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Estimation for Inverse Weibull distribution under Generalized Progressive Hybrid Censoring Scheme

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Abstract: In this paper, the statistical inference of the unknown parameters of a two-parameter inverse Weibull (IW) distribution based on the generalized Type-II progressive hybrid censoring scheme (GT-II PHCS) has been considered. The Bayes estimates for the IW parameters and the corresponding survival and hazard functions are obtained based on squared error loss (SEL) function by using the approximation form of Lindley (1980). Finally, a Monte Carlo simulation study is carried out to compare the performance of the maximum likelihood and the Bayesian estimates.

Keywords: Bayesian estimation; Inverse Weibull distribution; Lindley approximate; Maximum likelihood estimation; generalized progressive hybrid censoring sample.

1 Introduction

Consider an experiment in which n units are placed on life test. In progressive censoring schemes, m units complete failures are going to be observed. When the first failure is observed, R_1 of the $n-1$ surviving units are randomly selected and removed. At the second observed failure, R_2 of the $n-R_1-2$ surviving units are randomly selected and removed. The experiment finally terminates at the time of the m^{th} failure when all remaining $R_m = n - R_1 - \ldots - R_{m-1} - m$ surviving units are removed. The censoring numbers $\{R_i, i = 1,..,m-1\}$ are prefixed. We will denote the m ordered failure times thus observed by $X_{1:m:n}, \ldots, X_{m:m:n}$. It is evident that $n = m + \sum_{k=1}^{m} R_k$. The resulting m ordered values which are obtained from this type of censoring are referred to as progressively Type-II right censored order statistics. Several authors have studied progressive Type-II censoring and properties of order statistics arising from such a progressively censored life test. Some key references are are Aggarwala and Balakrishnan [1], Cramer and Iliopoulos [2], Raqab et al. [3], Mohie El-Din and Shafay [4], and Balakrishnan and Cohen [5].

The disadvantages of the progressive Type-II censoring scheme are that the time of the experiment can be very long if the units are highly reliable. Therefore, Kundu and Joarder [6] and Childs et al. [7] proposed a progressive hybrid censoring scheme (PHCS), in this life-testing the experiment is terminated at time $\min\{X_{m:m:n}, T\}$, where $T \in (0,\infty)$ pre-fixed in advance. Under PHCS, the time on experiment will be no more than T. Some recent studies on PHCS have been carried out by many authors including Lin et al. [8], Lin and Huang. [9], and Hemmati and Khorram [10]. On the other hand, the disadvantages of the PHCS is that it cannot be applied when very few failures may occur before time T. For this reason, Cho et al. [11] propose a GT-II PHCS which allows us to observe a pre-specified number of failures. So, the certain number of failures and their lifetimes are always provided under the GT-II PHCS. The life-testing experiment based on this censoring scheme can save both the total time on tests and the cost induced by failures of the units. Moreover, the efficiency of statistical estimation is increased due to more failed observations. The detailed description and its advantages will be described in the next section.

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In this paper, the underlying distribution is assumed to be the IW distribution, with probability density function (PDF), cumulative distribution function (CDF) and hazard rate function respectively given by

$$f(x|\alpha,\beta) = \alpha\beta x^{-(\beta+1)} \exp\left(-\alpha x^{-\beta}\right), \; x>0, \tag{1}$$

$$F(x|\alpha,\beta) = \exp\left(-\alpha x^{-\beta}\right), \quad x>0, \tag{2}$$

and

$$H(x|\alpha,\beta) = \frac{\alpha\beta x^{-(\beta+1)}}{1-\exp\left(\alpha x^{-\beta}\right)}, \quad x>0, \tag{3}$$

where $\alpha>0$ and $\beta>0$.

The IW distribution is more appropriate model than the Weibull distribution because the Weibull distribution does not provide a satisfactory parametric fit if the data indicate a non-monotone and unimodal hazard rate functions. The hazard rate function of IW distribution can be decreasing or increasing depending on the value of the shape parameter. The IW distribution is useful to model several data such as the time to breakdown of an insulating fluid subjected to the action of a constant tension and degradation of mechanical components such as pistons and crankshafts of diesel engines. Extensive work has been done on the IW distribution, see for example, Keller and Kamath [12], Erto and Rapone [13], Calabria and Pulcini [14], Maswadah [15] and for more details about the generalizations of IW distribution see [16]. In addition, many articles have considered IW distribution under different censoring schemes. Among others, Kundu and Howlader [17], Musleh and Helu [18], Sultan et al. [19] and Xiuyun and Zaizai [20].

The rest of this paper is organized as follows. In Section 2, the description of the model of the GT-II PHCS is presented. The maximum likelihood estimators (MLE) for the unknown parameters and the corresponding survival and hazard functions are derived in Section 3. In Section 4, Bayesian estimates under SEL functions using Lindley approximation [21] is provided. Finally, in Section 5, Monte Carlo simulation results and the analysis of data sets are presented.

2 The Model Description

Consider a life-testing experiment in which n identical units are put on test. Assume that $X_1, X_2, ..., X_n$ denote the corresponding lifetimes from a distribution with (CDF) $F(x|\alpha,\beta)$ and (PDF) $f(x|\alpha,\beta)$. GT-II PHCS may be described as follows. For $T\in(0,\infty)$ and integers $k,m\in\{1,2,...,n\}$ are pre-fixed such that $k<m$ with $R=(R_1,R_2,...,R_m)$ is also pre-fixed integers satisfying $n=m+R_1+\ldots+R_m$. At the time of first failure, R_1 of the remaining units are randomly removed. Similarly at the time of the second failure R_2, of the remaining units are removed and so on. This process continues until, immediately following the terminated time $T^*=\max\{X_{k:m:n},\min\{X_{m:m:n},T\}\}$, at this time all the remaining units are removed from the experiment. This GT-II PHCS modifies PHCS by allowing the experiment to continue beyond time T if very few failures had been observed up to time T. Under this scheme, the experimenter would ideally like to observe m failures, but is willing to accept a bare minimum of k failures. Let D denote the number of observed failures up to time T (see Fig. 1).

