Progress in Fractional Differentiation and Applications
An International Journal

http://dx.doi.org/10.18576/pfda/040105

Iterative Methods and their Applications to Banach Space Valued Functions in Abstract Fractional Calculus

George A. Anastassiou[1,*] *and Ioannis K. Argyros*[2]

[1] Department of Mathematical Sciences, University of Memphis, Memphis, TN 38152, U.S.A.
[2] Department of Mathematics Sciences, Cameron University, Lawton, Ok 73505, USA

Received: 3 Mar. 2017, Revised: 27 Apr. 2017, Accepted: 5 May 2017
Published online: 1 Jan. 2018

Abstract: Explicit iterative methods have been used extensively to generate a sequence approximating a solution of an equation on a Banach space setting. However, little attention has been given to the study of implicit iterative methods. We show a semi-local convergence analysis for a some general implicit and explicit iterative techniques. Illustrative applications are presented including Banach space valued functions of fractional calculus, such that all integrals are of Bochner-type.

Keywords: Explicit-Implicit iterative method, Banach space, semi-local convergence, Fractional derivatives and integrals, Bochner-type integral.

1 Introduction

Let B_1, B_2 stand for Banach space and let Ω stand for an open subset of B_1. We consider $U(z,\rho) := \{u \in B_1 : \|u - z\| < \rho\}$ and let $\overline{U}(z,\rho)$ denoting the closure of $U(z,\rho)$.

Many phenomena in several interdisciplinary areas can be expressed as

$$F(x) = 0 \tag{1.1}$$

using Mathematical Modeling [1]-[15], such that $F : \Omega \to B_2$ denotes a continuous operator. We recall that the solution x^* of (1.1) is sought in closed form, but only in particular cases it is attainable. Thus, it explains why most solution methods for such type of equations are usually iterative. We recall that there is a plethora of iterative methods for solving (1.1). These methods can be classified in two classes.

Explicit Methods [5,6,10,13,14]: Newton's method

$$x_{n+1} = x_n - F'(x_n)^{-1} F(x_n). \tag{1.2}$$

Secant method:

$$x_{n+1} = x_n - [x_{n-1}, x_n; F]^{-1} F(x_n), \tag{1.3}$$

where $[\cdot,\cdot;F]$ is a divided difference of order one on $\Omega \times \Omega$ [6,13,14].

Newton-like method:

$$x_{n+1} = x_n - E_n^{-1} F(x_n), \tag{1.4}$$

where $E_n = E(F)(x_n)$ and $E : \Omega \to \mathscr{L}(B_1, B_2)$ the space of bounded linear operators from B_1 into B_2. In [6], [10], [13], [14] and the references there in the reader can see other explicit methods.

Implicit Methods [5,8,10,14]:

$$F(x_n) + A_n(x_{n+1} - x_n) = 0 \tag{1.5}$$

$$x_{n+1} = x_n - A_n^{-1} F(x_n), \tag{1.6}$$

* Corresponding author e-mail: ganastss@memphis.edu

where $A_n = A(x_{n+1}, x_n) = A(F)(x_{n+1}, x_n)$ and $A : \Omega \times \Omega \to \mathscr{L}(B_1, B_2)$.

We recall that there exists plenty on local as well as semi-local convergence results for explicit methods [1]-[7], [9]-[14]. On the other hand, a little attention was devoted for the research on the convergence of implicit methods. The researchers, usually investigate the fixed point problem

$$P_z(x) = x, \tag{1.7}$$

where

$$P_z(x) = x + F(z) + A(x,z)(x-z) \tag{1.8}$$

or

$$P_z(x) = z - A(x,z)^{-1} F(z) \tag{1.9}$$

for methods (1.5) and (1.6), where $z \in \Omega$ is provided. If P denotes a contraction operator mapping a closed set into itself, then due to the contraction mapping principle [10], [13], [14], P_z admits a fixed point x_z^* which can be obtained utilizing the method of successive substitutions or Picard's method [14] defined for each fixed n by

$$y_{k+1,n} = P_{x_n}\left(y_{k,n}\right), \quad y_{0,n} = x_n, \;\; x_{n+1} = \lim_{k \to +\infty} y_{k,n}. \tag{1.10}$$

Below we discuss the analogous explicit methods

$$F(x_n) + A(x_n, x_n)(x_{n+1} - x_n) = 0 \tag{1.11}$$

$$x_{n+1} = x_n - A(x_n, x_n)^{-1} F(x_n) \tag{1.12}$$

$$F(x_n) + A(x_n, x_{n-1})(x_{n+1} - x_n) = 0 \tag{1.13}$$

and

$$x_{n+1} = x_n - A(x_n, x_{n-1})^{-1} F(x_n). \tag{1.14}$$

In our manuscript in Section 2, we investigate the semi-local convergence of the method (1.5) and (1.6), respectively. Section 3 deals with the semi-local convergence of the methods (1.11), (1.12), (1.13) and (1.14), respectively. Several applications to Abstract Fractional Calculus are presented in Section 4 on a certain Banach space valued function such that all the integrals are of Bochner-type [7].

2 Semi-Local Convergence for Implicit Methods

The following semi-local convergence analysis of method (1.6) is centered on the conditions (H):

(h_1) $F : \Omega \subset B_1 \to B_2$ is continuous and $A(F)(x,y) \in \mathscr{L}(B_1, B_2)$ for each $(x,y) \in \Omega \times \Omega$.

(h_2) There exist $l > 0$ and $\Omega_0 \subset B_1$ such that $A(F)(x,y)^{-1} \in \mathscr{L}(B_2, B_1)$ for each $(x,y) \in \Omega_0 \times \Omega_0$ and

$$\left\| A(F)(x,y)^{-1} \right\| \leq l^{-1}.$$

Set $\Omega_1 = \Omega \cap \Omega_0$.

(h_3) There exist real numbers $\alpha_1, \alpha_2, \alpha_3$ satisfying

$$0 \leq \alpha_2 \leq \alpha_1 \;\text{ and }\; 0 \leq \alpha_3 < 1$$

in such a way that for each $x, y \in \Omega_1$

$$\|F(x) - F(y) - A(F)(x,y)(x-y)\| \leq$$

$$l\left(\frac{\alpha_1}{2}\|x-y\| + \alpha_2\|y - x_0\| + \alpha_3\right)\|x-y\|.$$

(h_4) For each $x \in \Omega_0$ there exists $y \in \Omega_0$ fulfilling

$$y = x - A(y,x)^{-1} F(x).$$

(h_5) For $x_0 \in \Omega_0$ and $x_1 \in \Omega_0$ satisfying (h_4) there exists $\eta \geq 0$ fulfilling

$$\left\| A(F)(x_1,x_0)^{-1} F(x_0) \right\| \leq \eta.$$

(h_6) $h := \alpha_1 \eta \leq \frac{1}{2}(1-\alpha_3)^2$.
and
(h_7) $\overline{U}(x_0,t^*) \subset \Omega_0$, where

$$t^* = \begin{cases} \frac{1-\alpha_3-\sqrt{(1-\alpha_3)^2-2h}}{\alpha_1}, & \alpha_1 \neq 0 \\ \frac{1}{1-\alpha_3}\eta, & \alpha_1 = 0. \end{cases}$$

Thus, utilizing both the previous notation and conditions (H) we prove below the semi-local convergence result (1.6).

