=u_0^{(j)}$ for $j=0,1,...,\alpha_{n-1}$.
Since $\alpha_{n-1}+1<\alpha_n$, by (13), we can easily prove that $v(t)\in C^1[0,I]$. Differentiating (13), we have

$$D^jv(t)=\sum_{i=0}^{\lceil\alpha_n\rceil-\alpha_{n-1}-j-1}\frac{t^i}{i!}u_0^{(j+i+\alpha_{n-1})}+\prod_{l=1}^{j}(\alpha_n-\alpha_{n-1}-l)\int_0^t\frac{(t-s)^{\alpha_n-\alpha_{n-1}-1-j}}{\Gamma(\alpha_n-\alpha_{n-1})} f\left(s,\int_0^s\frac{(s-w)^{\alpha_{n-1}-\alpha_1-1}}{\Gamma(\alpha_{n-1}-\alpha_1)}v(w)dw,\cdots,\int_0^s\frac{(s-w)^{\alpha_{n-1}-\alpha_{n-2}-1}}{\Gamma(\alpha_{n-1}-\alpha_{n-2})}v(w)dw,v(s)\right)ds$$

and $D^jv(0)=D^{j+\alpha_{n-1}}u(0)=u_0^{(j+\alpha_{n-1})}$ for $j=0,1,...,\lceil\alpha_n\rceil-\alpha_{n-1}-1$.

$$D^{\lceil\alpha_n\rceil-\alpha_{n-1}-1}v(t)=u_0^{(\lceil\alpha_n\rceil-1)}+\prod_{l=1}^{\lceil\alpha_n\rceil-\alpha_{n-1}-1}(\alpha_n-\alpha_{n-1}-l)\int_0^t\frac{(t-s)^{\alpha_n-\lceil\alpha_n\rceil}}{\Gamma(\alpha_n-\alpha_{n-1})}$$

$$f\left(s,\int_0^s\frac{(s-w)^{\alpha_{n-1}-\alpha_1-1}}{\Gamma(\alpha_{n-1}-\alpha_1)}v(w)dw,,\cdots,\int_0^s\frac{(s-w)^{\alpha_{n-1}-\alpha_{n-2}-1}}{\Gamma(\alpha_{n-1}-\alpha_{n-2})}v(w)dw,v(s)\right)ds.$$

Since $v(t)\in C^1[0,I]$ and if $n>2,\alpha_{n-1}-\alpha_{n-2}<1$, then $u_0^{(\alpha_{n-1})}=0$, by Lemma 4, we have

$$\int_0^s\frac{(s-w)^{\alpha_{n-1}-\alpha_{n-2}-1}}{\Gamma(\alpha_{n-1}-\alpha_{n-2})}v(w)dw\in C^1[0,I].$$

and thus

$$f\left(s,\int_0^s\frac{(s-w)^{\alpha_{n-1}-\alpha_1-1}}{\Gamma(\alpha_{n-1}-\alpha_1)}v(w)dw,,\cdots,\int_0^s\frac{(s-w)^{\alpha_{n-1}-\alpha_{n-2}-1}}{\Gamma(\alpha_{n-1}-\alpha_{n-2})}v(w)dw,v(s)\right)\in C^1[0,I].$$

Since $f(0,...,0,u_0^{(\alpha_{n-1})})=0$, and using lemma 4, $D^{\lceil\alpha_n\rceil-\alpha_{n-1}-1}v(t)\in C^1[0,I]$. Therefore $v\in C^{\lceil\alpha_n\rceil-\alpha_{n-1}}[0,I]$ and $u\in C^{\lceil\alpha_n\rceil}[0,I]$.

