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Algorithmic Convergence on Banach Space Valued Functions in Abstract g-Fractional Calculus

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Abstract: The main aim of the paper is to develop suitable algorithms for solving equations on Banach spaces. Several applications of the semi-local convergence are given including Banach space valued functions of fractional calculus considering all integrals of Bochner-type.

Keywords: Iterative method, Banach space, semi-local convergence, fractional calculus, Bochner-type integral.

1 Introduction

Let B_1, B_2 denote the Banach space and let Ω be an open subset of B_1. Let also $U(z,\rho) := \{u \in B_1 : \|u - z\| < \rho\}$ and let $\overline{U}(z,\rho)$ stand for the closure of $U(z,\rho)$.

We recall that many problems in several disciplines can be reduced to

$$F(y) = 0 \tag{1.1}$$

utilising the Mathematical Modeling [1]-[17], where $F : \Omega \to B_2$ denotes a continuous operator. We recall that the solution y^* of (1.1) is sought in closed form, but this is obtained only in particular cases. Thus, most solution methods for these type of equations are usually iterative. For more details of iterative methods for finding the solution of equation (1.1), we suggest [2, 6, 7, 9 - 13, 15, 16].

Newton's method [6,7,11,15,16]:

$$y_{n+1} = y_n - F'(y_n)^{-1} F(y_n). \tag{1.2}$$

Secant method:

$$y_{n+1} = y_n - [y_{n-1}, y_n; F]^{-1} F(y_n), \tag{1.3}$$

where $[\cdot,\cdot;F]$ is a divided difference of order one on $\Omega \times \Omega$ [7,15,16].

Newton-like method:

$$y_{n+1} = y_n - E_n^{-1} F(y_n), \tag{1.4}$$

where $E_n = E(F)(y_n)$ and $E : \Omega \to \mathscr{L}(B_1, B_2)$ the space of bounded linear operators from B_1 into B_2. We recall that the readers can find other methods in [7], [11], [15], [16] and the references therein.

Below we investigate the new method defined for each $n = 0, 1, 2, \ldots$ as

$$y_{n+1} = G(y_n)$$

$$G(y_{n+1}) = G(y_n) - A_n^{-1} F(y_n), \tag{1.5}$$

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where $x_0 \in \Omega$ is an initial point, $G : B_3 \to \Omega$ (B_3 a Banach space), $A_n = A(F)(y_{n+1}, y_n) = A(y_{n+1}, y_n)$ and $A : \Omega \times \Omega \to \mathscr{L}(B_1, B_2)$. The approach (1.5) provides a sequence which we shall show converges to y^* under some Lipschitz-type conditions (see Section 2). Having in mind that the method (1.5) (and Section 2) is of independent interest, it is nevertheless designed especially to be utilized in g-abstract fractional calculus (in Section 3). We recall that in our knowledge such iterative methods have not yet appeared in connection to find the solutions of the equations in abstract fractional calculus.

In Section 2 of this manuscript we show the semi-local convergence of method (1.5). Section 3 provides several applications to abstract g-fractional calculus on a given Banach space valued functions provided that all the integrals are of Bochner-type ([8], [14]).

2 Analysis of Semi-Local Convergence

Below we give the semi-local convergence analysis of (1.5) utilising the conditions (M):

(m_1) $F_1 : \Omega \subset B_1 \to B_2$ is continuous, $G_1 : B_3 \to \Omega$ is continuous and $A_1(x,y) \in \mathscr{L}(B_1, B_2)$ for each $(x,y) \in \Omega \times \Omega$.

(m_2) There exist $\beta > 0$ and $\Omega_0 \subset B_1$ in such a way that $A_1(x,y)^{-1} \in \mathscr{L}(B_2, B_1)$ for each $(x,y) \in \Omega_0 \times \Omega_0$ and

$$\left\| A_1(x,y)^{-1} \right\| \leq \beta^{-1}.$$

Set $\Omega_1 = \Omega \cap \Omega_0$.

(m_3) There exists a continuous and nondecreasing function $\psi : [0,+\infty)^3 \to [0,+\infty)$ in such way that for each $x, y \in \Omega_1$

$$\| F_1(x) - F_1(y) - A_1(x,y)(G_1(x) - G_1(y)) \| \leq$$

$$\beta \psi(\|x-y\|, \|x-x_0\|, \|y-x_0\|) \|G_1(x) - G_1(y)\|.$$

(m_4) There exists a continuous and nondecreasing function $\psi_0 : [0,+\infty) \to [0,+\infty)$ in such a way that for each $x \in \Omega_1$

$$\|G_1(x) - G_1(x_0)\| \leq \psi_0(\|x-x_0\|) \|x-x_0\|.$$

(m_5) For $x_0 \in \Omega_0$ and $x_1 = G_1(x_0) \in \Omega_0$ there exists $\eta \geq 0$ such that

$$\left\| A_1(x_1, x_0)^{-1} F_1(x_0) \right\| \leq \eta.$$

(m_6) There exists $s > 0$ fulfilling

$$\psi(\eta, s, s) < 1,$$

$$\psi_0(s) < 1$$

and

$$\|G_1(x_0) - x_0\| \leq s \leq \frac{\eta}{1-q_0},$$

where $q_0 = \psi(\eta, s, s)$.

(m_7) $\overline{U}(x_0, s) \subset \Omega$.

Below, we show the semi-local convergence analysis for (1.5) by using the conditions (M) as well as the previous notation.

Theorem 1. *Suppose that the conditions (M) hold. Then, sequence $\{y_n\}$ provided by method (1.5) starting at $y_0 \in \Omega$ is well defined in $U(y_0, s)$, remains in $U(y_0, s)$ for each $n = 0, 1, 2, \ldots$ and converges to a solution $y^* \in \overline{U}(y_0, s)$ of equation $F_1(y) = 0$. The limit point y^* denotes the unique solution of equation $F_1(y) = 0$ in $\overline{U}(y_0, s)$.*

Proof. Using the definition of s and (m_5), we have $y_1 \in U(y_0, s)$. The proof rely on mathematical induction on k. Assuming that $\|y_k - y_{k-1}\| \leq q_0^{k-1}\eta$ and $\|y_k - y_0\| \leq s$.