Theorem 1. *Assume that the conditions (H) are fulfilled. Then, sequence $\{x_n\}$ generated by method (1.6) starting at $x_0 \in \Omega$ is well defined in $U(x_0,t^*)$, remains in $U(x_0,t^*)$ for each $n = 0,1,2,\ldots$ and converges to a solution $x^* \in \overline{U}(x_0,t^*)$ of equation $F(x) = 0$. Moreover, provided that (h_3) holds with A(F)(z,y) replacing A(F)(x,y) for each $z \in \Omega_1$, if $\alpha_1 \neq 0$, the equation $F(x) = 0$ possess a unique solution x^* in $\widetilde{U}$, such that*

$$\widetilde{U} = \begin{cases} \overline{U}(x_0,t^*) \cap \Omega_0, & \text{if } h = \frac{1}{2}(1-\alpha_3)^2 \\ U(x_0,t^{**}) \cap \Omega_0, & \text{if } h < \frac{1}{2}(1-\alpha_3)^2 \end{cases}$$

and, if $\alpha_1 = 0$, the solution x^* is unique in $\overline{U}\left(x_0, \frac{\eta}{1-\alpha_3}\right)$, where $t^{**} = \frac{1-\alpha_3+\sqrt{(1-\alpha_3)^2-2h}}{\alpha_1}$.

Proof. Case $\alpha_1 \neq 0$. Let g be scalar function on $\mathbb{R}$ by $g(t) = \frac{\alpha_1}{2}t^2 - (1-\alpha_3)t + \eta$ and majorizing sequence $\{t_n\}$ by

$$t_0 = 0, \ t_k = t_{k-1} + g(t_{k-1}) \ \text{ for each } k = 1,2,\ldots. \tag{2.1}$$

From (h_6) we conclude that g admits two positive roots t^* and t^{**}, $t^* \leq t^{**}$, and $t_k \leq t_{k+1}$. As a result, the sequence $\{t_k\}$ converges to t^*.

(a) Utilizing the mathematical induction on k, it can be proved that

$$\|x_{k+1} - x_k\| \leq t_{k+1} - t_k. \tag{2.2}$$

Therefore, (2.2) holds for $k = 0$ by (h_5) and (2.1), due to thee fact that $\|x_1 - x_0\| \leq \eta = t_1 - t_0$. Assume that for $1 \leq m \leq k$

$$\|x_m - x_{m-1}\| \leq t_m - t_{m-1}. \tag{2.3}$$

Then, we get $\|x_k - x_0\| \leq t_k - t_0 = t_k \leq t^*$ and $A(x_k,x_{k-1})$ is invertible by (h_2). We can write by method (1.6)

$$x_{k+1} - x_k = -A_k^{-1}\left(F(x_k) - F(x_{k-1}) - A_{k-1}(x_k - x_{k-1})\right). \tag{2.4}$$

From (2.3), (h_2), (h_3), (h_4), (2.1) and (2.4), we get in turn that

$$\|x_{k+1} - x_k\| = \left\|A_k^{-1}F(x_k)\right\| = \left\|A_k^{-1}\left(F(x_k) - F(x_{k-1}) - A_{k-1}(x_k - x_{k-1})\right)\right\|$$

$$\leq \left\|A_k^{-1}\right\| \|F(x_k) - F(x_{k-1}) - A_{k-1}(x_k - x_{k-1})\| \leq$$

$$l^{-1}l\left(\frac{\alpha_1}{2}\|x_k - x_{k-1}\| + \alpha_2\|x_{k-1} - x_0\| + \alpha_3\right)\|x_k - x_{k-1}\| \leq \tag{2.5}$$

$$\frac{\alpha_1}{2}(t_k - t_{k-1})^2 + \alpha_2(t_k - t_{k-1})t_{k-1} + \alpha_3(t_k - t_{k-1}) =$$

$$\frac{\alpha_1}{2}(t_k - t_{k-1})^2 + \alpha_2(t_k - t_{k-1})t_{k-1} + \alpha_3(t_k - t_{k-1}) - (t_k - t_{k-1}) + g(t_{k-1}) =$$

$$g(t_k) - (\alpha_1 - \alpha_2)(t_k - t_{k-1})t_{k-1} \leq$$

$$g(t_k) = t_{k+1} - t_k, \tag{2.6}$$

which finish the induction for (2.2).

Thus, we have for any k

$$\|x_{k+1}-x_k\| \leq t_{k+1}-t_k \tag{2.7}$$

and

$$\|x_k-x_0\| \leq t_k \leq t^*. \tag{2.8}$$

From both (2.7) and (2.8) we conclude that $\{x_k\}$ denotes a complete sequence in a Banach space B_1 and as such it converges to some $x^* \in \overline{U}(x_0,t^*)$ (due to the fact that $\overline{U}(x_0,t^*)$ denotes a closed set). By considering $k \to +\infty$, utilizing (h_1) and (h_2), we get $l^{-1}\lim_{k\to+\infty}\|F(x_k)\| = 0$, thus $F(x^*) = 0$.

Let $x^{**} \in \widetilde{U}$ fulfilling $F(x^{**}) = 0$. The next step is to prove by induction that

$$\|x^{**}-x_k\| \leq t^*-t_k \ \text{ for each } k = 0,1,2,\dots. \tag{2.9}$$

We conclude that the estimate (2.9) is valid for $k = 0$ according to the definition of x^{**} and $\widetilde{U}$. Assume that $\|x^{**}-x_k\| \leq t^*-t_k$. Thus, as in (2.5), we conclude that

$$\begin{aligned}
&\|x^{**}-x_{k+1}\| = \left\|x^{**}-x_k+A_k^{-1}F(x_k)-A_k^{-1}F(x^{**})\right\| = \\
&\left\|A_k^{-1}\left(A_k\left(x^{**}-x_k\right)+F(x_k)-F(x^{**})\right)\right\| \leq \\
&\left\|A_k^{-1}\right\| \left\|F(x^{**})-F(x_k)-A_k\left(x^{**}-x_k\right)\right\| \leq \\
&\left(\frac{\alpha_1}{2}\|x^{**}-x_k\|+\alpha_2\|x_k-x_0\|+\alpha_3\right)\|x^{**}-x_k\| \leq \\
&\left(\frac{\alpha_1}{2}(t^*-t_k)+\alpha_2 t_k+\alpha_3\right)(t^*-t_k) = \\
&\frac{\alpha_1}{2}(t^*)^2+\frac{\alpha_1}{2}(t_k)^2-\alpha_1 t_k t^*+\alpha_2(t^*-t_k)t_k+\alpha_3(t^*-t_k) = \\
&-\eta+(1-\alpha_3)t^*+\frac{\alpha_1}{2}t_k^2-\alpha_1 t_k t^*+\alpha_2 t_k t^*-\alpha_2 t_k^2+\alpha_3 t^*-\alpha_3 t_k \\
&= t^*-t_{k+1},
\end{aligned} \tag{2.10}$$

which completes the induction for (2.9). Thus, $\lim_{k\to+\infty} x_k = x^{**}$. But we showed that $\lim_{k\to+\infty} x_k = x^*$, so $x^{**} = x^*$.