Theorem 3.*Let $I>0,\alpha_{n-1}\notin N,\alpha_n\in N$ and $\lceil\alpha_{n-1}\rceil+1\le\alpha_n$. Suppose that $u_0^{(\lceil\alpha_{n-1}\rceil)}=0$ and $f:[0,I]\times R^{n-1}\to R$ is continuous. Then a function $u\in C^{\alpha_n}[0,I]$ is a solution of the initial value problem (1)-(2) if and only if*

$$u(t)=\sum_{i=0}^{\lceil\alpha_{n-1}\rceil-1}\frac{t^i}{i!}u_0^{(i)}+\int_0^t\frac{(t-s)^{\lceil\alpha_{n-1}\rceil-1}}{(\lceil\alpha_{n-1}\rceil-1)!}v(s)ds,t\in[0,I], \tag{15}$$

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where $v \in C[0,I]$ is a solution of the integral equation

$$v(t) = \sum_{i=0}^{\alpha_n-\lceil\alpha_{n-1}\rceil-1} \frac{t^i}{i!}u_0^{(i+\lceil\alpha_{n-1}\rceil)} + \int_0^t \frac{(t-s)^{\alpha_n-\lceil\alpha_{n-1}\rceil-1}}{\Gamma(\alpha_n-\lceil\alpha_{n-1}\rceil)} f\left(s, \int_0^s \frac{(s-w)^{\lceil\alpha_{n-1}\rceil-\alpha_1-1}}{\Gamma(\lceil\alpha_{n-1}\rceil-\alpha_1)}v(w)dw,,\cdots, \int_0^s \frac{(s-w)^{\lceil\alpha_{n-1}\rceil-\alpha_{n-1}-1}}{\Gamma(\lceil\alpha_{n-1}\rceil-\alpha_{n-1})}v(w)dw\right)ds. \quad (16)$$

*Proof.*Using lemma 3, we have

$$D^{\alpha_n-\lceil\alpha_{n-1}\rceil}u^{(\lceil\alpha_{n-1}\rceil)}(t) = D^{\alpha_n}u(t) = f(t,u(t),D^{\alpha_1}u(t),\cdots,D^{\alpha_{n-1}}u(t)). \quad (17)$$

Applying the Riemann-Liouville integral $J^{\alpha_n-\lceil\alpha_{n-1}\rceil}$ for both sides of (17) and making the substitution $v(t) = u^{(\lceil\alpha_{n-1}\rceil)}(t)$, we obtain (15) and (16). In order to prove the converse, let $v \in C[0,I]$ be a solution of (16). By (15), it is easy to see that $u^{(\lceil\alpha_{n-1}\rceil)}(t) = v(t)$ and $u^{(j)}(0) = u_0^{(j)}$ for $j = 0,1,...,\lceil\alpha_{n-1}\rceil$.
Since $\lceil\alpha_{n-1}\rceil+1 < \alpha_n$, by (16), we can easily prove that $v(t) \in C^1[0,I]$. Differentiating (16), we have

$$D^jv(t) = \sum_{i=0}^{\alpha_n-\lceil\alpha_{n-1}\rceil-j-1} \frac{t^i}{i!}u_0^{(j+i+\lceil\alpha_{n-1}\rceil)} + \prod_{l=1}^{j}(\alpha_n-\lceil\alpha_{n-1}\rceil-l)\int_0^t \frac{(t-s)^{\alpha_n-\lceil\alpha_{n-1}\rceil-1-j}}{\Gamma(\alpha_n-\lceil\alpha_{n-1}\rceil)} f\left(s, \int_0^s \frac{(s-w)^{\lceil\alpha_{n-1}\rceil-\alpha_1-1}}{\Gamma(\lceil\alpha_{n-1}\rceil-\alpha_1)}v(w)dw,\cdots, \int_0^s \frac{(s-w)^{\lceil\alpha_{n-1}\rceil-\alpha_{n-1}-1}}{\Gamma(\lceil\alpha_{n-1}\rceil-\alpha_{n-1})}v(w)dw\right)ds$$

and $D^jv(0) = D^{j+\lceil\alpha_{n-1}\rceil}u(0) = u_0^{(j+\lceil\alpha_{n-1}\rceil)}$ for $j = 0,1,...,\alpha_n-\lceil\alpha_{n-1}\rceil-1$.