We get by (1.5), $(m_2) - (m_5)$ in turn that

$$\|G_1(y_{k+1}) - G_1(y_k)\| = \left\| A_k^{-1} F_1(y_k) \right\| =$$

$$\left\| A_k^{-1}(F_1(y_k) - F_1(y_{k-1}) - A_{k-1}(G_1(y_k) - G_1(y_{k-1}))) \right\| y$$

$$\leq \left\| A_k^{-1} \right\| \|F_1(y_k) - F_1(y_{k-1}) - A_{k-1}(G_1(y_k) - G_1(y_{k-1}))\| \leq$$

$$\beta^{-1}\beta\psi\left(\|y_k-y_{k-1}\|,\|y_{k-1}-y_0\|,\|y_k-x_0\|\right)\|G_1(y_k)-G_1(y_{k-1})\|\leq$$

$$\psi(\eta,s,s)\|G_1(y_k)-G_1(y_{k-1})\|=q_0\|G_1(y_k)-G_1(y_{k-1})\|\leq q_0^k\|y_1-y_0\|\leq q_0^k\eta \tag{2.1}$$

and by (m_6)

$$\|y_{k+1}-y_0\|=\|G_1(y_k)-y_0\|\leq\|G_1(y_k)-G_1(y_0)\|+\|G_1(y_0)-y_0\|$$
$$\leq\psi_0(\|y_k-y_0\|)\|y_k-y_0\|+\|G_1(y_0)-y_0\|$$
$$\leq\psi_0(s)s+\|G_1(y_0)-y_0)\leq s.$$

Thus,the induction is finished. In addition, (2.1) implies that for $m=0,1,2,\ldots$

$$\|y_{k+m}-y_k\|\leq\frac{1-q_0^m}{1-q_0}q_0^k\eta.$$

It implies from the preceding inequation that sequence $\{G_1(y_k)\}$ is complete in a Banach space B_1, thus it converges to some $y^*\in\overline{U}(y_0,s)$ (since $\overline{U}(_0,s)$ is a closed ball). By considering $k\to+\infty$ in (2.1) we obtain $F_1(y^*)=0$. From(1.5) we conclude that $G_1(y^*)=y^*$. To prove the uniqueness part, let $y^{**}\in U(y_0,s)$ be a solution of $F_1(y)=0$ and $G_1(y^{**})=y^{**}$. From (1.5), we conclude that

$$\|y^{**}-G_1(y_{k+1})\|=\left\|y^{**}-G_1(y_k)+A_k^{-1}F_1(y_k)-A_k^{-1}F_1(y^{**})\right\|\leq$$

$$\left\|A_k^{-1}\right\|\|F_1(y^{**})-F_1(y_k)-A_k(G_1(y^{**})-G_1(y_k))\|\leq$$

$$\beta^{-1}\beta\psi_0\left(\|y^{**}-y_k\|,\|y_{k+1}-y_0\|,\|y_k-y_0\|\right)\|G_1(y^{**})-G_1(y_k)\|\leq$$

$$q_0\|G_1(y^{**})-G_1(y_k)\|\leq q_0^{k+1}\|y^{**}-y_0\|,$$

so $\lim_{k\to+\infty}y_k=y^{**}$. We proved that $\lim_{k\to+\infty}y_k=y^*$, so $y^*=y^{**}$.

Remark.(1) Condition (m_2) can become part of (m_3) by taking into account the followings

$(m_3)'$ There exists a continuous and nondecreasing function $\varphi:[0,+\infty)^3\to[0,+\infty)$ in such a way that for each $x,y\in\Omega_1$

$$\left\|A_1(x,y)^{-1}\left[F_1(x)-F_1(y)-A_1(x,y)(G_1(x)-G_1(y))\right]\right\|\leq$$

$$\varphi(\|x-y\|,\|x-x_0\|,\|y-x_0\|)\|G_1(x)-G_1(y)\|.$$

Notice that

$$\varphi(u_1,u_2,u_3)\leq\psi(u_1,u_2,u_3)$$

for each $u_1\geq0$, $u_2\geq0$ and $u_3\geq0$. Similarly, a function φ_1 can replace ψ_1 for the uniqueness of the solution part. These changes are of Mysovskii-type [6], [11], [15] and influence the weaking of the convergence criterion in (m_6), error bounds as well as the precision of s.

(2) Assume that there exist $\beta>0$, $\beta_1>0$ and $L\in\mathscr{L}(B_1,B_2)$ with $L^{-1}\in\mathscr{L}(B_2,B_1)$ in such a way that

$$\left\|L^{-1}\right\|\leq\beta^{-1}$$

$$\|A_1(x,y)-L\|\leq\beta_1$$

and

$$\beta_2:=\beta^{-1}\beta_1<1.$$

Thus, it follows from the Banach lemma on invertible operators [11], and

$$\left\|L^{-1}\right\|\|A_1(x,y)-L\|\leq\beta^{-1}\beta_1=\beta_2<1$$

that $A_1(x,y)^{-1}\in\mathscr{L}(B_2,B_1)$. Let $\beta=\frac{\beta^{-1}}{1-\beta_2}$. As a result, under these replacements, condition (m_2) is implied, therefore it can be dropped from the conditions (M).

Remark. We conclude that the results especially of Theorem 1 can apply in abstract g-fractional calculus as given in Section 3. We can use the results of say Theorem 1 in the examples described in Section 3 by specializing function ψ. Particularly, for (3.4), we choose for $u_1 \geq 0$, $u_2 \geq 0$, $u_3 \geq 0$

$$\psi(u_1,u_2,u_3) = \frac{\lambda \mu_1^{(n+1)\alpha}}{\beta\Gamma((n+1)\alpha)((n+1)\alpha+1)},$$

if $|g_1(x) - g_1(y)| \leq \mu_1$ for each $x,y \in [a,b]$;

$$\psi(u_1,u_2,u_3) = \frac{\lambda \mu_2^{(n+1)\alpha}}{\beta\Gamma((n+1)\alpha)((n+1)\alpha+1)},$$

if $|g_1(x) - g_1(y)| \leq \xi_2 \|x-y\|$ for each $x,y \in [a,b]$ and $\mu_2 = \xi_2 |b-a|$;

$$\psi(u_1,u_2,u_3) = \frac{\lambda \mu_3^{(n+1)\alpha}}{\beta\Gamma((n+1)\alpha)((n+1)\alpha+1)},$$

if $|g_1(x)| \leq \xi_3$ for each $x,y \in [a,b]$ and $\mu_3 = 2\xi_3$.

We recall that other choices of function ψ are also valid.

We conclude that with these choices of function ψ and $f_1 = F_1$ and $g_1 = G_1$, fundamental condition (m_3) is fulfilled, which justifies our definition of method (1.5). Similar choices for the remaining examples of Section 3 can be provided.