Case $\alpha_1 = 0$. Then, we have by (h_3) that $\alpha_2 = 0$ and estimate (2.5) gives

$$\|x_{k+1}-x_k\| \leq \alpha_3\|x_k-x_{k-1}\| \leq \dots \leq \alpha_3^k\|x_1-x_0\| \leq \alpha_3^k\eta \tag{2.11}$$

and

$$\begin{aligned}
\|x_{k+1}-x_0\| &\leq \|x_{k+1}-x_k\|+\|x_k-x_{k-1}\|+\dots+\|x_1-x_0\| \\
&\leq \frac{1-\alpha_3^{k+1}}{1-\alpha_3}\eta < \frac{\eta}{1-\alpha_3}.
\end{aligned} \tag{2.12}$$

Then, it follows from (2.11) and (2.12) that

$$\|x_{k+i}-x_k\| \leq \frac{1-\alpha_3^i}{1-\alpha_3}\alpha_3^k\eta, \tag{2.13}$$

so sequence $\{x_k\}$ is complete and x^* is a solution of $F(x) = 0$. Next, the uniqueness part emerges from (2.10) for $\alpha_1 = \alpha_2 = 0$, since

$$\|x^{**}-x_{k+1}\| \leq \alpha_3\|x^{**}-x_k\| \leq \alpha_3^{k+1}\|x^{**}-x_0\| \leq \alpha_3^{k+1}\frac{\eta}{1-\alpha_3}, \tag{2.14}$$

which implies that $\lim_{k\to+\infty} x_k = x^{**}$.

Remark.(1) Condition (h_2) can be incorporated in (h_3) as

(h_3') There exist real numbers $\overline{\alpha}_1, \overline{\alpha}_2, \overline{\alpha}_3$ satisfying $0 \le \overline{\alpha}_2 \le \overline{\alpha}_1$ and $0 \le \overline{\alpha}_3 < 1$ such that for each $x, y \in \Omega$

$$\left\| A(x,y)^{-1} [F(x) - F(y) - A(x,y)(x-y)] \right\| \le$$

$$\left((\overline{\alpha}_1/2) \|x-y\| + \overline{\alpha}_2 \|y - x_0\| + \overline{\alpha}_3 \right) \|x-y\|.$$

As a result, (h_3') will replace (h_2) and (h_3) in Theorem 1 for $\alpha_1 = \overline{\alpha}_1$, $\overline{\alpha}_2 = \alpha_2$, $\overline{\alpha}_3 = \alpha_3$ and $\Omega_0 = \Omega$. Moreover, notice that $\overline{\alpha}_1 \le \alpha_1$, $\overline{\alpha}_2 \le \alpha_1$ and $\overline{\alpha}_3 \le \alpha_3$, which has a role in the sufficient convergence criterion (h_6), error bounds and the precision of t^* and t^{**}. The condition (h_3) is of Mysowksii-type [10].

(2) Suppose that there exist $l_0 > 0$, $\alpha_4 > 0$ and $L \in \mathscr{L}(B_1, B_2)$ with $L^{-1} \in \mathscr{L}(B_2, B_1)$ such that $\left\| L^{-1} \right\| \le l_0^{-1}$

$$\|A(F)(x,y) - L\| \le \alpha_4 \text{ for each } x, y \in \Omega$$

and

$$\alpha_5 := l_0^{-1} \alpha_4 < 1.$$

As a result, due to the Banach lemma on invertible operators we have [6], [8], [10], [13], [14] and

$$\left\| L^{-1} \right\| \|A(F)(x,y) - L\| \le l_0^{-1} \alpha_4 = \alpha_5 < 1$$

that $A(F)(x,y)^{-1} \in \mathscr{L}(B_2, B_1)$. Set $l^{-1} = \frac{l_0^{-1}}{1-\alpha_5}$, then the condition (h_2) is implied, therefore it can be dropped from the conditions (H).

(3) Definitely, (1.5) converges under the conditions (H), due to the fact that (1.6) implies (1.5).

(4) Let $R > 0$ and define $R_0 = \sup\{t \in [0, R) : U(x_0, R_0) \subseteq D\}$. Set $\Omega_0 = \overline{U}(x_0, R_0)$. Condition (h_3) can be extended, if the additional term $a_2 \|x - x_0\|$ is inserted inside the parenthesis at the right hand side for some $a_2 \ge 0$. Then, the conclusions of Theorem 1 are valid in this more general setting, if $a_3 = a_2 R_0 + \alpha_3$ replaces α_3 in conditions (h_6) and (h_7).

(5) Regarding the solvability of equations (1.6) (or (1.5)), we wanted to leave condition (h_4) as uncluttered as possible in conditions (H).

Below, we prove the solvability of method (1.5) utilizing a stronger version of the contraction mapping principle and based on the conditions (C) :

$(c_1) = (h_1)$.

(c_2) There exist $\gamma_0 \in [0,1)$, $\gamma_1 \in [0, +\infty)$, $\gamma_2 \in [0,1)$, $x_0 \in \Omega$ such that for each $x, y, z \in \Omega$

$$\|I + A(x,z) - A(y,z)\| \le \gamma_0,$$

$$\|A(x,z) - A(y,z)\| \le \gamma_1 \|x - y\|$$

$$\|F(z) + A(x_0,z)(x_0 - z)\| \le \begin{cases} \gamma_2 \|x_0 - z\| & \text{for } x_0 \ne z \\ \|F(x_0)\| & \text{for } x_0 = z \end{cases}$$

(c_3)

$$\gamma_0 + \gamma_1 \|x_0\| + \gamma_2 \le 1 \text{ for } \gamma_2 \ne 0,$$

$$\gamma_0 + \gamma_1 \|x_0\| < 1 \text{ for } \gamma_2 = 0,$$

$$\|F(x_0)\| \le \frac{(1 - (\gamma_0 + \gamma_1 \|x_0\|))^2}{\gamma_1} \text{ for } \gamma_1 \ne 0,$$

$$\gamma_0 < 1 \text{ for } \gamma_1 = 0$$

and

(c_4) $\overline{U}(x_0, r) \subseteq \Omega$, where

$$\frac{\|F(x_0)\|}{1 - (\gamma_0 + \gamma_1 \|x_0\|)} \le r < \frac{1 - (\gamma_0 + \gamma_1 \|x_0\|)}{\gamma_1} \text{ for } \gamma_1 \ne 0,$$

$$\frac{\|F(x_0)\|}{1 - \gamma_0} \le r \text{ for } \gamma_1 = 0,$$

$$r < \frac{1 - (\gamma_0 + \gamma_1 \|x_0\|)}{\gamma_1} \text{ for } z = x_0,\ \gamma_1 \ne 0.$$