$$D^{\alpha_n-\lceil\alpha_{n-1}\rceil-1}v(t) = u_0^{(\alpha_n-1)} + \prod_{l=1}^{\alpha_n-\lceil\alpha_{n-1}\rceil-1}(\alpha_n-\lceil\alpha_{n-1}\rceil-l)\frac{1}{\Gamma(\alpha_n-\lceil\alpha_{n-1}\rceil)}$$

$$\int_0^t f\left(s, \int_0^s \frac{(s-w)^{\lceil\alpha_{n-1}\rceil-\alpha_1-1}}{\Gamma(\lceil\alpha_{n-1}\rceil-\alpha_1)}v(w)dw,,\cdots, \int_0^s \frac{(s-w)^{\lceil\alpha_{n-1}\rceil-\alpha_{n-1}-1}}{\Gamma(\lceil\alpha_{n-1}\rceil-\alpha_{n-1})}v(w)dw\right)ds.$$

Since $v(t) \in C^1[0,I]$ and $v(0) = u_0^{(\lceil\alpha_{n-1}\rceil)} = 0$, by Lemma 4, we have

$$\int_0^s \frac{(s-w)^{\lceil\alpha_{n-1}\rceil-\alpha_{n-1}-1}}{\Gamma(\lceil\alpha_{n-1}\rceil-\alpha_{n-1})}v(w)dw \in C^1[0,I].$$

and thus

$$f\left(s, \int_0^s \frac{(s-w)^{\lceil\alpha_{n-1}\rceil-\alpha_1-1}}{\Gamma(\lceil\alpha_{n-1}\rceil-\alpha_1)}v(w)dw,,\cdots, \int_0^s \frac{(s-w)^{\lceil\alpha_{n-1}\rceil-\alpha_{n-1}-1}}{\Gamma(\lceil\alpha_{n-1}\rceil-\alpha_{n-1})}v(w)dw\right) \in C^1[0,I].$$

Then $D^{\alpha_n-\lceil\alpha_{n-1}\rceil-1}v(t) \in C^1[0,I]$. Therefore $v \in C^{\alpha_n-\lceil\alpha_{n-1}\rceil}[0,I]$ and $u \in C^{\alpha_n}[0,I]$.

We can easily see that lemma 1 and lemma 2 are more general than theorem 1 in the case $n = 2$. By making counterexamples, we show that lemma 1 and lemma 2 are incomplete. Firstly we present a counterexample of lemma 1. Set $I = 1, \alpha_1 = 1.8, \alpha_2 = 2.2, u_0^{(i)} = 0, i = 0,1,2$ and

$$f(t,s) = \frac{\Gamma(1.6)}{2\Gamma(1.4)}t^{0.4} + \frac{[\Gamma(1.6)\Gamma(1.8)]^{0.5}}{2\Gamma(1.4)}s^{0.5}.$$

Then it is easy to see that $v(t) = t^{0.6}$ is the solution of the equation(6). By the equation (5), we have

$$u(t) = \frac{\Gamma(1.6)}{\Gamma(3.6)}t^{2.6}.$$

It is clear that $u \notin C^3[0,1]$. Thus lemma 1 is incomplete. Secondly we give a counterexample of lemma 2. Set $I = 1, \alpha_1 = 1.4, \alpha_2 = 1.5, u_0^{(i)} = 0, i = 0,1$ and

$$f(t,s) = \frac{\Gamma(1.2)}{2\Gamma(1.1)}(t^{0.1}+s^{0.5}).$$

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Then we can easily see that $v(t) = t^{0.2}$ is the solution of (8). By (7), we have

$$u(t) = \frac{\Gamma(1.2)}{\Gamma(2.6)} t^{1.6}.$$

It is evident that $u \notin C^2[0,1]$. Thus lemma 2 is incomplete.