3 X-Valued Modified g-Fractional Calculus Applications

In this section we consider with Banach space $(X, \|\cdot\|)$ valued functions f_1 of real domain $[a,b]$. Besides, all integrals considered in this section are of Bochner-type, see [14]. For the definition of the derivatives of f_1, see [17].

Let $0 < \alpha \leq 1$, $m = \lceil \alpha \rceil = 1$ ($\lceil \cdot \rceil$ ceiling of number), g_1 is strictly increasing and $g_1 \in C^1([a,b])$, $g_1^{-1} \in C([g_1(a), g_1(b)])$. Suppose that $f_1 \in C^1([a,b], X)$. Below we used the notations from [5].

I) The expression of the X-valued right generalized g-fractional derivative of f of order α is written as:

$$\left(D_{b-;g_1}^{\alpha} f_1\right)(x) := \frac{-1}{\Gamma(1-\alpha)} \int_x^b (g_1(\tau) - g_1(x))^{-\alpha} g_1'(\tau) \left(f_1 \circ g_1^{-1}\right)'(g_1(\tau))\, d\tau, \tag{3.1}$$

$a \leq x \leq b$.

If $0 < \alpha < 1$, then $\left(D_{b-;g_1}^{\alpha} f_1\right) \in C([a,b], X)$[4].

Besides, we introduce the following definitions

$$\left(D_{b-;g_1}^{1} f_1\right)(x) := -\left(\left(f_1 \circ g^{-1}\right)' \circ g_1\right)(x), \tag{3.2}$$

$$\left(D_{b-;g_1}^{0} f_1\right)(x) := f_1(x), \ \forall\, x \in [a,b].$$

For $g_1 = id$, we have

$$D_{b-;g_1}^{\alpha} f_1(x) = D_{b-;id}^{\alpha} f_1(x) = D_{b-}^{\alpha} f_1(x), \tag{3.3}$$

the classical X-valued right Caputo fractional derivative [3].

Let us denote

$$D_{b-;g_1}^{n\alpha} := D_{b-;g_1}^{\alpha} D_{b-;g_1}^{\alpha} \ldots D_{b-;g_1}^{\alpha} \quad (n \text{ times}),\ n \in \mathbb{N}. \tag{3.4}$$

The expression of X-valued right generalized fractional Riemann-Liouville integral is written as

$$\left(I_{b-;g_1}^{\alpha} f_1\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (g_1(\tau) - g_1(x))^{\alpha-1} g_1'(\tau) f_1(\tau)\, d\tau, \ \ a \leq x \leq b. \tag{3.5}$$

We have

$$I_{b-;g_1}^{n\alpha} := I_{b-;g_1}^{\alpha} I_{b-;g_1}^{\alpha} \ldots I_{b-;g_1}^{\alpha} \quad (n \text{ times}). \tag{3.6}$$

The following X-valued modified g_1-right generalized Taylor's formula will be used

Theorem 2.([5]) *Let $f_1 \in C^1([a,b],X)$, $g_1 \in C^1([a,b])$, strictly increasing, fulfilling $g_1^{-1} \in C^1([g_1(a), g_1(b)])$. Assume that $F_k := D_{b-;g_1}^{k\alpha} f_1$, $k = 1,...,n$, obey $F_k \in C^1([a,b],X)$, and $F_{n+1} \in C([a,b],X)$, where $0 < \alpha \leq 1$, $n \in \mathbb{N}$. As a result we have*

$$f_1(x) - f_1(b) = \sum_{i=1}^{n} \frac{(g_1(b) - g_1(x))^{i\alpha}}{\Gamma(i\alpha+1)} \left(D_{b-;g_1}^{i\alpha} f_1\right)(b) + \tag{3.7}$$

$$\frac{1}{\Gamma((n+1)\alpha)} \int_x^b (g_1(\tau) - g_1(x))^{(n+1)\alpha-1} g'(\tau) \left(D_{b-;g_1}^{(n+1)\alpha} f_1\right)(\tau) d\tau,$$

$\forall\, x \in [a,b]$.

In this section we discuss a more general case. Let $f_1 \in C^1([a,b],X)$. The definition of X-valued right generalized g_1-fractional derivative is given by

$$\left(D_{y-;g_1}^{\alpha} f_1\right)(x) := \frac{-1}{\Gamma(1-\alpha)} \int_x^y (g_1(\tau) - g_1(x))^{-\alpha} g_1'(\tau) \left(f_1 \circ g_1^{-1}\right)'(g_1(\tau)) d\tau, \tag{3.8}$$

all $a \leq x \leq y$; $y \in [a,b]$,

$$\left(D_{y-;g_1}^{1} f_1\right)(x) := -\left(\left(f_1 \circ g_1^{-1}\right)' \circ g_1\right)(x), \ \forall\, x \in [a,b]. \tag{3.9}$$

In the same manner we define:

$$\left(D_{x-;g_1}^{\alpha} f_1\right)(y) := \frac{-1}{\Gamma(1-\alpha)} \int_y^x (g_1(\tau) - g_1(y))^{-\alpha} g_1'(\tau) \left(f_1 \circ g_1^{-1}\right)'(g_1(\tau)) d\tau, \tag{3.10}$$

all $a \leq y \leq x$; $x \in [a,b]$,

$$\left(D_{x-;g_1}^{1} f_1\right)(y) := -\left(\left(f_1 \circ g_1^{-1}\right)' \circ g_1\right)(y), \ \forall\, y \in [a,b]. \tag{3.11}$$

When $0 < \alpha < 1$, $D_{y-;g_1}^{\alpha} f_1$ and $D_{x-;g_1}^{\alpha} f_1$ are continuous functions on $[a,b]$, see [5]. We recall that by convention we have

$$\begin{array}{l} \left(D_{y-;g_1}^{\alpha} f_1\right)(x) = 0, \ \text{for } x > y \\ \text{and} \\ \left(D_{x-;g_1}^{\alpha} f_1\right)(y) = 0, \ \text{for } y > x \end{array} \tag{3.12}$$

Denote by

$$F_k^y := D_{y-;g_1}^{k\alpha} f_1, \ \ F_k^x := D_{x-;g}^{k\alpha} f_1, \ \forall\, x,y \in [a,b]. \tag{3.13}$$

We suppose that

$$F_k^y, F_k^x \in C^1([a,b],X), \text{ and } F_{n+1}^y, F_{n+1}^x \in C([a,b],X), \tag{3.14}$$

$k = 1,...,n$, $\forall\, x,y \in [a,b]$; $0 < \alpha < 1$.