Under GT-II PHCS described above, we have one of the following types of observations:

1. Suppose that the k^{th} failure occurs after T, then the experiment terminates at $X_{k:m:n}$ and we will observe $\{X_{1:m:n}<\ldots<X_{D:m:n}<<X_{D+1:n}<\ldots<X_{k:n}\}$.
2. Suppose that the k^{th} failure occurs before T and the m^{th} failure occurs after T then the experiment terminates at T and we will observe $\{X_{1:m:n}<\ldots<X_{k:m:n}<X_{k+1:m:n}<\ldots<X_{D:m:n}\}$.
3. Suppose that the m^{th} failure occurs before T, then the experiment terminates at $X_{m:m:n}$ and we will observe $\{X_{1:m:n}<\ldots<X_{k:m:n}<X_{k+1:m:n}<\ldots<X_{m:m:n}\}$.

Given a GT-II PHCS, the joint density function for three different cases are as follows:

$$f_{\underline{\mathbf{X}}}(\underline{\mathbf{x}}) = \left[\prod_{i=1}^{D^*}\sum_{j=i}^{m}\left(R_j^*+1\right)\right]\prod_{i=1}^{D^*} f\left(x_{i:D^*:n}\right)\left[\bar{F}(x_{i:D^*:n})\right]^{R_i^*}\left[\bar{F}(T)\right]^{R_t^*}, \tag{4}$$

where

$$D^* = \begin{cases} k \text{ if } T<X_{k:m:n}<X_{m:m:n}, \\ D \text{ if } X_{k:m:n}\le T<X_{m:m:n}, \\ m \text{ if } X_{k:m:n}<X_{m:m:n}\le T, \end{cases} \tag{5}$$

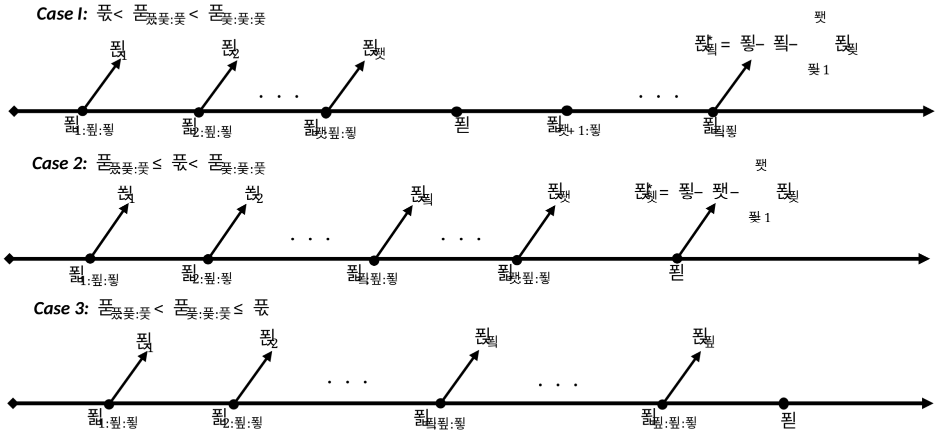

Fig. 1. Schematic representation of generalized progressive hybrid censoring scheme

Fig. 1: ???

$$R^* = \begin{cases} \left(R_1,\ldots,R_D,0,\ldots,0,R_k^* = n-k-\sum_{j=1}^{D} R_j\right) & \text{if } T < X_{k:m:n} < X_{m:m:n}, \\ (R_1,\ldots,R_D) & \text{if } X_{k:m:n} \le T < X_{m:m:n}, \\ (R_1,\ldots,R_m) & \text{if } X_{k:m:n} < X_{m:m:n} \le T, \end{cases} \tag{6}$$

with R_t^* is the number of surviving units that are removed at T, given by

$$R_t^* = \begin{cases} 0 & \text{if } T < X_{k:m:n} < X_{m:m:n}, \\ n-D-\sum_{j=1}^{D} R_j & \text{if } X_{k:m:n} \le T < X_{m:m:n}, \\ 0 & \text{if } X_{k:m:n} < X_{m:m:n} \le T, \end{cases} \tag{7}$$

and

$$\underline{\mathbf{x}} = \begin{cases} (x_{1:m:n},\ldots,x_{D:m:n},x_{D+1:n},\ldots,x_{k:n}) & \text{if } T < X_{k:m:n} < X_{m:m:n}, \\ (x_{1:m:n},\ldots,x_{D:m:n}) & \text{if } X_{k:m:n} \le T < X_{m:m:n}, \\ (x_{1:m:n},\ldots,x_{m:m:n}) & \text{if } X_{k:m:n} < X_{m:m:n} \le T. \end{cases} \tag{8}$$

Upon using (2) and (1) in (4), the likelihood function of α,β based on GT-II PHCS can be obtained as

$$\begin{aligned} L(\alpha,\beta) &\propto \prod_{i=1}^{D^*} \alpha\beta x_{i:D^*:n}^{-(\beta+1)} \exp\left(-\alpha x_{i:D^*:n}^{-\beta}\right) \left[1-\exp\left(-\alpha x_{i:D^*:n}^{-\beta}\right)\right]^{R_i^*} \left[1-\exp\left(-\alpha T^{-\beta}\right)\right]^{R_t^*} \\ &\propto (\alpha\beta)^{D^*} \exp\left(-\alpha \sum_{i=1}^{D^*} x_i^{\beta}\right) \left[1-\exp\left(-\alpha T^{-\beta}\right)\right]^{R_t^*} \prod_{i=1}^{D^*} x_i^{(\beta+1)} \left[1-\exp\left(-\alpha x_i^{\beta}\right)\right]^{R_i^*}, \end{aligned} \tag{9}$$

where $x_i = x_{i:D^*:n}^{-1}$ for simplicity of notation.