Theorem 2.*Assume that the conditions (C) are satisfied. Then, for each $n=0,1,2,\ldots$ equation (1.5) is unique solvable. Moreover, if $A_n^{-1}\in\mathscr{L}(B_2,B_1)$, then equation (1.6) is also uniquely solvable for each $n=0,1,2,\ldots$*

*Proof.*The proof is based on the contraction mapping principle. Let $x,y\in U(x_0,r)$. Then, using (1.8) we have in turn by (c_2) that

$$\begin{aligned}\|P_z(x)-P_z(y)\| &= \|(I+A(x,z)-A(y,z))(x-y)-(A(x,z)-A(y,z))z\|\\ &\le \|I+A(x,z)-A(y,z)\|\,\|x-y\|+\|A(x,z)-A(y,z)\|\,\|z\|\\ &\le \gamma_0\|x-y\|+\gamma_1(\|z-x_0\|+\|x_0\|)\|x-y\|\end{aligned}$$

$$\le \varphi(\|x-x_0\|)\|x-y\|, \tag{2.15}$$

where

$$\varphi(t)=\begin{cases}\gamma_0+\gamma_1(t+\|x_0\|) & \text{for } z\neq x_0\\ \gamma_0+\gamma_1\|x_0\| & \text{for } z=x_0.\end{cases} \tag{2.16}$$

Wee observe that $\varphi(t)\in[0,1)$ for $t\in[0,r]$ by the choice of r in (c_4). In addition we have

$$\|P_z(x)-x_0\|\le\|P_z(x)-P_z(x_0)\|+\|P_z(x_0)-x_0\|. \tag{2.17}$$

If $z=x_0$ in (2.17), then we get by $(c_3),(c_4)$ and (2.15) that

$$\left\|P_{x_0}(x)-x_0\right\|\le\varphi(\|x-x_0\|)\|x-x_0\|+\|F(x_0)\|$$

$$\le(\gamma_0+\gamma_1\|x_0\|)r+\|F(x_0)\|\le r. \tag{2.18}$$

The existence of $x_1\in U(x_0,r)$ solving (1.5) for $n=0$ is now obtained by the contraction mapping principle, (2.15) and (2.18).

In addition, if $z\neq x_0$, the last condition in (c_3), $(c_3),(c_4)$ and (2.17) give instead of (2.18) that

$$\|P_z(x)-x_0\|\le\varphi(\|x-x_0\|)\|x-x_0\|+\gamma_2\|x-x_0\|$$

$$\le(\gamma_0+\gamma_1\|x_0\|+\gamma_2)r\le r. \tag{2.19}$$

From (2.15), (2.19) and the contraction mapping principle, we prove the unique solvability of (1.5) and the existence of a unique sequence $\{x_n\}$ for each $n=0,1,2,\ldots$ We conclude that, the equation (1.6) is also uniquely solvable by the preceding proof and the condition $A_n^{-1}\in\mathscr{L}(B_2,B_1)$.

Remark.(a) The gamma conditions can be weakened, if γ_i are replaced by functions $\gamma_i(t)$, $i=0,1,2,3$. Then, γ_i will appear as $\gamma_i(\|x-x_0\|)$ and $\gamma_i(r)$ in the conditions (C).

(b) Section 2 has an interest independent of Section 4. However, the results especially of Theorem 1 can apply in Abstract Fractional Calculus as suggested in Section 4. As an example crucial condition (h_3) is satisfied in (4.8), if we choose $\alpha_2=\alpha_3=0$ and $l\alpha_1=\frac{c}{2}$, where c is defined in (4.8). Similar choices can be given for the rest of the special cases of (h_3) appearing in Section 4.

3 Semi-Local Convergence for Explicit Methods

Theorem 1 is general enough so it can be utilized to investigate the semi-local convergence of methods (1.11), (1.12), (1.13) and (1.14), respectively. In particular, for the investigation of (1.12) (and consequently of (1.11)), we utilized the conditions (H') :

(h_1') $F:\Omega\subset B_1\to B_2$ is continuous and $A(F)(x,x)\in\mathscr{L}(B_1,B_2)$ for each $x\in\Omega$.

(h_2') There exist $l>0$ and $\Omega_0\subset B_1$ such that $A(F)(x,x)^{-1}\in\mathscr{L}(B_2,B_1)$ and

$$\left\|A(F)(x,x)^{-1}\right\|\le l^{-1}.$$

Set $\Omega_1=\Omega\cap\Omega_0$.

(h_3') There exist real numbers $\gamma_1, \alpha_2, \gamma_3$ satisfying

$$0 \leq \alpha_2 \leq \gamma_1 \text{ and } 0 \leq \gamma_3$$

such that for each $x, y \in \Omega_1$

$$\|F(x) - F(y) - A(F)(y,y)(x-y)\| \leq$$

$$l\left(\frac{\gamma_1}{2}\|x-y\| + \alpha_2\|y-x_0\| + \gamma_3\right)\|x-y\|.$$

(h_4') For each $x, y \in \Omega_1$ and some $\gamma_4 \geq 0$, $\gamma_5 \geq 0$

$$\|A(x,y) - A(y,y)\| \leq l\gamma_4$$

or

$$\|A(x,y) - A(y,y)\| \leq l\gamma_5\|x-y\|.$$

Set $\alpha_1 = \gamma_1 + \gamma_5$ and $\alpha_3 = \gamma_3 + \gamma_4$, if the second inequation holds or $\alpha_1 = \gamma_1$ and $\alpha_3 = \gamma_3 + \gamma_4$, if the first inequation holds. Further, suppose $0 \leq \alpha_3 < 1$.

(h_5') There exist $x_0 \in \Omega_0$ and $\eta \geq 0$ such that $A(F)(x_0,x_0)^{-1} \in \mathscr{L}(B_2, B_1)$ and

$$\left\|A(F)(x_0,x_0)^{-1} F(x_0)\right\| \leq \eta.$$

$(h_6') = (h_6)$
and
$(h_7') = (h_7)$.

Then, we can present the following semi-local convergence of method (1.12) utilizing the conditions (H') and the preceding notation.

Proposition 1. *Assume that the conditions (H') are fulfilled. Then, sequence $\{x_n\}$ generated by method (1.12) starting at $x_0 \in \Omega$ is well defined in $U(x_0,t^*)$, remains in $U(x_0,t^*)$ for each $n = 0,1,2,\ldots$ and converges to a solution $x^* \in \overline{U}(x_0,t^*)$ of equation $F(x) = 0$. Moreover, if $\alpha_1 \neq 0$, the equation $F(x) = 0$ posses a unique solution x^* in $\widetilde{U}$, where*

$$\widetilde{U} = \begin{cases} \overline{U}(x_0,t^*) \cap \Omega_0, & \text{if } h = \frac{1}{2}(1-\alpha_3)^2 \\ U(x_0,t^{**}) \cap \Omega_0, & \text{if } h < \frac{1}{2}(1-\alpha_3)^2 \end{cases}$$

and, if $\alpha_1 = 0$, the solution x^* is unique in $\overline{U}\left(x_0, \frac{\eta}{1-\alpha_3}\right)$, where t^* and t^{**} are given in Theorem 1.