We also observe that $(0 < \alpha < 1)$ ([8])

$$\left\| \left(D_{b-;g_1}^{\alpha} f_1\right)(x) \right\| \leq \frac{1}{\Gamma(1-\alpha)} \int_x^b (g_1(\tau) - g_1(x))^{-\alpha} g_1'(\tau) \left\| \left(f_1 \circ g_1^{-1}\right)'(g_1(\tau)) \right\| d\tau \leq \tag{3.15}$$

$$\frac{\left\| \left(f_1 \circ g_1^{-1}\right)' \circ g_1 \right\|_{\infty,[a,b]}}{\Gamma(1-\alpha)} \int_x^b (g_1(\tau) - g_1(x))^{-\alpha} g_1'(\tau) d\tau =$$

$$\frac{\left\| \left(f_1 \circ g_1^{-1}\right)' \circ g_1 \right\|_{\infty,[a,b]}}{\Gamma(1-\alpha)} \frac{(g_1(b) - g_1(x))^{1-\alpha}}{1-\alpha} =$$

$$\frac{\left\| \left(f_1 \circ g_1^{-1}\right)' \circ g_1 \right\|_{\infty,[a,b]}}{\Gamma(2-\alpha)} (g_1(b) - g_1(x))^{1-\alpha}, \ \forall\, x \in [a,b].$$

We have show that

$$\left\| \left(D_{b-;g_1}^{\alpha} f_1\right)(x) \right\| \leq \frac{\left\| \left(f_1 \circ g_1^{-1}\right)' \circ g_1 \right\|_{\infty,[a,b]}}{\Gamma(2-\alpha)} (g_1(b) - g_1(x))^{1-\alpha} \tag{3.16}$$

$$\leq \frac{\left\| \left(f_1 \circ g_1^{-1}\right)' \circ g_1 \right\|_{\infty,[a,b]}}{\Gamma(2-\alpha)} \left(g_1(b) - g_1(a)\right)^{1-\alpha}, \ \forall\, x,y \in [a,b].$$

Clearly here we have

$$\left(D_{b-;g_1}^{\alpha} f_1\right)(b) = 0, \ 0 < \alpha < 1. \tag{3.17}$$

As a particular case we have

$$\left(D_{x-;g_1}^{\alpha} f_1\right)(x) = \left(D_{y-;g_1}^{\alpha} f_1\right)(y) = 0, \ \forall\, x,y \in [a,b]; \ 0 < \alpha < 1. \tag{3.18}$$

By (3.7) we derive

$$f_1(x) - f_1(y) = \sum_{i=2}^{n} \frac{\left(g_1(y) - g_1(x)\right)^{i\alpha}}{\Gamma(i\alpha+1)} \left(D_{y-;g_1}^{i\alpha} f_1\right)(y) + \tag{3.19}$$

$$\frac{1}{\Gamma((n+1)\alpha)} \int_x^y \left(g_1(\tau) - g_1(x)\right)^{(n+1)\alpha-1} g_1'(\tau) \left(D_{y-;g_1}^{(n+1)\alpha} f_1\right)(\tau)\, d\tau,$$

$\forall\, x < y$; $x,y \in [a,b]$; $0 < \alpha < 1$, and also it holds:

$$f_1(y) - f_1(x) = \sum_{i=2}^{n} \frac{\left(g_1(x) - g_1(y)\right)^{i\alpha}}{\Gamma(i\alpha+1)} \left(D_{x-;g_1}^{i\alpha} f_1\right)(x) + \tag{3.20}$$

$$\frac{1}{\Gamma((n+1)\alpha)} \int_y^x \left(g_1(\tau) - g_1(y)\right)^{(n+1)\alpha-1} g_1'(\tau) \left(D_{x-;g_1}^{(n+1)\alpha} f_1\right)(\tau)\, d\tau,$$

$\forall\, y < x$; $x,y \in [a,b]$; $0 < \alpha < 1$.

We define also the following X-valued linear operator

$$\left(A_1(f_1)\right)(x,y) :=$$

$$\begin{cases} \sum_{i=2}^{n} \frac{(g_1(y)-g_1(x))^{i\alpha-1}}{\Gamma(i\alpha+1)} \left(D_{y-;g_1}^{i\alpha} f_1\right)(y) - \left(D_{y-;g_1}^{(n+1)\alpha} f_1(x)\right) \frac{(g_1(y)-g_1(x))^{(n+1)\alpha-1}}{\Gamma((n+1)\alpha+1)}, & x < y, \\ \sum_{i=2}^{n} \frac{(g_1(x)-g_1(y))^{i\alpha-1}}{\Gamma(i\alpha+1)} \left(D_{x-;g}^{i\alpha} f_1\right)(x) - \left(D_{x-;g_1}^{(n+1)\alpha} f_1(y)\right) \frac{(g_1(x)-g_1(y))^{(n+1)\alpha-1}}{\Gamma((n+1)\alpha+1)}, & x > y, \\ f_1'(x), \ \text{when } x = y, \end{cases} \tag{3.21}$$

$\forall\, x,y \in [a,b]$; $0 < \alpha < 1$.

We may suppose that (see [12], p. 3)

$$\left\| \left(A_1(f_1)\right)(x,x) - \left(A_1(f_1)\right)(y,y) \right\| = \left\| f_1'(x) - f_1'(y) \right\|$$

$$= \left\| \left(f_1' \circ g_1^{-1}\right)(g_1(x)) - \left(f_1' \circ g_1^{-1}\right)(g_1(y)) \right\| \leq \Phi \left| g_1(x) - g_1(y) \right|, \tag{3.22}$$

$\forall\, x,y \in [a,b]$; with $\Phi > 0$.