3 Existence of Solutions

In this section, based on the result of Section 2, the existence of smooth solutions of the two-term Caputo-type fractional differential equation (3)-(4) is established. In order to avoid the repetition of the proof process for our theorems, we state the following lemma.

Lemma 5.*Let $I > 0$, B be a convex bounded closed subset of $C[0,I]$ and $T : B \to C[0,I]$ is defined by*

$$Tv(t) = P(t) + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f\left(s, \frac{1}{\Gamma(\gamma)} \int_0^s (s-w)^{\gamma-1} v(w) dw\right) ds,$$

where $\beta, \gamma > 0$ and $P(t) \in C[0,I]$, $f(t,s) : [0,I] \times R \to R$ are continuous. If $T(B) \subset B$, then T has at least one fixed point in B.

*Proof.*Similar to the proof of theorem 6.1 in [1], we can prove this result by using Schauder fixed point theorem.

With the help of lemma 5, the initial value problem (3)-(4) is reduced to the problem for finding a bounded, convex and closed subset B in Y such that $T(B) \subset B$.

Theorem 4.*Let $k, I > 0$ and suppose that the hypotheses of theorem 1 hold. Define*

$$G = \left\{ (t,v) : t \in [0,I], |v| \le \frac{I^{\lceil \alpha_1 \rceil - \alpha_1}}{\Gamma(\lceil \alpha_1 \rceil - \alpha_1 + 1)} \left(k + \sum_{i=0}^{\lceil \alpha_2 \rceil - \lceil \alpha_1 \rceil - 1} \frac{I^i}{i!} \left| u_0^{(i+\lceil \alpha_1 \rceil)} \right| \right) \right\}$$

and $M := sup_{(t,v)\in G} f(t,v)$. Then the initial value problem (3)-(4) has at least one solution in $C^{\lceil \alpha_2 \rceil}[0,h]$, where h is defined by

$$h := \begin{cases} I & if\ M = 0 \\ min\left\{ I, \left(k\Gamma(\lceil \alpha_2 \rceil - \lceil \alpha_1 \rceil + 1)/M\right)^{\frac{1}{\lceil \alpha_2 \rceil - \lceil \alpha_1 \rceil}} \right\} & else. \end{cases}$$

*Proof.*We introduce the function P and the set B defined by

$$P(t) := \sum_{i=0}^{\lceil \alpha_2 \rceil - \lceil \alpha_1 \rceil - 1} \frac{t^i}{i!} u_0^{(i+\lceil \alpha_1 \rceil)}$$

and $B := \{v \in C[0,h] : \| P - v \| \le k\}$, where $\| \cdot \|$ is the supremum norm. In order to prove our desired result, using lemma 5 and theorem 1, we need to prove that $T(B) \subset B$ where T is defined by

$$Tv(t) = \sum_{i=0}^{\lceil \alpha_2 \rceil - \lceil \alpha_1 \rceil - 1} \frac{t^i}{i!} u_0^{(i+\lceil \alpha_1 \rceil)} + \int_0^t \frac{(t-s)^{\alpha_2 - \lceil \alpha_1 \rceil - 1}}{\Gamma(\alpha_2 - \lceil \alpha_1 \rceil)} f\left(s, \frac{1}{\Gamma(\lceil \alpha_1 \rceil - \alpha_1)} \int_0^s (s-w)^{\lceil \alpha_1 \rceil - \alpha_1 - 1} v(w) dw\right) ds.$$

For $v \in B$ and $t \in [0,h]$, we obtain

$$|v(t)| \le \| v \| \le k + \| P \| \le k + \sum_{i=0}^{\lceil \alpha_2 \rceil - \lceil \alpha_1 \rceil - 1} \frac{I^i}{i!} \left| u_0^{(i+\lceil \alpha_1 \rceil)} \right|.$$