3 Maximum likelihood estimation

The corresponding log-likelihood function (ℓ) is obtained from (9) as

$$\begin{aligned} \ell = D^* \ln(\alpha\beta) - \alpha \sum_{i=1}^{D^*} x_i^{\beta} + R_t^* \ln\left[1-\exp\left(-\alpha T^{-\beta}\right)\right] \\ + (\beta+1)\sum_{i=1}^{D^*} \ln(x_i) + R_i^* \ln\left[1-\exp\left(-\alpha x_i^{\beta}\right)\right]. \end{aligned} \tag{10}$$

By taking derivatives of (10)with respect to β and α and equating them to zero

$$\frac{\partial \ell}{\partial \alpha} = \frac{D^*}{\alpha} - \sum_{i=1}^{D^*} x_i^{\beta} + R_t^* u_t T^{\beta} + \sum_{i=1}^{D^*} R_i^* x_i^{\beta} u_i \tag{11}$$

$$\frac{\partial \ell}{\partial \beta} = \frac{D^*}{\beta} - \alpha \sum_{i=1}^{D^*} x_i^{\beta} \ln(x_i) + \sum_{i=1}^{D^*} \ln(x_i) + \\ + \alpha \sum_{i=1}^{D^*} R_i^* x_i^{\beta} u_i \ln(x_i) + \alpha R_t^* u_t T^{\beta} \ln(T), \tag{12}$$

where $u_i = [y_i - 1]^{-1}$, $u_t = [y_t - 1]^{-1}$, with $y_i = \exp\left(\alpha x_i^{\beta}\right)$, and $y_t = \exp\left(\alpha T^{\beta}\right)$. We obtain the MLE of α and β as

$$\widehat{\alpha}_{ML}(\beta) = \frac{D^*}{\sum_{i=1}^{D^*} x_i^{\beta} - R_t^* u_t T^{\beta} + \sum_{i=1}^{D^*} R_i^* x_i^{\beta} u_i} \tag{13}$$

$$\widehat{\beta}_{ML} = \frac{D^*}{\widehat{\alpha}_{ML}(\beta)\left[\sum_{i=1}^{D^*} x_i^{\beta} \ln(x_i) - \sum_{i=1}^{D^*} R_i^* x_i^{\beta} u_i \ln(x_i) - R_t^* u_t T^{\beta} \ln(T)\right]}. \tag{14}$$

We can evaluate the MLE of α and β by solving these two likelihood equations using numerical technique.

By using the invariance property of MLE, the MLE of the corresponding survival and hazard functions are then given, respectively, by

$$\widehat{S}_{ML}(t) = 1 - \exp\left[-\widehat{\alpha}_{ML} t^{-\widehat{\beta}_{ML}}\right], t > 0 \tag{15}$$

$$\widehat{H}_{ML}(t) = \widehat{\alpha}_{ML} \widehat{\beta}_{ML} t^{-\left(\widehat{\beta}_{ML}+1\right)} \left[\exp\left(\widehat{\alpha}_{ML} t^{-\widehat{\beta}_{ML}}\right) - 1\right]^{-1}, t > 0\,. \tag{16}$$

4 Bayesian estimations

In this section, the SEL function is used to obtain the Bayes estimates of the unknown parameters α and β. The Bayes estimates are considered under the assumption of independent gamma priors of and with the following joint density

$$\pi(\alpha, \beta) \propto \alpha^{a_1 - 1} \exp(-b_1 \alpha) \beta^{a_2 - 1} \exp(-b_2 \beta), \tag{17}$$

where $a_1,\ b_1,\ a_2,\ b_2$ are positive constants.

Based on the likelihood function (9) and the joint prior density (17), the joint posterior distribution of α, β and the data is

$$\pi^*(\alpha, \beta) = \pi(\alpha, \beta) L(\alpha, \beta) \Big/ \int_0^{\infty}\int_0^{\infty} \pi(\alpha, \beta) L(\alpha, \beta)\, d\alpha d\beta \\ \propto \alpha^{D^* + a_1 - 1} \beta^{D^* + a_2 - 1} \exp\left[-\alpha\left(\sum_{i=1}^{D^*} x_i^{\beta} + b_1\right)\right] \exp(-b_2 \beta) \\ \times \left[1 - \exp\left(-\alpha T^{-\beta}\right)\right]^{R_t^*} \prod_{i=1}^{D^*} x_i^{(\beta+1)} \left[1 - \exp\left(-\alpha x_i^{\beta}\right)\right]^{R_i^*}. \tag{18}$$

By using (18), the Bayesian estimator of the function of the parameters α and β, say $G(\alpha, \beta)$ under SEL function is given by

$$\widehat{G}_B(\alpha, \beta) = \frac{\int_0^{\infty}\int_0^{\infty} G(\alpha, \beta) \pi^*(\alpha, \beta)\, d\alpha d\beta}{\int_0^{\infty}\int_0^{\infty} G(\alpha, \beta) \pi(\alpha, \beta)\, d\alpha d\beta} = \frac{\int_0^{\infty}\int_0^{\infty} G(\alpha, \beta) \exp[Q(\alpha, \beta)]\, d\alpha d\beta}{\int_0^{\infty}\int_0^{\infty} \exp[Q(\alpha, \beta)]\, d\alpha d\beta}. \tag{19}$$

where $Q(\alpha, \beta) = \ln[\pi^*(\alpha, \beta)] = \ln[L(\alpha, \beta)] + \ln[\pi(\alpha, \beta)] = \ell + \rho(a_1, b_1)$. It is not possible to compute the ratio of the two integrals given by (19) in a closed form. Therefore, in such situation, we suggest using Lindley's approximation to obtain the Bayes estimates of the unknown parameters.