We estimate and conclude that

i) case $x < y$:

$$\left\| f_1(x) - f_1(y) - \left(A_1(f_1)\right)(x,y)\left(g_1(x) - g_1(y)\right) \right\| =$$

$$\left\| \frac{1}{\Gamma((n+1)\alpha)} \int_x^y \left(g_1(\tau) - g_1(x)\right)^{(n+1)\alpha-1} g_1'(\tau) \left(D_{y-;g_1}^{(n+1)\alpha} f_1\right)(\tau)\, d\tau - \right.$$

$$\left. \left(D_{y-;g_1}^{(n+1)\alpha} f_1(x)\right) \frac{\left(g_1(y) - g_1(x)\right)^{(n+1)\alpha}}{\Gamma((n+1)\alpha+1)} \right\| \tag{3.23}$$

(by [1], p. 426, Theorem 11.43)

$$= \frac{1}{\Gamma((n+1)\alpha)} \cdot$$

$$\left\| \int_x^y \left(g_1(\tau) - g_1(x)\right)^{(n+1)\alpha-1} g_1'(\tau) \left(\left(D_{y-;g_1}^{(n+1)\alpha} f_1\right)(\tau) - \left(D_{y-;g_1}^{(n+1)\alpha} f_1\right)(x) \right) d\tau \right\|$$

(by [8])

$$\leq \frac{1}{\Gamma((n+1)\alpha)}\cdot$$

$$\int_x^y (g_1(\tau) - g_1(x))^{(n+1)\alpha-1} g_1'(\tau) \left\| \left(D_{y-;g_1}^{(n+1)\alpha} f_1\right)(\tau) - \left(D_{y-;g_1}^{(n+1)\alpha} f_1\right)(x) \right\| d\tau. \tag{3.24}$$

We suppose that

$$\left\| \left(D_{y-;g_1}^{(n+1)\alpha} f_1\right)(\tau) - \left(D_{y-;g_1}^{(n+1)\alpha} f_1\right)(x) \right\| \leq \lambda_1 |g_1(\tau) - g_1(x)|, \tag{3.25}$$

$\forall\, t,x,y \in [a,b] : y \geq t \geq x;\ \lambda_1 > 0)$

$$\leq \frac{\lambda_1}{\Gamma((n+1)\alpha)} \int_x^y (g_1(\tau) - g_1(x))^{(n+1)\alpha-1} g_1'(\tau)(g_1(\tau) - g_1(x))\, d\tau = \tag{3.26}$$

$$\frac{\lambda_1}{\Gamma((n+1)\alpha)} \int_x^y (g_1(\tau) - g_1(x))^{(n+1)\alpha} g_1'(\tau)\, d\tau =$$

$$\frac{\lambda_1}{\Gamma((n+1)\alpha)} \frac{(g_1(y) - g_1(x))^{(n+1)\alpha+1}}{((n+1)\alpha+1)}. \tag{3.27}$$

We have proved that

$$\| f_1(x) - f_1(y) - (A_1(f_1))(x,y)(g_1(x) - g_1(y)) \| \leq$$

$$\frac{\lambda_1}{\Gamma((n+1)\alpha)} \frac{(g_1(y) - g_1(x))^{(n+1)\alpha+1}}{((n+1)\alpha+1)}, \tag{3.28}$$

for any $x,y \in [a,b] : x < y;\ 0 < \alpha < 1$.

ii) case $x > y$:

$$\| f_1(x) - f_1(y) - (A_1(f_1))(x,y)(g_1(x) - g_1(y)) \| =$$

$$\| f_1(y) - f_1(x) - (A_1(f_1))(x,y)(g_1(y) - g_1(x)) \| = \tag{3.29}$$

$$\left\| \frac{1}{\Gamma((n+1)\alpha)} \int_y^x (g_1(\tau) - g_1(y))^{(n+1)\alpha-1} g_1('\tau) \left(D_{x-;g_1}^{(n+1)\alpha} f_1\right)(\tau)\, d\tau - \right.$$

$$\left. \left(D_{x-;g_!}^{(n+1)\alpha} f_1\right)(y) \frac{(g_1(x) - g_1(y))^{(n+1)\alpha}}{\Gamma((n+1)\alpha+1)} \right\| \tag{3.30}$$

$$= \frac{1}{\Gamma((n+1)\alpha)}\cdot$$

$$\left\| \int_y^x (g_1(\tau) - g_1(y))^{(n+1)\alpha-1} g_1'(\tau) \left(\left(D_{x-;g_1}^{(n+1)\alpha} f_1\right)(\tau) - \left(D_{x-;g_1}^{(n+1)\alpha} f_1\right)(y) \right) d\tau \right\| \leq$$

$$\frac{1}{\Gamma((n+1)\alpha)} \int_y^x (g_1(t) - g_1(y))^{(n+1)\alpha-1} g_1'(t) \left\| D_{x-;g_1}^{(n+1)\alpha} f_1(t) - D_{x-;g_1}^{(n+1)\alpha} f_1(y) \right\| dt$$

We suppose that

$$\left\| D_{x-;g_1}^{(n+1)\alpha} f_1(\tau) - D_{x-;g_1}^{(n+1)\alpha} f_1(y) \right\| \leq \lambda_2 |g_1(\tau) - g_1(y)|, \tag{3.31}$$

$\forall\, t,y,x \in [a,b] : x \geq t \geq y;\ \lambda_2 > 0)$

$$\leq \frac{\lambda_2}{\Gamma((n+1)\alpha)} \int_y^x (g_1(\tau) - g_1(y))^{(n+1)\alpha-1} g_1'(\tau)(g_1(\tau) - g_1(y))\, d\tau = \tag{3.32}$$

$$\frac{\lambda_2}{\Gamma((n+1)\alpha)} \int_y^x (g_1(t) - g_1(y))^{(n+1)\alpha} g_1'(\tau)\, d\tau =$$

$$\frac{\lambda_2}{\Gamma((n+1)\alpha)} \frac{(g_1(x)-g_1(y))^{(n+1)\alpha+1}}{((n+1)\alpha+1)}.$$

We show that

$$\|f_1(x)-f_1(y)-(A_1(f_1))(x,y)(g_1(x)-g_1(y))\| \leq$$

$$\frac{\lambda_2}{\Gamma((n+1)\alpha)} \frac{(g_1(x)-g_1(y))^{(n+1)\alpha+1}}{((n+1)\alpha+1)}, \tag{3.33}$$

$\forall\, x,y \in [a,b] : x > y;\ 0 < \alpha < 1.$

Conclusion 1*Set* $\lambda = \max(\lambda_1, \lambda_2)$. *We have proved that*

$$\|f_1(x)-f_1(y)-(A_1(f_1))(x,y)(g_1(x)-g_1(y))\| \leq$$

$$\frac{\lambda}{\Gamma((n+1)\alpha)} \frac{|g_1(x)-g_1(y)|^{(n+1)\alpha+1}}{((n+1)\alpha+1)}, \tag{3.34}$$

$\forall\, x,y \in [a,b];\ 0 < \alpha < 1,\ n \in \mathbb{N}.$

(3.34) becomes trivially true when $x = y$)).
One may suppose that

$$\frac{\lambda}{\Gamma((n+1)\alpha)} < 1. \tag{3.35}$$

Now using (3.22) and (3.34), we can apply our numerical methods presented in this article to to find the solutions of $f(x) = 0$.