Proof. Use in the proof of Theorem 1 instead of estimate (2.5) the analogous estimate

$$\|F(x_k)\| = \|F(x_k) - F(x_{k-1}) - A(x_{k-1},x_{k-1})(x_k - x_{k-1})\| =$$

$$\|[F(x_k) - F(x_{k-1}) - A(x_k,x_{k-1})(x_k - x_{k-1})] +$$

$$(A(x_k,x_{k-1}) - A(x_{k-1},x_{k-1}))(x_k - x_{k-1})\|$$

$$\leq l\left(\frac{\gamma_1}{2}\|x_k - x_{k-1}\| + \alpha_2\|x_{k-1} - x_0\| + \gamma_3\right)\|x_k - x_{k-1}\| +$$

$$\|A(x_k,x_{k-1}) - A(x_{k-1},x_{k-1})\|\,\|x_k - x_{k-1}\| \leq$$

$$l\left(\frac{\alpha_1}{2}(t_k - t_{k-1})^2 + \alpha_2(t_k - t_{k-1})t_{k-1} + \alpha_3(t_k - t_{k-1})\right),$$

where we used again that $\|x_k - x_{k-1}\| \leq t_k - t_{k-1}$, $\|x_{k-1} - x_0\| \leq t_{k-1}$ and the condition (h_4').

Remark. Comments similar to Remark 2 (1)-(3) can be reported but for method (1.11) and method (1.12) instead of method (1.5) and method (1.6), respectively.

Similarly, for (1.13) and (1.14), we use the conditions (H'') :
$(h''_1) = (h_1)$
$(h''_2) = (h_2)$
(h''_3) There exist real numbers $\alpha_1, \alpha_2, \gamma_3$ fulfilling

$$0 \leq \alpha_2 \leq \alpha_1 \text{ and } 0 \leq \gamma_3$$

such that for each $x, y \in \Omega_1$

$$\|F(x) - F(y) - A(F)(x,y)(x-y)\| \leq$$
$$l\left(\frac{\alpha_1}{2}\|x-y\| + \alpha_2\|y-x_0\| + \gamma_3\right)\|x-y\|.$$

(h''_4) For each $x, y, z \in \Omega_1$ and some $\gamma_3 \geq 0$

$$\|A(z,y) - A(y,x)\| \leq l\delta_3.$$

Set $\alpha_3 = \gamma_3 + \delta_3$ and further suppose $0 \leq \alpha_3 < 1$.
(h''_5) There exist $x_{-1} \in \Omega$, $x_0 \in \Omega$ and $\eta \geq 0$ such that $A(F)(x_0, x_{-1})^{-1} \in \mathscr{L}(B_2, B_1)$ and

$$\left\|A(F)(x_0, x_{-1})^{-1} F(x_0)\right\| \leq \eta.$$

$(h''_6) = (h_6)$
and
$(h''_7) = (h_7)$.
As a result, we show the following semi-local convergence of method (1.14) taking into account the conditions (H'') and the previous notation.

Proposition 2. *Assume that the conditions (H'') are satisfied. Then, sequence $\{x_n\}$ generated by method (1.14) starting at $x_0 \in \Omega$ is well defined in $U(x_0, t^*)$, remains in $U(x_0, t^*)$ for each $n = 0, 1, 2, \ldots$ and converges to a solution $x^* \in \overline{U}(x_0, t^*)$ of equation $F(x) = 0$. Moreover, if $\alpha_1 \neq 0$, the equation $F(x) = 0$ possess a unique solution x^* in $\widetilde{U}$, where*

$$\widetilde{U} = \begin{cases} \overline{U}(x_0, t^{**}) \cap \Omega_0, & \text{if } h = \frac{1}{2}(1-\alpha_3)^2 \\ U(x_0, t^{**}) \cap \Omega_0, & \text{if } h < \frac{1}{2}(1-\alpha_3)^2 \end{cases}$$

and, if $\alpha_1 = 0$, the solution x^* is unique in $\overline{U}\left(x_0, \frac{\eta}{1-\alpha_3}\right)$, where t^* and t^{**} are given in Theorem 1.

Proof. As in Proposition 1, utilize in the proof of Theorem 1 instead of estimate (2.5) the analogous estimate

$$\|F(x_k)\| =$$
$$\|F(x_k) - F(x_{k-1}) - A(x_k, x_{k-1})(x_k - x_{k-1})$$
$$+ (A(x_k, x_{k-1}) - A(x_{k-1}, x_{k-2}))(x_k - x_{k-1})\| \leq$$
$$\|F(x_k) - F(x_{k-1}) - A(x_k, x_{k-1})(x_k - x_{k-1})\| +$$
$$\|A(x_k, x_{k-1}) - A(x_{k-1}, x_{k-2})\|\|x_k - x_{k-1}\|$$
$$\leq l\left(\frac{\alpha_1}{2}\|x_k - x_{k-1}\| + \alpha_2\|x_{k-1} - x_0\| + \gamma_3\right)\|x_k - x_{k-1}\| + l\delta_3\|x_k - x_{k-1}\|$$
$$\leq l\left(\frac{\alpha_1}{2}(t_k - t_{k-1})^2 + \alpha_2(t_k - t_{k-1})t_{k-1} + \alpha_3(t_k - t_{k-1})\right),$$

where we used again that $\|x_k - x_{k-1}\| \leq t_k - t_{k-1}$, $\|x_{k-1} - x_0\| \leq t_{k-1}$ and (h''_4).

Remark. Similar results to Remark 2 (1)-(3) can be obtained but for methods (1.13) and (1.14) instead of methods (1.5) and(1.6), respectively.

4 Applications to X-valued Fractional Calculus

In this section we deal with Banach space $(X, \|\cdot\|)$ valued functions g of real domain $[0,a]$, $a>0$. All integrals here are of Bochner-type, see [7,12]. Once more, the derivatives of g are defined similarly to numerical ones, see [15], pp. 83-86 and p. 93.

Below we apply our Newton like numerical methods to X-valued fractional calculus.

Our aim is to solve

$$g(x)=0. \tag{4.1}$$

I) Let $1<\nu<2$, i.e. $\lceil \nu \rceil = 2$ ($\lceil\cdot\rceil$ ceiling of number); $x,y\in[0,a]$, $a>0$, and $g\in C^2([0,a],X)$.

The left X-valued Caputo fractional derivatives (see [4]) is written as

$$\left(D^{\nu}_{*y}g\right)(x) := \frac{1}{\Gamma(2-\nu)}\int_y^x (x-t)^{1-\nu} g''(t)\,dt, \tag{4.2}$$

when $x\geq y$, and

$$\left(D^{\nu}_{*x}g\right)(y) := \frac{1}{\Gamma(2-\nu)}\int_x^y (y-t)^{1-\nu} g''(t)\,dt, \tag{4.3}$$

when $y\geq x$, where Γ is the gamma function.