4.1 Lindley's approximation

For the two parameter case, $(\alpha,\beta)=(\theta_1,\theta_2)$, Lindley's approximation of (19) takes the form

$$\widehat{G}_B(\theta)=G(\theta)+\frac{1}{2}\sum g_{ij}\sigma_{ij}+\sum G_j\rho_j+\frac{1}{2}L_{30}\sigma_{11}G_1+\frac{1}{2}L_{21}(2\sigma_{12}G_1+\sigma_{11}G_2)$$
$$+\frac{1}{2}L_{21}(\sigma_{22}G_1+2\sigma_{12}G_2)+\frac{1}{2}L_{03}\sigma_{22}G_2, \tag{20}$$

where $L_{ij}=\frac{\partial^{i+j}L(\theta)}{\partial^i\theta_1\partial^j\theta_2}$, $i,j=0,\ 1,\ 2,\ 3$ and $i+j=3$, $g_i=\frac{\partial G(\theta)}{\partial\theta_i}$, $g_{ij}=\frac{\partial^2 G(\theta)}{\partial\theta_1\partial\theta_2}$ for $i,j=1,\ 2$, $G_k=\sum_j g_j\sigma_{kj}$, with σ_{ij} being the $(i,j)th$ elements of the inverse of the Fisher information matrix.

Evaluation of all functions in Eq. (20) at the MLE of (θ_1,θ_2), produces the approximation $\widehat{G}_B(\theta)$ to (19). Now, to apply Lindley's approximation (20), we obtain the quantities

$$L_{20}=-\frac{\partial^2\ell}{\partial\alpha^2}=\frac{D^*}{\widehat{\alpha}^2}+R_t^*T^{2\widehat{\beta}}u_t^2y_t+\sum_{i=1}^{D^*}R_i^*x_i^{2\widehat{\beta}}u_i^2y_i,$$

$$L_{20}=-\frac{\partial^2\ell}{\partial\beta^2}=\frac{D^*}{\widehat{\beta}^2}+\widehat{\alpha}\sum_{i=1}^{D^*}x_i^{\widehat{\beta}}\ln(x_i)^2-\widehat{\alpha}R_t^*T^{\widehat{\beta}}\ln(T)^2u_t\left[1-\widehat{\alpha}T^{\widehat{\beta}}u_ty_t\right]$$
$$-\widehat{\alpha}\sum_{i=1}^{D^*}R_i^*x_i^{\widehat{\beta}}\ln(x_i)^2u_i\left[1-\widehat{\alpha}x_i^{\widehat{\beta}}u_iy_i\right],$$

$$L_{11}=-\frac{\partial^2\ell}{\partial\alpha\partial\beta}=\sum_{i=1}^{D^*}x_i^{\widehat{\beta}}\ln(x_i)-R_t^*T^{\widehat{\beta}}u_t\ln(T)\left[1-\widehat{\alpha}T^{\widehat{\beta}}u_ty_t\right]$$
$$-\sum_{i=1}^{D^*}R_i^*x_i^{\widehat{\beta}}u_i\ln(x_i)\left[1-\widehat{\alpha}x_i^{\widehat{\beta}}u_iy_i\right],$$

$$L_{30}=\frac{\partial^3\ell}{\partial\alpha^3}=\frac{2D^*}{\widehat{\alpha}^3}-R_t^*T^{3\widehat{\beta}}u_t^2y_t\left[1-2u_iy_t\right]-\sum_{i=1}^{D^*}R_i^*x_i^{3\widehat{\beta}}u_i^2y_i\left[1-2u_iy_i\right],$$

$$L_{03}=\frac{\partial^3\ell}{\partial\beta^3}=\frac{2D^*}{\widehat{\beta}^3}-\widehat{\alpha}\sum_{i=1}^{D^*}x_i^{\widehat{\beta}}\ln(x_i)^3+\widehat{\alpha}\sum_{i=1}^{D^*}R_i^*\ln(x_i)^2$$
$$\times\left[x_i^{\widehat{\beta}}u_i\ln(x_i)-\left(1-\widehat{\alpha}x_i^{\widehat{\beta}}u_iy_i\right)-\widehat{\alpha}x_i^{2\widehat{\beta}}u_i^2y_i\ln x_i\left(2+\widehat{\alpha}x_i^{\widehat{\beta}}-2\widehat{\alpha}x_i^{\widehat{\beta}}u_iy_i\right)\right]+\widehat{\alpha}R_t^*\ln(T)^2$$
$$\times\left[T^{\widehat{\beta}}u_t\ln(T)-\left(1-\widehat{\alpha}T^{\widehat{\beta}}u_ty_t\right)-\widehat{\alpha}T^{2\widehat{\beta}}u_t^2y_t\ln(T)\left(2+\widehat{\alpha}T^{\widehat{\beta}}-2\widehat{\alpha}T^{\widehat{\beta}}u_ty_t\right)\right]$$