To have $(n+1)\alpha+1 \geq 2$, we need to take $1 > \alpha \geq \frac{1}{n+1}$, where $n \in \mathbb{N}$.
Below we provide some forms of g_1:

$$\begin{aligned}
&g_1(x) = e^x,\ \ x \in [a,b] \subset \mathbb{R},\\
&g_1(x) = \sin x,\\
&g_1(x) = \tan x,\\
&\text{where } x \in \left[-\tfrac{\pi}{2}+\varepsilon, \tfrac{\pi}{2}-\varepsilon\right],\ \varepsilon > 0 \text{ small.}
\end{aligned} \tag{3.36}$$

II) The expression of X-valued left generalized g_1-fractional derivative of f_1 of order α is [5]:

$$\left(D^{\alpha}_{a+;g_1} f_1\right)(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^x (g_1(x)-g_1(\tau))^{-\alpha} g_1'(\tau) \left(f_1 \circ g_1^{-1}\right)'(g_1(\tau))\, d\tau, \tag{3.37}$$

$\forall\, x \in [a,b]$.

If $0 < \alpha < 1$, then $\left(D^{\alpha}_{a+;g_1} f_1\right) \in C([a,b], X)$ (see [5]).
Also, we define

$$D^{1}_{a+;g_1} f_1(x) = \left(\left(f_1 \circ g_1^{-1}\right)' \circ g_1\right)(x), \tag{3.38}$$

$$D^{0}_{a+;g_1} f_1(x) = f_1(x),\ \ \forall\, x \in [a,b].$$

When $g_1 = id$, then

$$D^{\alpha}_{a+;g_!} f_1 = D^{\alpha}_{a+;id} f_1 = D^{\alpha}_{*a} f_1,$$

the classical X-valued left Caputo fractional derivative [4].

Let

$$D^{n\alpha}_{a+;g_1} := D^{\alpha}_{a+;g_1} D^{\alpha}_{a+;g_1} \ldots D^{\alpha}_{a+;g_1} \quad (n \text{ times}),\ n \in \mathbb{N}. \tag{3.39}$$

The expression of the X-valued left generalized fractional Riemann-Liouville integral (see [5]) is written below

$$\left(I^{\alpha}_{a+;g_1} f_1\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (g_1(x)-g_1(\tau))^{\alpha-1} g_1'(\tau)\, f_1(t)\, d\tau,\ \ a \leq x \leq b. \tag{3.40}$$

Also denote by

$$I^{n\alpha}_{a+;g_1} := I^{\alpha}_{a+;g_1} I^{\alpha}_{a+;g_1} \ldots I^{\alpha}_{a+;g_1} \quad (n \text{ times}). \tag{3.41}$$

The following X-valued modified g_1-left generalized Taylor's formula will be used.

Theorem 3.([5]) *Let $0 < \alpha \leq 1$, $n \in \mathbb{N}$, $f_1 \in C^1([a,b],X)$, $g_1 \in C^1([a,b])$, strictly increasing, fulfilling $g_1^{-1} \in C^1([g_1(a), g_1(b)])$. Let $F_k := D_{a+;g}^{k\alpha} f_1$, $k = 1,\dots,n$, that fulfill $F_k \in C^1([a,b],X)$, and $F_{n+1} \in C([a,b],X)$. Then*

$$f_1(x) - f_1(a) = \sum_{i=1}^{n} \frac{(g_1(x) - g_1(a))^{i\alpha}}{\Gamma(i\alpha+1)} \left(D_{a+;g_1}^{i\alpha} f_1\right)(a) + \tag{3.42}$$

$$\frac{1}{\Gamma((n+1)\alpha)} \int_a^x (g_1(x) - g_1(\tau))^{(n+1)\alpha-1} g_1'(\tau) \left(D_{a+;g_1}^{(n+1)\alpha} f_1\right)(\tau)\, d\tau,$$

$\forall\, x \in [a,b]$.

Let $f_1 \in C^1([a,b],X)$. The expression of X-valued left generalized g-fractional derivative is written as

$$\left(D_{y+;g_1}^{\alpha} f_1\right)(x) = \frac{1}{\Gamma(1-\alpha)} \int_y^x (g_1(x) - g_1(\tau))^{-\alpha} g_1'(\tau) \left(f_1 \circ g_1^{-1}\right)'(g_1(\tau))\, d\tau, \tag{3.43}$$

for any $y \leq x \leq b$; $x,y \in [a,b]$,

$$\left(D_{y+;g_1}^{1} f\right)(x) = \left(f_1 \circ g^{-1}\right)'(g_1(x)),\ \forall\, x \in [a,b]. \tag{3.44}$$

In a similar manner, we define

$$\left(D_{x+;g_1}^{\alpha} f-1\right)(y) = \frac{1}{\Gamma(1-\alpha)} \int_x^y (g_1(y) - g_1(\tau))^{-\alpha} g_1'(\tau) \left(f_1 \circ g_1^{-1}\right)'(g_1(\tau))\, d\tau, \tag{3.45}$$

for any $x \leq y \leq b$; $x,y \in [a,b]$,

$$\left(D_{x+;g_1}^{1} f_1\right)(y) = \left(f_1 \circ g-1^{-1}\right)'(g_1(y)),\ \forall\, y \in [a,b]. \tag{3.46}$$

When $0 < \alpha < 1$, $D_{y+;g_1}^{\alpha} f_1$ and $D_{x+;g_1}^{\alpha} f_1$ are continuous functions on $[a,b]$, see [5]. We recall that

$$\begin{gathered}\left(D_{y+;g_1}^{\alpha} f_1\right)(x) = 0,\ \text{when } x < y,\\ \text{and}\\ \left(D_{x+;g_1}^{\alpha} f_1\right)(y) = 0,\ \text{when } y < x.\end{gathered} \tag{3.47}$$

Let consider

$$G_k^y := D_{y+;g_1}^{k\alpha} f_1,\quad G_k^x := D_{x+;g_1}^{k\alpha} f_1,\ \forall\, x,y \in [a,b]. \tag{3.48}$$

We suppose that

$$G_k^y, G_k^x \in C^1([a,b],X),\ \text{and } G_{n+1}^y, G_{n+1}^x \in C([a,b],X), \tag{3.49}$$

$k = 1,\dots,n$, $\forall\, x,y \in [a,b]$; $0 < \alpha < 1$.