We define also the X-valued fractional linear operator

$$(A_0(g))(x,y) := \begin{cases} g'(y)+\left(D^{\nu}_{*y}g\right)(x)\cdot\frac{(x-y)^{\nu-1}}{\Gamma(\nu+1)}, & x>y,\\ g'(x)+\left(D^{\nu}_{*x}g\right)(y)\cdot\frac{(y-x)^{\nu-1}}{\Gamma(\nu+1)}, & y>x,\\ 0, & x=y. \end{cases} \tag{4.4}$$

By X-valued left fractional Caputo Taylor's formula (see [4]) we get that

$$g(x)-g(y)=g'(y)(x-y)+\frac{1}{\Gamma(\nu)}\int_y^x (x-t)^{\nu-1} D^{\nu}_{*y}g(t)\,dt, \quad \text{for } x>y, \tag{4.5}$$

and

$$g(y)-g(x)=g'(x)(y-x)+\frac{1}{\Gamma(\nu)}\int_x^y (y-t)^{\nu-1} D^{\nu}_{*x}g(t)\,dt, \quad \text{for } x<y, \tag{4.6}$$

equivalently, it holds

$$g(x)-g(y)=g'(x)(x-y)-\frac{1}{\Gamma(\nu)}\int_x^y (y-t)^{\nu-1} D^{\nu}_{*x}g(t)\,dt, \quad \text{for } x<y. \tag{4.7}$$

Our aim is to show that

$$\|g(x)-g(y)-(A_0(g))(x,y)\cdot(x-y)\| \leq c\cdot\frac{(x-y)^2}{2}, \tag{4.8}$$

for any $x,y\in[0,a]$, $0<c<1$.

When $x=y$, (4.8) becomes trivial.

We suppose $x\neq y$. The following cases can be distinguish:

1) $x>y$: We report that

$$\|g(x)-g(y)-(A_0(g))(x,y)\cdot(x-y)\| = \tag{4.9}$$

$$\left\| g'(y)(x-y)+\frac{1}{\Gamma(\nu)}\int_y^x (x-t)^{\nu-1}\left(D^{\nu}_{*y}g\right)(t)\,dt - \right.$$

$$\left.\left(g'(y)+\left(D^{\nu}_{*y}g\right)(x)\cdot\frac{(x-y)^{\nu-1}}{\Gamma(\nu+1)}\right)(x-y)\right\| =$$

$$\left\|\frac{1}{\Gamma(\nu)}\int_y^x (x-t)^{\nu-1}\left(D^{\nu}_{*y}g\right)(t)\,dt - \left(D^{\nu}_{*y}g\right)(x)\frac{(x-y)^{\nu}}{\Gamma(\nu+1)}\right\| = \tag{4.10}$$

$$\left\|\frac{1}{\Gamma(\nu)}\int_y^x (x-t)^{\nu-1}\left(D^{\nu}_{*y}g\right)(t)\,dt - \frac{1}{\Gamma(\nu)}\int_y^x (x-t)^{\nu-1}\left(D^{\nu}_{*y}g\right)(x)\,dt\right\| = \tag{4.11}$$

(by [1], p. 426, Theorem 11.43)

$$\frac{1}{\Gamma(\nu)}\left\|\int_y^x (x-t)^{\nu-1}\left(\left(D_{*y}^{\nu}g\right)(t)-\left(D_{*y}^{\nu}g\right)(x)\right)dt\right\| \leq$$

(by [7])

$$\frac{1}{\Gamma(\nu)}\int_y^x (x-t)^{\nu-1}\left\|\left(D_{*y}^{\nu}g\right)(t)-\left(D_{*y}^{\nu}g\right)(x)\right\|dt =: (\xi), \tag{4.12}$$

(assume that

$$\left\|\left(D_{*y}^{\nu}g\right)(t)-\left(D_{*y}^{\nu}g\right)(x)\right\| \leq \lambda_1 |t-x|^{2-\nu}, \tag{4.13}$$

for any $t,x,y \in [0,a] : x \geq t \geq y$, where $\lambda_1 < \Gamma(\nu)$, i.e. $\rho_1 := \frac{\lambda_1}{\Gamma(\nu)} < 1$).

Therefore

$$(\xi) \leq \frac{\lambda_1}{\Gamma(\nu)}\int_y^x (x-t)^{\nu-1}(x-t)^{2-\nu}dt \tag{4.14}$$

$$= \frac{\lambda_1}{\Gamma(\nu)}\int_y^x (x-t)\,dt = \frac{\lambda_1}{\Gamma(\nu)}\frac{(x-y)^2}{2} = \rho_1\frac{(x-y)^2}{2}. \tag{4.15}$$

We have proved that

$$\|g(x)-g(y)-(A_0(g))(x,y)\cdot(x-y)\| \leq \rho_1\frac{(x-y)^2}{2}, \tag{4.16}$$

where $0 < \rho_1 < 1$, and $x > y$.

2) $x < y$: We report that

$$\|g(x)-g(y)-(A_0(g))(x,y)\cdot(x-y)\| = \tag{4.17}$$

$$\left\|g'(x)(x-y)-\frac{1}{\Gamma(\nu)}\int_x^y (y-t)^{\nu-1}D_{*x}^{\nu}g(t)\,dt-\right.$$

$$\left.\left(g'(x)+\left(D_{*x}^{\nu}g\right)(y)\cdot\frac{(y-x)^{\nu-1}}{\Gamma(\nu+1)}\right)(x-y)\right\| =$$

$$\left\|-\frac{1}{\Gamma(\nu)}\int_x^y (y-t)^{\nu-1}D_{*x}^{\nu}g(t)\,dt+\left(D_{*x}^{\nu}g\right)(y)\frac{(y-x)^{\nu}}{\Gamma(\nu+1)}\right\| = \tag{4.18}$$

$$\left\|\frac{1}{\Gamma(\nu)}\int_x^y (y-t)^{\nu-1}D_{*x}^{\nu}g(t)\,dt-\left(D_{*x}^{\nu}g\right)(y)\frac{(y-x)^{\nu}}{\Gamma(\nu+1)}\right\| = \tag{4.19}$$

$$\frac{1}{\Gamma(\nu)}\left\|\int_x^y (y-t)^{\nu-1}D_{*x}^{\nu}g(t)\,dt-\frac{1}{\Gamma(\nu)}\int_x^y (y-t)^{\nu-1}\left(D_{*x}^{\nu}g\right)(y)\,dt\right\| =$$

$$\frac{1}{\Gamma(\nu)}\left\|\int_x^y (y-t)^{\nu-1}\left(D_{*x}^{\nu}g(t)-D_{*x}^{\nu}g(y)\right)dt\right\| \leq \tag{4.20}$$

$$\frac{1}{\Gamma(\nu)}\int_x^y (y-t)^{\nu-1}\left\|D_{*x}^{\nu}g(t)-D_{*x}^{\nu}g(y)\right\|dt$$

(by assumption,

$$\left\|D_{*x}^{\nu}g(t)-D_{*x}^{\nu}g(y)\right\| \leq \lambda_2|t-y|^{2-\nu}, \tag{4.21}$$

for any $t,y,x \in [0,a] : y \geq t \geq x$).