$$L_{21}=\frac{\partial^3\ell}{\partial\alpha^2\partial\beta}=-\sum_{i=1}^{D^*}R_i^*\left[x_i^{2\widehat{\beta}}u_i^2y_i\ln(x_i)\left(-2\widehat{\alpha}x_i^{\widehat{\beta}}u_iy_i+\widehat{\alpha}x_i^{\widehat{\beta}}+1\right)+x_i^{2\widehat{\beta}}\ln(x_i)u_i^2y_i\right]$$
$$-R_t^*\left[T^{2\widehat{\beta}}u_t^2y_t\ln(T)\left(-2\widehat{\alpha}T^{\widehat{\beta}}u_ty_t+\widehat{\alpha}T^{\widehat{\beta}}+1\right)+T^{2\widehat{\beta}}\ln(T)u_t^2y_t\right],$$

and

$$L_{12}=\frac{\partial^3\ell}{\partial\alpha\partial\beta^2}=-\sum_{i=1}^{D^*}x_i^{\widehat{\beta}}\ln(x_i)^2+\sum_{i=1}^{D^*}R_i^*\ln(x_i)^2\left[x_i^{\widehat{\beta}}u_i\ln(x_i)\left(1-2\widehat{\alpha}x_i^{\widehat{\beta}}u_iy_i\right)\right.$$
$$\left.-\widehat{\alpha}x_i^{2\widehat{\beta}}u_i^2y_i\left(2+\widehat{\alpha}x_i^{\widehat{\beta}}-2\widehat{\alpha}x_i^{\widehat{\beta}}u_iy_i\right)\right]-R_t^*\ln(T)^2\left[T^{\widehat{\beta}}u_t\ln(T)\right.$$
$$\left.\times\left(1-2\widehat{\alpha}T^{\widehat{\beta}}u_ty_t\right)-\widehat{\alpha}T^{2\widehat{\beta}}u_t^2y_t\left(2+\widehat{\alpha}T^{\widehat{\beta}}-2\widehat{\alpha}T^{\widehat{\beta}}u_ty_t\right)\right].$$

The elements of the variance covariance matrix σ_{ij} , can be obtained as

$$\sigma_{11} = \frac{L_{02}}{L_{02}L_{20} - L_{11}^2},\ \sigma_{22} = \frac{L_{20}}{L_{02}L_{20} - L_{11}^2}, \text{and } \sigma_{12} = \sigma_{21} = \frac{-L_{11}}{L_{02}L_{20} - L_{11}^2}.$$

Based on the joint prior function (17), we obtain

$$\rho(\alpha,\beta) = \ln \pi(\alpha,\beta) \propto (a_1 - 1)\ln\alpha + (a_2 - 1)\ln\beta - (b_1\alpha + b_2\beta),$$

hence

$$\rho_1 = \frac{\partial\rho(\alpha,\beta)}{\partial\alpha} = \frac{(a_1 - 1)}{\alpha} - b_1 \text{ and } \rho_2 = \frac{\partial\rho(\alpha,\beta)}{\partial\beta} = \frac{(a_2 - 1)}{\beta} - b_2.$$

Now, we derive the Bayes estimators for the unknown parameters (α,β), the survival and hazard functions under the square error loss function.

For Bayes estimators for the unknown parameter α, we set $G(\alpha,\beta) = \alpha$, then

$$g_1 = 1, g_2 = 0, \text{ and } g_{11} = g_{21} = g_{12} = g_{22} = 0.$$

Similarly, for Bayes estimator for the unknown parameter β, we set $G(\alpha,\beta) = \beta$, then

$$g_2 = 1, g_1 = 0 \text{ and } g_{11} = g_{21} = g_{12} = g_{22} = 0.$$

Proceeding similarly, the Bayes estimate for the reliability function by set $G(\alpha,\beta) = 1 - y$ where $y = \exp\left(-\alpha t^{-\beta}\right)$, then

$$g_1 = t^{-\widehat{\beta}}y,\ g_2 = -\widehat{\alpha}t^{-\widehat{\beta}}\ln(t)y,\ g_{11} = -t^{-2\widehat{\beta}}y, g_{22} = \widehat{\alpha}\ln(t)^2 t^{-\widehat{\beta}}y\left[1 - \widehat{\alpha}t^{-\widehat{\beta}}\right],$$

and

$$g_{12} = g_{21} = x^{-\widehat{\beta}}\ln(t)y\left[\widehat{\alpha}t^{-\widehat{\beta}} - 1\right].$$

The Bayes estimate for the reliability function by set $G(\alpha,\beta) = \frac{\alpha\beta t^{-(\beta+1)}}{1-y}$, then

$$g_1 = -\frac{\widehat{\alpha}\widehat{\beta}t^{-2\widehat{\beta}-1}y}{(1-y)^2}, g_2 = \frac{\widehat{\alpha}^2\widehat{\beta}t^{-2\widehat{\beta}-1}\ln(t)y}{(1-y)^3} - \frac{\widehat{\alpha}\widehat{\beta}t^{-\widehat{\beta}-1}\ln(t)}{(1-y)},$$

$$g_{11} = \frac{2\widehat{\alpha}\widehat{\beta}t^{-3\widehat{\beta}-1}y^2}{(1-y)^3} + \frac{\widehat{\alpha}\widehat{\beta}t^{-3\widehat{\beta}-1}y}{(1-y)^2},$$

$$g_{12} = g_{21} = \frac{2\widehat{\alpha}\widehat{\beta}t^{-2\widehat{\beta}-1}\ln(t)y}{(1-y)^2} - \frac{2\widehat{\alpha}^2\widehat{\beta}t^{-3\widehat{\beta}-1}\ln(t)y^2}{(1-y)^3} - \frac{\widehat{\alpha}^2\widehat{\beta}t^{-3\widehat{\beta}-1}\ln(t)y}{(1-y)^2},$$

and

$$g_{22} = \frac{\widehat{\alpha}\widehat{\beta}t^{-\widehat{\beta}-1}\ln(t)^2}{(1-y)} - \frac{3\widehat{\alpha}^2\widehat{\beta}t^{-2\widehat{\beta}-1}\ln(t)^2 y}{(1-y)^2} + \frac{2\widehat{\alpha}^3\widehat{\beta}t^{-3\beta-1}\ln(t)^2 y^2}{(1-y)^3} + \frac{\widehat{\alpha}^3\widehat{\beta}t^{-3\widehat{\beta}-1}\ln(t)^2 y}{(1-y)^3}.$$