We also report that ($0 < \alpha < 1$) (by [8])

$$\begin{aligned}\left\|\left(D_{a+;g_1}^{\alpha} f_1\right)(x)\right\| &\leq \frac{1}{\Gamma(1-\alpha)} \int_a^x (g_1(x) - g_1(t))^{-\alpha} g_1'(t) \left\|\left(f_1 \circ g_1^{-1}\right)'(g_1(t))\right\| dt \leq \\ &\frac{\left\|\left(f_1 \circ g_1^{-1}\right)' \circ g_1\right\|_{\infty,[a,b]}}{\Gamma(1-\alpha)} \int_a^x (g_1(x) - g_1(\tau))^{-\alpha} g_1'(\tau)\, d\tau = \\ &\frac{\left\|\left(f_1 \circ g^{-1}\right)' \circ g_1\right\|_{\infty,[a,b]}}{\Gamma(1-\alpha)} \frac{(g_1(x) - g_1(a))^{1-\alpha}}{1-\alpha} = \\ &\frac{\left\|\left(f_1 \circ g_1^{-1}\right)' \circ g_1\right\|_{\infty,[a,b]}}{\Gamma(2-\alpha)} (g_1(x) - g_1(a))^{1-\alpha}.\end{aligned} \tag{3.50}$$

We concluded that

$$\left\|\left(D_{a+;g_1}^{\alpha} f_1\right)(x)\right\| \leq \frac{\left\|\left(f_1 \circ g_1^{-1}\right)' \circ g_1\right\|_{\infty,[a,b]}}{\Gamma(2-\alpha)} (g_1(x) - g_1(a))^{1-\alpha}$$

$$\leq \frac{\left\| \left(f_1 \circ g_1^{-1} \right)' \circ g_1 \right\|_{\infty,[a,b]}}{\Gamma(2-\alpha)} \left(g_1(b) - g_1(a) \right)^{1-\alpha}, \quad \forall\, x \in [a,b]. \tag{3.51}$$

We report that

$$\left(D^{\alpha}_{a+;g_1} f_1 \right)(a) = 0, \quad 0 < \alpha < 1, \tag{3.52}$$

and

$$\left(D^{\alpha}_{y+;g_1} f_1 \right)(y) = \left(D^{\alpha}_{x+;g_1} f_1 \right)(x) = 0, \quad \forall\, x,y \in [a,b];\ 0 < \alpha < 1. \tag{3.53}$$

From (3.42) we conclude

$$f_1(x) - f_1(y) = \sum_{i=2}^{n} \frac{\left(g_1(x) - g_1(y) \right)^{i\alpha}}{\Gamma(i\alpha+1)} \left(D^{i\alpha}_{y+;g_1} f_1 \right)(y) +$$
$$\frac{1}{\Gamma((n+1)\alpha)} \int_y^x \left(g_1(x) - g_1(t) \right)^{(n+1)\alpha-1} g_1'(t) \left(D^{(n+1)\alpha}_{y+;g_1} f_1 \right)(\tau)\, d\tau, \tag{3.54}$$

for any $x > y : x,y \in [a,b]$; $0 < \alpha < 1$, we have

$$f_1(y) - f_1(x) = \sum_{i=2}^{n} \frac{\left(g_1(y) - g_1(x) \right)^{i\alpha}}{\Gamma(i\alpha+1)} \left(D^{i\alpha}_{x+;g_1} f \right)(x) + \tag{3.55}$$
$$\frac{1}{\Gamma((n+1)\alpha)} \int_x^y \left(g_1(y) - g_1(\tau) \right)^{(n+1)\alpha-1} g_1'(\tau) \left(D^{(n+1)\alpha}_{x+;g_1} f_1 \right)(\tau)\, d\tau,$$

for any $y > x : x,y \in [a,b]$; $0 < \alpha < 1$.

Below we give the definition of the following X-valued linear operator

$$\left(A_2(f) \right)(x,y) :=$$

$$\begin{cases} \sum_{i=2}^{n} \frac{(g_1(x)-g_1(y))^{i\alpha-1}}{\Gamma(i\alpha+1)} \left(D^{i\alpha}_{y+;g_1} f_1 \right)(y) + \left(D^{(n+1)\alpha}_{y+;g_1} f_1 \right)(x) \frac{(g_1(x)-g_1(y))^{(n+1)\alpha-1}}{\Gamma((n+1)\alpha+1)}, & x > y, \\ \sum_{i=2}^{n} \frac{(g_1(y)-g_1(x))^{i\alpha-1}}{\Gamma(i\alpha+1)} \left(D^{i\alpha}_{x+;g_1} f_1 \right)(x) + \left(D^{(n+1)\alpha}_{x+;g_1} f_1 \right)(y) \frac{(g_1(y)-g_1(x))^{(n+1)\alpha-1}}{\Gamma((n+1)\alpha+1)}, & y > x, \\ f_1'(x), \quad \text{when } x = y, \end{cases} \tag{3.56}$$

$\forall\, x,y \in [a,b]$; $0 < \alpha < 1$.

We suppose that (see [12], p. 3)

$$\left\| \left(A_2(f_1) \right)(x,x) - \left(A_2(f_1) \right)(y,y) \right\| = \left\| f_1'(x) - f_1'(y) \right\| \tag{3.57}$$
$$\leq \Phi^* \left| g_1(x) - g_1(y) \right|, \quad \forall\, x,y \in [a,b];$$

with $\Phi^* > 0$.