Then we have

$$\int_0^t \frac{(t-s)^{\lceil \alpha_1 \rceil - \alpha_1 - 1} |v(s)|}{\Gamma(\lceil \alpha_1 \rceil - \alpha_1)} ds \le \frac{I^{\lceil \alpha_1 \rceil - \alpha_1}}{\Gamma(\lceil \alpha_1 \rceil - \alpha_1 + 1)} \left(k + \sum_{i=0}^{\lceil \alpha_2 \rceil - \lceil \alpha_1 \rceil - 1} \frac{I^i}{i!} \left| u_0^{(i+\lceil \alpha_1 \rceil)} \right| \right).$$

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$$|Tv(t)-P(t)| \leq \int_0^t \frac{(t-s)^{\alpha_2-\lceil\alpha_1\rceil-1}}{\Gamma(\alpha_2-\lceil\alpha_1\rceil)}\left|f\left(s,\int_0^s \frac{(s-w)^{\lceil\alpha_1\rceil-\alpha_1-1}v(w)}{\Gamma(\lceil\alpha_1\rceil-\alpha_1)}dw\right)\right|ds$$
$$\leq \int_0^t \frac{M(t-s)^{\alpha_2-\lceil\alpha_1\rceil-1}}{\Gamma(\alpha_2-\lceil\alpha_1\rceil)} \leq \frac{Mh^{\alpha_2-\lceil\alpha_1\rceil}}{\Gamma(\alpha_2-\lceil\alpha_1\rceil+1)} \leq k,$$

,which implies that $T(B) \subset B$.

Theorem 5. *Let $k>0$ and suppose that the hypotheses of theorem 2 hold. Define*

$$G := \left\{(t,v) : t \in [0,I], |v| \leq k + \sum_{i=0}^{\lceil\alpha_2\rceil-\alpha_1-1} \frac{I^i}{i!}\left|u_0^{(i+\alpha_1)}\right|\right\}$$

and $M := sup_{(t,v)\in G} f(t,v)$. Then the initial value problem (3)-(4) has at least one solution in $C^{\lceil\alpha_2\rceil}[0,h]$, where h is defined by

$$h := \begin{cases} I & if\, M=0 \\ min\left\{I, \left(k\Gamma(\lceil\alpha_2\rceil-\alpha_1+1)/M\right)^{\frac{1}{\lceil\alpha_2\rceil-\alpha_1}}\right\} & else. \end{cases}$$

Proof. Similar to the proof of theorem 4, we can prove this result.

Theorem 6. *Let $k>0$ and suppose that the hypotheses of theorem 3 hold. Define*

$$G = \left\{(t,v) : t \in [0,I], |v| \leq \frac{I^{\lceil\alpha_1\rceil-\alpha_1}}{\Gamma(\lceil\alpha_1\rceil-\alpha_1+1)}\left(k + \sum_{i=0}^{\alpha_2-\lceil\alpha_1\rceil-1} \frac{I^i}{i!}\left|u_0^{(i+\lceil\alpha_1\rceil)}\right|\right)\right\}$$

and $M := sup_{(t,v)\in G} f(t,v)$. Then the initial value problem (3)-(4) has at least one solution in $C^{\alpha_2}[0,h]$, where h is defined by

$$h := \begin{cases} I & if\, M=0 \\ min\left\{I, \left(k\Gamma(\alpha_2-\lceil\alpha_1\rceil+1)/M\right)^{\frac{1}{\alpha_2-\lceil\alpha_1\rceil}}\right\} & else. \end{cases}$$

Proof. Similar to the proof of theorem 4, we can prove this result.

Remark. Similar to the method of [7], the existence of global solutions to the initial value problem (3)-(4) can be obtained.

Remark. The existence results for the two-term fractional differential equation (3)-(4) presented in this section can be easily generalized to the multi-term fractional differential equation (1)-(2).

4 Conclusion

We identified new sufficient conditions for the existence of smooth solutions of multi-term Caputo-type fractional differential equations which were established by using Schauder fixed point theorem. In particular, by making a counterexample, it was pointed out that the previous result for existence of smooth solutions was incomplete.

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