5 Numerical Results

5.1 Monte Carlo Simulation

In this section, a Monte Carlo simulation study is carried out to compare the performance of the ML and the Bayesian estimates under different sampling schemes. We used different values for n, m, k and T to generate 1000 generalized progressive hybrid censored samples from the Pareto distribution (with $\alpha = 0.5$ and $\beta = 2$). For comparison, we computed the estimated risk (ER) for each estimate by using the root mean square error and also computed the estimated bias (EB) for each estimate. Tables 1-4 present the values of EB and ER of the ML and Bayesian estimates for α, β, $S(t = 0.5)$ and $H(t = 0.5)$, respectively.

We perform a Monte Carlo Simulation study using different sample sizes (n), different effective samples sizes (m,k) and the following two censoring schemes

1.Scheme 1: $R_i = \frac{2(n-m)}{m}$ if i is odd and $R_i = 0$ if i is even.
2.Scheme 2: $R_i = \frac{2(n-m)}{m}$ if i is even and $R_i = 0$ if i is odd.

All Bayesian results are computed based on two different choices of the hyperparameters (a_1, b_1, a_2, b_2), namely,

1.Informative prior (IP) : $a_1 = 3$, $b_1 = 4$, $a_2 = 4$ and $b_2 = 2$.
2.Noninformative prior (NIP) : $a_1 = 0$, $b_1 = 0$, $a_2 = 0$ and $b_2 = 0$.

Table 1: The values of EB and ER of the ML and Bayesian estimates for α.

				$\widehat{\alpha}_{ML}$		$\widehat{\alpha}_B$			
						IP		NIP	
n	m	k	$Scheme$	ER	EB	ER	EB	ER	EB
				$T = 0.500$					
15	10	7	1	0.94829	0.26655	0.75507	0.23397	0.79968	0.24283
			2	0.91334	0.24646	0.69048	0.23405	0.72675	0.23542
20	10	7	1	0.88054	0.22858	0.64527	0.22426	0.66924	0.22568
			2	1.00065	0.29804	0.84126	0.26682	0.90407	0.29466
25	10	7	1	0.98846	0.28963	0.80330	0.24802	0.86170	0.26939
			2	0.94137	0.25981	0.63573	0.21729	0.64758	0.21933
30	20	15	1	0.97958	0.28358	0.86236	0.25268	0.90047	0.26976
			2	0.96395	0.27355	0.77990	0.21935	0.81422	0.22920
40	20	15	1	0.91260	0.24352	0.64196	0.20948	0.66710	0.20969
			2	0.94416	0.26252	0.78621	0.21590	0.80913	0.22202
50	20	15	1	0.90507	0.24059	0.71223	0.21251	0.73047	0.21411
			2	0.87371	0.22444	0.66507	0.20629	0.68166	0.20660
				$T = 1.500$					
15	10	7	1	0.76801	0.10630	0.57479	0.07372	0.61940	0.08258
			2	0.73306	0.08622	0.51020	0.07380	0.54647	0.07518
20	10	7	1	0.70026	0.06834	0.46499	0.06503	0.48896	0.06543
			2	0.82037	0.13779	0.66098	0.10658	0.72379	0.13441
25	10	7	1	0.80818	0.12938	0.62302	0.08778	0.68142	0.10914
			2	0.76109	0.09956	0.45570	0.05807	0.46730	0.05908
30	20	15	1	0.79930	0.12333	0.68208	0.09243	0.72019	0.10951
			2	0.78368	0.11330	0.59963	0.05910	0.63394	0.06895
40	20	15	1	0.73232	0.08327	0.46168	0.04923	0.48682	0.04944
			2	0.76388	0.10227	0.60593	0.05565	0.62885	0.06177
50	20	15	1	0.72479	0.08034	0.53195	0.05226	0.55019	0.05386
			2	0.69343	0.06419	0.48479	0.04604	0.50138	0.04635
				$T = \infty$					
15	10	7	1	0.54869	0.09657	0.35548	0.06399	0.40009	0.07285
			2	0.51375	0.07649	0.29089	0.06407	0.32716	0.06545
20	10	7	1	0.48095	0.05861	0.24568	0.05477	0.26965	0.05570
			2	0.60106	0.12806	0.44167	0.09685	0.50448	0.12468
25	10	7	1	0.58887	0.11965	0.40370	0.07805	0.46211	0.09941
			2	0.54178	0.08983	0.23688	0.04918	0.24799	0.04935
30	20	15	1	0.57999	0.11360	0.46277	0.08270	0.50088	0.09978
			2	0.56436	0.10357	0.38031	0.04937	0.41463	0.05922
40	20	15	1	0.51301	0.07354	0.24237	0.03950	0.26751	0.03971
			2	0.54457	0.09254	0.38662	0.04592	0.40954	0.05204
50	20	15	1	0.50548	0.07061	0.31264	0.04253	0.33088	0.04413
			2	0.47412	0.05446	0.26548	0.03631	0.28207	0.03662

5.2 Conclusions and discussion

From Tables 1-4, it can be seen that the performance of the ML estimators is quite close to that of the Bayesian estimators based noninformative priors, as expected. Thus, if we have no prior information on the unknown parameters, then it is

Table 2: The values of EB and ER of the ML and Bayesian estimates for β.