We conclude

i) case of $x > y$:

$$\left\| f_1(x) - f_1(y) - \left(A_2(f_1) \right)(x,y) \left(g_1(x) - g_1(y) \right) \right\| = \tag{3.58}$$
$$\left\| \frac{1}{\Gamma((n+1)\alpha)} \int_y^x \left(g_1(x) - g_1(\tau) \right)^{(n+1)\alpha-1} g_1'(\tau) \left(D^{(n+1)\alpha}_{y+;g_1} f_1 \right)(\tau)\, d\tau - \right.$$
$$\left. \left(D^{(n+1)\alpha}_{y+;g_1} f_1 \right)(x) \frac{\left(g_1(x) - g_1(y) \right)^{(n+1)\alpha}}{\Gamma((n+1)\alpha+1)} \right\|$$

(using[1], p. 426)

$$= \frac{1}{\Gamma((n+1)\alpha)} \cdot$$
$$\left\| \int_y^x \left(g_1(x) - g_1(\tau) \right)^{(n+1)\alpha-1} g_1'(\tau) \left(\left(D^{(n+1)\alpha}_{y+;g_1} f_1 \right)(\tau) - \left(D^{(n+1)\alpha}_{y+;g_1} f_1 \right)(x) \right) d\tau \right\| \tag{3.59}$$

(by [8])

$$\leq \frac{1}{\Gamma((n+1)\alpha)} \cdot$$

$$\int_y^x (g_1(x) - g_1(\tau))^{(n+1)\alpha-1} g_1'(\tau) \left\| \left(D_{y+;g_1}^{(n+1)\alpha} f_1\right)(\tau) - \left(D_{y+;g_1}^{(n+1)\alpha} f_1\right)(x) \right\| d\tau$$

(we suppose here that

$$\left\| \left(D_{y+;g_1}^{(n+1)\alpha} f_1\right)(\tau) - \left(D_{y+;g_1}^{(n+1)\alpha} f_1\right)(x) \right\| \leq \rho_1 |g_1(\tau) - g_1(x)|, \tag{3.60}$$

$\forall\, t,x,y \in [a,b] : x \geq t \geq y;\ \rho_1 > 0$)

$$\leq \frac{\rho_1}{\Gamma((n+1)\alpha)} \int_y^x (g_1(x) - g_1(\tau))^{(n+1)\alpha-1} g_1'(\tau)(g_1(x) - g_1(\tau))\, d\tau =$$

$$\frac{\rho_1}{\Gamma((n+1)\alpha)} \int_y^x (g_1(x) - g_1(\tau))^{(n+1)\alpha} g_1'(\tau)\, d\tau =$$

$$\frac{\rho_1}{\Gamma((n+1)\alpha)} \frac{(g_1(x) - g_1(y))^{(n+1)\alpha+1}}{((n+1)\alpha+1)}. \tag{3.61}$$

We show that

$$\| f_1(x) - f_1(y) - (A_2(f_1))(x,y)(g_1(x) - g_1(y)) \| \leq$$

$$\frac{\rho_1}{\Gamma((n+1)\alpha)} \frac{(g_1(x) - g_1(y))^{(n+1)\alpha+1}}{((n+1)\alpha+1)}, \tag{3.62}$$

$\forall\, x,y \in [a,b] : x > y;\ 0 < \alpha < 1.$

ii) case of $y > x$:

$$\| f_1(x) - f_1(y) - (A_2(f_!))(x,y)(g_1(x) - g_1(y)) \| = \tag{3.63}$$

$$\| f_1(y) - f_1(x) - (A_2(f_1))(x,y)(g_1(y) - g_1(x)) \| =$$

$$\left\| \frac{1}{\Gamma((n+1)\alpha)} \int_x^y (g_1(y) - g_1(\tau))^{(n+1)\alpha-1} g_1'(\tau) \left(D_{x+;g_1}^{(n+1)\alpha} f_1\right)(\tau)\, d\tau - \right.$$

$$\left. \left(D_{x+;g_1}^{(n+1)\alpha} f_1\right)(y) \frac{(g_1(y) - g_1(x))^{(n+1)\alpha}}{\Gamma((n+1)\alpha+1)} \right\|$$

$$= \frac{1}{\Gamma((n+1)\alpha)} \cdot$$

$$\left\| \int_x^y (g_1(y) - g_1(t))^{(n+1)\alpha-1} g_1'(\tau) \left(\left(D_{x+;g_1}^{(n+1)\alpha} f_1\right)(\tau) - \left(D_{x+;g_1}^{(n+1)\alpha} f_1\right)(y) \right) d\tau \right\|$$

$$\leq \frac{1}{\Gamma((n+1)\alpha)} \cdot$$

$$\int_x^y (g_1(y) - g_1(\tau))^{(n+1)\alpha-1} g_1'(\tau) \left\| \left(D_{x+;g_1}^{(n+1)\alpha} f_1\right)(\tau) - \left(D_{x+;g_1}^{(n+1)\alpha} f_1\right)(y) \right\| d\tau. \tag{3.64}$$

We suppose here that

$$\left\| \left(D_{x+;g_1}^{(n+1)\alpha} f_1\right)(\tau) - \left(D_{x+;g_1}^{(n+1)\alpha} f_1\right)(y) \right\| \leq \rho_2 |g_1(\tau) - g_1(y)|, \tag{3.65}$$

$\forall\, t,y,x \in [a,b] : y \geq t \geq x;\ \rho_2 > 0$)

$$\leq \frac{\rho_2}{\Gamma((n+1)\alpha)} \int_x^y (g_1(y) - g_1(\tau))^{(n+1)\alpha-1} g_1'(\tau)(g_1(y) - g_1(\tau))\, d\tau =$$

$$\frac{\rho_2}{\Gamma((n+1)\alpha)} \frac{(g_1(y) - g_1(x))^{(n+1)\alpha+1}}{((n+1)\alpha+1)}. \tag{3.66}$$

We proved that

$$\left\| f_1(x) - f_1(y) - (A_2(f_1))(x,y)(g_1(x) - g_1(y)) \right\| \leq$$

$$\frac{\rho_2}{\Gamma((n+1)\alpha)} \frac{(g_1(y) - g_1(x))^{(n+1)\alpha+1}}{((n+1)\alpha+1)},$$

$\forall\, x,y \in [a,b] : y > x;\ 0 < \alpha < 1.$

Conclusion 2*Set* $\rho = \max(\rho_1, \rho_2)$. *Thus,*

$$\left\| f_1(x) - f_1(y) - (A_2(f_1))(x,y)(g_1(x) - g_1(y)) \right\| \leq$$

$$\frac{\rho}{\Gamma((n+1)\alpha)} \frac{|g_1(x) - g_1(y)|^{(n+1)\alpha+1}}{((n+1)\alpha+1)}, \tag{3.67}$$

$\forall\, x,y \in [a,b];\ 0 < \alpha < 1.$

(When $x = y$ we notice that (3.67) becomes trivially true.)

One may suppose that

$$\frac{\rho}{\Gamma((n+1)\alpha)} < 1. \tag{3.68}$$

Taking into account (3.57) and (3.67), we can use the developed numerical methods shown in this article to find the solution of $f_1(x) = 0$.

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