$$\leq \frac{1}{\Gamma(\nu)}\int_x^y (y-t)^{\nu-1}\lambda_2|t-y|^{2-\nu}dt$$

$$= \frac{\lambda_2}{\Gamma(\nu)}\int_x^y (y-t)^{\nu-1}(y-t)^{2-\nu}dt \tag{4.22}$$

$$= \frac{\lambda_2}{\Gamma(\nu)}\int_x^y (y-t)\,dt = \frac{\lambda_2}{\Gamma(\nu)}\frac{(x-y)^2}{2}.$$

Assuming also $\rho_2 := \frac{\lambda_2}{\Gamma(\nu)} < 1$ (i.e. $\lambda_2 < \Gamma(\nu)$), we found that

$$\|g(x) - g(y) - (A_0(g))(x,y)\cdot(x-y)\| \leq \rho_2 \frac{(x-y)^2}{2}, \quad \text{for } x<y. \tag{4.23}$$

Conclusion: Choosing $\lambda := \max(\lambda_1, \lambda_2)$ and $\rho := \frac{\lambda}{\Gamma(\nu)} < 1$, we show that

$$\|g(x) - g(y) - (A_0(g))(x,y)\cdot(x-y)\| \leq \rho \frac{(x-y)^2}{2}, \quad \text{for any } x,y \in [0,a]. \tag{4.24}$$

This represents a condition utilized to solve numerically $g(x) = 0$.

II) Let $n-1 < \nu < n$, $n \in \mathbb{N} - \{1\}$, i.e. $\lceil \nu \rceil = n$; $x,y \in [0,a]$, $a>0$, and $g \in C^n([0,a], X)$.

We define the following X-valued right Caputo fractional derivatives (see [3]),

$$D^{\nu}_{x-} g(y) = \frac{(-1)^n}{\Gamma(n-\nu)} \int_y^x (z-y)^{n-\nu-1} g^{(n)}(z)\,dz, \quad \text{for } y \leq x, \tag{4.25}$$

and

$$D^{\nu}_{y-} g(x) = \frac{(-1)^n}{\Gamma(n-\nu)} \int_x^y (z-x)^{n-\nu-1} g^{(n)}(z)\,dz, \quad \text{for } x \leq y. \tag{4.26}$$

By X-valued right Caputo fractional Taylor's formula (see [3]) we have

$$g(x) - g(y) = \sum_{k=1}^{n-1} \frac{g^{(k)}(y)}{k!}(x-y)^k + \frac{1}{\Gamma(\nu)} \int_x^y (z-x)^{\nu-1} \left(D^{\nu}_{y-} g\right)(z)\,dz, \tag{4.27}$$

when $x \leq y$, and

$$g(y) - g(x) = \sum_{k=1}^{n-1} \frac{g^{(k)}(x)}{k!}(y-x)^k + \frac{1}{\Gamma(\nu)} \int_y^x (z-y)^{\nu-1} \left(D^{\nu}_{x-} g\right)(z)\,dz, \tag{4.28}$$

when $x \geq y$.

The fractional linear operator is defined as

$$(A_0(g))(x,y) := \begin{cases} \sum_{k=1}^{n-1} \frac{g^{(k)}(x)}{k!}(y-x)^k - \left(D^{\nu}_{x-} g\right)(y)\cdot \frac{(x-y)^{\nu-1}}{\Gamma(\nu+1)}, & x>y, \\ \sum_{k=1}^{n-1} \frac{g^{(k)}(y)}{k!}(x-y)^k - \left(D^{\nu}_{y-} g\right)(x)\cdot \frac{(y-x)^{\nu-1}}{\Gamma(\nu+1)}, & y>x, \\ 0, \quad x=y. \end{cases} \tag{4.29}$$

Our aim is to show that

$$\|g(x) - g(y) - (A_0(g))(x,y)\cdot(x-y)\| \leq c \cdot \frac{|x-y|^n}{n}, \tag{4.30}$$

for any $x,y \in [0,a]$, $0<c<1$.

When $x=y$ (4.30) becomes trivial.

We suppose that $x \neq y$. We have the following cases:

1) $x>y$: We report that

$$\|(g(x) - g(y)) - (A_0(g))(x,y)\cdot(x-y)\| = \tag{4.31}$$

$$\|(g(y) - g(x)) - (A_0(g))(x,y)\cdot(y-x)\| =$$

$$\left\| \left(\sum_{k=1}^{n-1} \frac{g^{(k)}(x)}{k!}(y-x)^k + \frac{1}{\Gamma(\nu)} \int_y^x (z-y)^{\nu-1} \left(D^{\nu}_{x-} g\right)(z)\,dz \right) - \right.$$

$$\left. \left(\sum_{k=1}^{n-1} \frac{g^{(k)}(x)}{k!}(y-x)^{k-1} - \left(D^{\nu}_{x-} g\right)(y)\cdot \frac{(x-y)^{\nu-1}}{\Gamma(\nu+1)} \right)(y-x) \right\| =$$

$$\left\| \frac{1}{\Gamma(\nu)} \int_y^x (z-y)^{\nu-1} \left(D^{\nu}_{x-} g\right)(z)\,dz + \left(D^{\nu}_{x-} g\right)(y) \frac{(x-y)^{\nu-1}}{\Gamma(\nu+1)}(y-x) \right\| = \tag{4.32}$$

$$\left\| \frac{1}{\Gamma(\nu)} \int_y^x (z-y)^{\nu-1} \left(D_{x-}^{\nu} g\right)(z)\,dz - \left(D_{x-}^{\nu} g\right)(y) \frac{(x-y)^{\nu}}{\Gamma(\nu+1)} \right\| =$$

$$\frac{1}{\Gamma(\nu)} \left\| \int_y^x (z-y)^{\nu-1} \left(D_{x-}^{\nu} g\right)(z)\,dz - \int_y^x (z-y)^{\nu-1} \left(D_{x-}^{\nu} g\right)(y)\,dz \right\| =$$

$$\frac{1}{\Gamma(\nu)} \left\| \int_y^x (z-y)^{\nu-1} \left(\left(D_{x-}^{\nu} g\right)(z) - \left(D_{x-}^{\nu} g\right)(y) \right) dz \right\| \leq \tag{4.33}$$

$$\frac{1}{\Gamma(\nu)} \int_y^x (z-y)^{\nu-1} \left\| \left(D_{x-}^{\nu} g\right)(z) - \left(D_{x-}^{\nu} g\right)(y) \right\| dz$$

(we suppose that

$$\left\| \left(D_{x-}^{\nu} g\right)(z) - \left(D_{x-}^{\nu} g\right)(y) \right\| \leq \lambda_1 |z-y|^{n-\nu}, \tag{4.34}$$