						$\widehat{\beta}_B$			
				$\widehat{\beta}_{ML}$		*IP*		*NIP*	
n	*m*	*k*	*Scheme*	*ER*	*EB*	*ER*	*EB*	*ER*	*EB*
				$T = 0.500$					
15	10	7	1	1.26977	0.04180	0.93262	0.02840	0.93414	0.02970
			2	1.31342	0.04879	0.94547	0.03049	0.97035	0.03227
20	10	7	1	1.35665	0.05733	1.01362	0.03733	1.03541	0.04021
			2	1.33986	0.05343	0.94663	0.02534	0.97446	0.02582
25	10	7	1	1.36763	0.05934	0.93430	0.02576	0.96166	0.02644
			2	1.42906	0.07377	1.05722	0.04331	1.08147	0.04707
30	20	15	1	1.32749	0.05076	1.00009	0.01964	1.01873	0.01993
			2	1.36101	0.05775	0.96208	0.01911	0.97948	0.01915
40	20	15	1	1.41346	0.06980	0.97042	0.02750	0.98417	0.02847
			2	1.28968	0.04384	0.98345	0.01720	0.99641	0.01741
50	20	15	1	1.32850	0.05099	0.94583	0.01973	0.95731	0.01989
			2	1.37070	0.05984	0.93249	0.02042	0.94183	0.02075
				$T = 1.500$					
15	10	7	1	1.22109	0.03472	0.88394	0.02131	0.88547	0.02262
			2	1.26474	0.04171	0.89679	0.02340	0.92167	0.02519
20	10	7	1	1.30797	0.05025	0.96494	0.03025	0.98673	0.03312
			2	1.29119	0.04635	0.89795	0.01825	0.92579	0.01874
25	10	7	1	1.31895	0.05226	0.88562	0.01867	0.91298	0.01936
			2	1.38038	0.06669	1.00854	0.03623	1.03280	0.03999
30	20	15	1	1.27881	0.04368	0.95141	0.01256	0.97005	0.01285
			2	1.31234	0.05067	0.91340	0.01203	0.93080	0.01207
40	20	15	1	1.36478	0.06272	0.92174	0.02042	0.93549	0.02138
			2	1.24100	0.03676	0.93477	0.01012	0.94774	0.01033
50	20	15	1	1.27982	0.04391	0.89716	0.01265	0.90863	0.01281
			2	1.32202	0.05276	0.88381	0.01334	0.89315	0.01367
				$T = \infty$					
15	10	7	1	0.93115	0.02557	0.59400	0.01217	0.59552	0.01347
			2	0.97480	0.03256	0.60684	0.01426	0.63173	0.01605
20	10	7	1	1.01803	0.04111	0.67500	0.02111	0.69679	0.02398
			2	1.00124	0.03720	0.60800	0.00911	0.63584	0.00959
25	10	7	1	1.02901	0.04311	0.59567	0.00953	0.62303	0.01021
			2	1.09044	0.05755	0.71860	0.02708	0.74285	0.03085
30	20	15	1	0.98887	0.03453	0.66146	0.00341	0.68010	0.00371
			2	1.02239	0.04153	0.62346	0.00288	0.64085	0.00292
40	20	15	1	1.07483	0.05357	0.63180	0.01127	0.64554	0.01224
			2	0.95106	0.02762	0.64482	0.00097	0.65779	0.00118
50	20	15	1	0.98987	0.03477	0.60721	0.00350	0.61869	0.00366
			2	1.03208	0.04362	0.59387	0.00419	0.60321	0.00452

always better to use the ML rather than the Bayesian estimators, because the Bayesian estimators are computationally more expensive. Also, the Bayesian method with informative priors is the best method for estimation under all different censoring schemes. Moreover, mean-squared error decreases when *n* and *m* increase.

Table 3: The values of EB and ER of the ML and Bayesian estimates for $S(t = 0.5)$.

						$\widehat{S}_B(t)$			
				$\widehat{S}_{ML}(t)$		IP		NIP	
n	m	k	$Scheme$	ER	EB	ER	EB	ER	EB
				$T = 0.500$					
15	10	7	1	0.00103	0.00048	0.00052	0.00046	0.00103	0.00103
			2	0.00202	0.00123	0.00126	0.00123	0.00202	0.00202
20	10	7	1	0.00823	0.00115	0.00606	0.00114	0.00823	0.00823
			2	0.00904	0.00457	0.00615	0.00057	0.00904	0.00904
25	10	7	1	0.00051	0.00044	0.00042	0.00043	0.00051	0.00051
			2	0.00263	0.00101	0.00205	0.00099	0.00263	0.00263
30	20	15	1	0.00467	0.00026	0.00422	0.00026	0.00467	0.00467
			2	0.00150	0.00044	0.00110	0.00044	0.00150	0.00150
40	20	15	1	0.00473	0.00046	0.00375	0.00045	0.00473	0.00473
			2	0.00104	0.00045	0.00048	0.00044	0.00104	0.00104
50	20	15	1	0.00053	0.00033	0.00039	0.00032	0.00053	0.00053
			2	0.00056	0.00031	0.00036	0.00031	0.00056	0.00056
				$T = 1.500$					
15	10	7	1	0.00052	0.00062	0.00040	0.00058	0.00052	0.00052
			2	0.00132	0.00118	0.00097	0.00116	0.00116	0.00116
20	10	7	1	0.00154	0.00520	0.00445	0.00066	0.00543	0.00635
			2	0.00215	0.00042	0.00125	0.00040	0.00215	0.00215
25	10	7	1	0.00320	0.00031	0.00293	0.00030	0.00312	0.00312
			2	0.00056	0.00059	0.00036	0.00057	0.00056	0.00056
30	20	15	1	0.00066	0.00128	0.00033	0.00128	0.00066	0.00066
			2	0.00224	0.00082	0.00118	0.00079	0.00224	0.00224
40	20	15	1	0.01017	0.00066	0.00736	0.00064	0.01017	0.01017
			2	0.00112	0.00128	0.00030	0.00115	0.00030	0.00030
50	20	15	1	0.00364	0.00030	0.00287	0.00029	0.00364	0.00364
			2	0.00103	0.00161	0.00056	0.00179	0.00056	0.00056
				$T = \infty$					
15	10	7	1	0.03853	0.01534	0.00096	0.07431	0.00096	0.00096
			2	0.03746	0.01503	0.00099	0.07080	0.00099	0.00099
20	10	7	1	0.03295	0.01459	0.00112	0.06693	0.00112	0.00112
			2	0.03299	0.01444	0.00114	0.06597	0.00114	0.00114
25	10	7	1	0.03405	0.01430	0.00120	0.06398	0.00120	0.00120
			2	0.00751	0.00730	0.00126	0.02240	0.00126	0.00126
30	20	15	1	0.00985	0.00668	0.00096	0.02128	0.00108	0.00108
			2	0.01191	0.00610	0.00041	0.02049	0.00077	0.00077
40	20	15	1	0.01260	0.00587	0.00023	0.02031	0.00063	0.00063
			2	0.01317	0.00548	0.00003	0.01968	0.00043	0.00043
50	20	15	1	0.00236	0.00186	0.00113	0.00808	0.00236	0.00236
			2	0.00228	0.00155	0.00090	0.00725	0.00200	0.00200

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Table 4: The values of EB and ER of the ML and Bayesian estimates for $H(t=0.5)$.

				$\widehat{H}_{ML}(t)$		$\widehat{H}_B(t)$ IP		$\widehat{H}_B(t)$ NIP	
n	m	k	*Scheme*	ER	EB	ER	EB	ER	EB
				$T=0.500$					
15	10	7	1	0.08357	0.00323	0.08276	0.00293	0.08303	0.00316
			2	0.08348	0.00298	0.08267	0.00298	0.08292	0.00320
20	10	7	1	0.08342	0.00301	0.08267	0.00301	0.08284	0.00321
			2	0.08330	0.00303	0.08260	0.00303	0.08272	0.00322
25	10	7	1	0.08328	0.00304	0.08264	0.00304	0.08291	0.00322
			2	0.08327	0.00305	0.08259	0.00305	0.08285	0.00323
30	20	15	1	0.08300	0.00316	0.08243	0.00316	0.08273	0.00331
			2	0.08316	0.00310	0.08253	0.00310	0.08286	0.00327
40	20	15	1	0.08328	0.00308	0.08267	0.00308	0.08295	0.00324
			2	0.08298	0.00316	0.08247	0.00316	0.08269	0.00330
50	20	15	1	0.08317	0.00312	0.08263	0.00312	0.08294	0.00327
			2	0.08327	0.00309	0.08277	0.00309	0.08305	0.00323
				$T=1.500$					
15	10	7	1	0.07522	0.00290	0.07449	0.00264	0.07473	0.00285
			2	0.07514	0.00268	0.07440	0.00268	0.07463	0.00288
20	10	7	1	0.07508	0.00271	0.07441	0.00271	0.07455	0.00289
			2	0.07497	0.00273	0.07434	0.00273	0.07445	0.00290
25	10	7	1	0.07495	0.00274	0.07437	0.00274	0.07462	0.00290
			2	0.07494	0.00275	0.07433	0.00275	0.07457	0.00291
30	20	15	1	0.07470	0.00284	0.07419	0.00284	0.07446	0.00298
			2	0.07484	0.00279	0.07428	0.00279	0.07458	0.00294
40	20	15	1	0.07495	0.00277	0.07440	0.00277	0.07466	0.00292
			2	0.07468	0.00285	0.07422	0.00285	0.07442	0.00297
50	20	15	1	0.07485	0.00281	0.07437	0.00281	0.07464	0.00294
			2	0.07494	0.00278	0.07450	0.00278	0.07475	0.00290
				$T=\infty$					
15	10	7	1	0.06393	0.00247	0.06331	0.00224	0.06352	0.00242
			2	0.06387	0.00228	0.06324	0.00228	0.06343	0.00245
20	10	7	1	0.06382	0.00230	0.06325	0.00230	0.06337	0.00246
			2	0.06373	0.00232	0.06319	0.00232	0.06328	0.00246
25	10	7	1	0.06371	0.00233	0.06322	0.00233	0.06342	0.00246
			2	0.06370	0.00234	0.06318	0.00234	0.06338	0.00247
30	20	15	1	0.06349	0.00241	0.06306	0.00241	0.06329	0.00253
			2	0.06362	0.00237	0.06314	0.00237	0.06339	0.00250
40	20	15	1	0.06371	0.00236	0.06324	0.00236	0.06346	0.00248
			2	0.06348	0.00242	0.06309	0.00242	0.06326	0.00253
50	20	15	1	0.06362	0.00238	0.06321	0.00238	0.06345	0.00250
			2	0.06370	0.00236	0.06332	0.00236	0.06353	0.00247

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