$\lambda_1 > 0$, for all $x,z,y \in [0,a]$, with $x \geq z \geq y$)

$$\leq \frac{\lambda_1}{\Gamma(\nu)} \int_y^x (z-y)^{\nu-1} (z-y)^{n-\nu}\,dz = \tag{4.35}$$

$$= \frac{\lambda_1}{\Gamma(\nu)} \int_y^x (z-y)^{n-1}\,dz = \frac{\lambda_1}{\Gamma(\nu)} \frac{(x-y)^n}{n}$$

(assume $\lambda_1 < \Gamma(\nu)$, i.e. $\rho_1 := \frac{\lambda_1}{\Gamma(\nu)} < 1$)

$$= \rho_1 \frac{(x-y)^n}{n}.$$

We have reported, when $x > y$, that

$$\| g(x) - g(y) - (A_0(g))(x,y) \cdot (x-y) \| \leq \rho_1 \frac{(x-y)^n}{n}. \tag{4.36}$$

2) $y > x$: We conclude that

$$\| g(x) - g(y) - (A_0(g))(x,y) \cdot (x-y) \| =$$

$$\left\| \left(\sum_{k=1}^{n-1} \frac{g^{(k)}(y)}{k!} (x-y)^k + \frac{1}{\Gamma(\nu)} \int_x^y (z-x)^{\nu-1} \left(D_{y-}^{\nu} g\right)(z)\,dz \right) - \right.$$

$$\left. \left(\sum_{k=1}^{n-1} \frac{g^{(k)}(y)}{k!} (x-y)^{k-1} - \left(D_{y-}^{\nu} g\right)(x) \cdot \frac{(y-x)^{\nu-1}}{\Gamma(\nu+1)} \right)(x-y) \right\| = \tag{4.37}$$

$$\left\| \frac{1}{\Gamma(\nu)} \int_x^y (z-x)^{\nu-1} \left(D_{y-}^{\nu} g\right)(z)\,dz - \left(D_{y-}^{\nu} g\right)(x) \frac{(y-x)^{\nu}}{\Gamma(\nu+1)} \right\| = \tag{4.38}$$

$$\left\| \frac{1}{\Gamma(\nu)} \int_x^y (z-x)^{\nu-1} \left(D_{y-}^{\nu} g\right)(z)\,dz - \frac{1}{\Gamma(\nu)} \int_x^y (z-x)^{\nu-1} \left(D_{y-}^{\nu} g\right)(x)\,dz \right\| =$$

$$\frac{1}{\Gamma(\nu)} \left\| \int_x^y (z-x)^{\nu-1} \left(\left(D_{y-}^{\nu} g\right)(z) - \left(D_{y-}^{\nu} g\right)(x) \right) dz \right\| \leq \tag{4.39}$$

$$\frac{1}{\Gamma(\nu)} \int_x^y (z-x)^{\nu-1} \left\| \left(D_{y-}^{\nu} g\right)(z) - \left(D_{y-}^{\nu} g\right)(x) \right\| dz$$

(we suppose that

$$\left\| \left| \left(D_{y-}^{\nu} g\right)(z) - \left(D_{y-}^{\nu} g\right)(x) \right| \right\| \leq \lambda_2 |z-x|^{n-\nu}, \tag{4.40}$$

$\lambda_2 > 0$, for all $y,z,x \in [0,a]$ with $y \geq z \geq x$)

$$\leq \frac{\lambda_2}{\Gamma(\nu)} \int_x^y (z-x)^{\nu-1} (z-x)^{n-\nu}\,dz = \tag{4.41}$$

$$\frac{\lambda_2}{\Gamma(\nu)} \int_x^y (z-x)^{n-1}\,dz = \frac{\lambda_2}{\Gamma(\nu)} \frac{(y-x)^n}{n}.$$

Suppose now that $\lambda_2 < \Gamma(\nu)$, that is $\rho_2 := \frac{\lambda_2}{\Gamma(\nu)} < 1$.

We have reported, for $y > x$, that

$$\|g(x) - g(y) - (A_0(g))(x,y) \cdot (x-y)\| \leq \rho_2 \frac{(y-x)^n}{n}. \tag{4.42}$$

Set $\lambda := \max(\lambda_1, \lambda_2)$, and

$$0 < \rho := \frac{\lambda}{\Gamma(\nu)} < 1. \tag{4.43}$$

5 Conclusion

We show that

$$\|g(x) - g(y) - (A_0(g))(x,y) \cdot (x-y)\| \leq \rho \frac{|x-y|^n}{n}, \text{ for any } x, y \in [0,a]. \tag{4.44}$$

In the particular case of $1 < \nu < 2$, we conclude that

$$\|g(x) - g(y) - (A_0(f))(x,y) \cdot (x-y)\| \leq \rho \frac{(x-y)^2}{2}, \tag{4.45}$$

for any $x, y \in [0,a]$, $0 < \rho < 1$.

Thus, this represents a condition required to solve numerically $g(x) = 0$.

References

[1] C. D. Aliprantis and K. C. Border, *Infinite Dimensional Analysis*, Springer, New York, 2006.
[2] S. Amat, S. Busquier and S. Plaza, Chaotic dynamics of a third-order Newton-type method, J. Math. Anal. Appl. **366**, 164–174 (2010).
[3] G. A. Anastassiou, *Strong right fractional calculus for Banach space valued functions, Rev. Proyecc.* **36**, (2017), 149-186 (2017).
[4] G. A. Anastassiou, A strong fractional calculus theory for Banach space valued functions, *Nonlin. Funct. Anal. Appl.* (Korea), accepted for publication, (2017).
[5] I. K. Argyros, A unifying local-semilocal convergence analysis and applications for two-point Newton-like methods in Banach space, *J. Math. Anal. Appl.* **298**, 374-397 (2004).
[6] I. K. Argyros and A. Magrenan, *Iterative Methods and Their Dynamics With Applications*, CRC Press, New York, 2017.
[7] Bochner integral, *Encyclopedia of Mathematics*, URL: http://www.encyclopediaofmath.org/index.php?title=Bochner_integral&oldid=38659.
[8] M. Edelstein, On fixed and periodic points under contractive mappings, *J. London Math. Soc.* **37**, 74-79 (1962).
[9] J. A. Ezquerro, J. M. Gutierrez, M. A. Hernandez, N. Romero and M. J. Rubio, The Newton Method: From Newton to Kantorovich (Spanish), *Gac. R. Soc. Mat. Esp.* **13**, 53-76 (2010).
[10] L. V. Kantorovich and G. P. Akilov, *Functional Analysis in Normed Spaces*, Pergamon Press, New York, 1982.
[11] A. Magrenan, A new tool to study real dynamics: The convergence plane, *Appl. Math. Comput.* **248**, 215-224 (2014).
[12] J. Mikusinski, *The Bochner Integral*, Academic Press, New York, 1978.
[13] F. A. Potra and V. Ptak, *Nondiscrete induction and iterative processes*, Pitman Publ., London, 1984.
[14] P. D. Proinov, New general convergence theory for iterative processes and its applications to Newton-Kantorovich type theorems, *J. Complex.* **26**, 3–42 (2010).
[15] G. E. Shilov, *Elementary Functional Analysis*, Dover Publications, Inc., New York, 1996.