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Existence of Solutions to a Three-point Entrainment of Frequency Problem

Mesliza Mohamed[1,*] *and Bevan Thompson*[2]

[1] Fakulti Sains Komputer dan Matematik, Universiti Teknologi MARA Cawangan Pahang, 26400 Bandar Tun Abdul Razak, Jengka, Pahang, Malaysia
[2] Department of Mathematics,The University of Queensland, Queensland 4072, Australia

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Abstract: In this paper, we deal with the existence of solutions to the frequency problem of a perturbed system, $x' - A(t)x = \varepsilon f(x, \sin wt, \cos wt, \varepsilon)$ with three-point boundary condition. The topological technique is used to obtain existence theorem. Two examples are given to illustrate our results.

Keywords: Boundary value problems, first-order, local degree.

1 Introduction

Consider the three-point (boundary value problem) BVP

$$x' - A(t)x = \varepsilon f(x, \sin wt, \cos wt, \varepsilon), 0 \le t \le 1, \tag{1}$$

$$Mx(0) + Nx(\eta) + Rx(1) = \ell, \tag{2}$$

where M, N and R are constant square matrices of order n, $A(t)$ is an $n \times n$ matrix with continuous entries, $f : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ is a continuous function and $\varepsilon \in \mathbb{R}$ such that $|\varepsilon| < \varepsilon_0$, $\ell \in \mathbb{R}^n$, $\eta \in (0,1)$, and w is a function of ε such that $w = (1+\varepsilon\mu(\varepsilon))^{-1}$, $\mu(\varepsilon)$ is a twice differentiable function of ε such that $\mu(0) = \mu_0 \neq 0$. Let $\tau = wt = (1+\varepsilon\mu(\varepsilon))^{-1}t$ and $\frac{d\tau}{dt} = (1+\varepsilon\mu(\varepsilon))^{-1}$.

The existence of solutions to two-point, three-point, four-point or multipoint BVPs for ODEs at resonance have been studied by a number of authors (see, for example [4], [12] [46], [47], [14], [15], [16], [17], [18], [23], [24], [25], [26], [27], [44], [36], [19], [20], [21], [31], [40], [43], [45], [48]). A great amount of work has been completed on the existence of solutions to BVPs for nonlinear systems of first-order ODEs at resonance which involve a small parameter (see, for example [5], [29], [30] and [41]). The resonance case for systems of first-order difference and differential equations has been considered by several authors (see for example Agarwal [1], Agarwal and O'Regan [2], Agarwal and Sambandham [3], Etheridge and Rodriguez [13], Rodriguez [37,38,39] and [42]). In these cases, resonance happens where the associated linear homogeneous BVP admits nontrivial solutions.

Recently, Mohamed et al. [32,34] established the existence of solutions at resonance for the following nonlinear boundary conditions

$$x' - A(t)x = H(t,x,\varepsilon) = \varepsilon F(t,x,\varepsilon) + E(t),\ 0 \le t \le 1, \tag{3}$$

$$Mx(0) + Nx(\eta) + Rx(1) = \ell + \varepsilon g(x(0), x(\eta), x(1)), \tag{4}$$

where M, N and R are constant square matrices of order n, $A(t)$ is an $n \times n$ matrix with continuous entries, $E : [0,1] \to \mathbb{R}$ is continuous, $F : [0,1] \times \mathbb{R}^n \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ is a continuous function where $\varepsilon_0 > 0$, $\ell \in \mathbb{R}^n$, $\eta \in (0,1)$ and $g : \mathbb{R}^{3n} \to \mathbb{R}^n$ is continuous. In [32], they applied a version of Brouwer's fixed point theorem which is due to Miranda (see Piccinini, Stampacchia and Vidossich [35]) to prove the existence of solutions to (3), (4). In [34], by employing the implicit function theorem sufficient conditions for the existence of solutions to (3), (4) are established.

In addition, Mohamed et al. [33] used the Theorem of Borsuk to show the existence of solutions to (3) with boundary condition

$$Mx(0) + Nx(\eta) + Rx(1) = 0. \tag{5}$$

* Corresponding author e-mail: mesliza@pahang.uitm.edu.my

A result for computing the local degree of polynomials whose terms of highest order have no common linear factors is also presented.

In [9,10], Cronin gives examples of the entrainment of frequency problem for a singularly perturbed system for the case of periodic boundary conditions. In this paper, we establish analogues of these results for three-point and two-point BVPs adapting the approach of Cronin [6,7].

By the chain rule, $\frac{dx}{d\tau}=\frac{dx}{dt}\frac{dt}{d\tau}$, the system (1) becomes

$$\frac{dx}{d\tau}=(1+\varepsilon\mu(\varepsilon))A(t)x+(1+\varepsilon\mu(\varepsilon))\varepsilon f(x,\sin\tau,\cos\tau,\varepsilon)$$

or

$$\frac{dx}{d\tau}=A(t)x+\varepsilon\Big[\mu(\varepsilon)A(t)x+(1+\varepsilon\mu(\varepsilon))f(x,\sin\tau,\cos\tau,\varepsilon)\Big].$$

Let

$$\begin{aligned}&F(x,\sin\tau,\cos\tau,\varepsilon)\\ &=\mu(\varepsilon)A(t)x+(1+\varepsilon\mu(\varepsilon))f(x,\sin\tau,\cos\tau,\varepsilon).\end{aligned}$$

Then the system (1) becomes

$$\frac{dx}{d\tau}=A(t)x+\varepsilon F(x,\sin\tau,\cos\tau,\varepsilon). \tag{6}$$

We assume the following:

Assumption (D1). Let $f(x,\sin\tau,\cos\tau,\varepsilon)=h(x,\varepsilon)+k(\sin\tau,\cos\tau,\varepsilon)$.

Our goal is to solve the following problem: Let $\overline{x}(t)$ be the solution of the BVP

$$\frac{dx}{d\tau}=A(t)x \tag{7}$$

subject to boundary conditions (2). The problem is to show that for $\varepsilon>0$, ε sufficiently small, there exists a solution of the BVP (6), (2) close to $\overline{x}(t)$ where $\overline{x}(t)$ is the solution of (7), (2). This is called the resonance or entrainment of frequency problem. The entrainment of frequency has been observed in physical systems for centuries and has been studied mathematically for over a hundred years. The phenomenon can be described as follows: ‘Suppose a periodic force is impressed upon a physical system which has a natural frequency of oscillation and that the natural frequency is different from the frequency of the impresses force. If the system then oscillates with the frequency impresses force, we say that the entrainment occurs ’(see [9]). For example, the entrainment of frequency occurs in a model of cardiac Punkinje fiber given by Cronin [9].

2 Preliminary Results

We recall the following results of [32].

Lemma 1. *Consider the system*

$$x'=A(t)x \tag{8}$$

where $A(t)$ is an $n\times n$ matrix with continuous entries on the interval [0,1]. Let $Y(t)$ be a fundamental matrix of (8). Then the solution of (8) which satisfies the initial condition

$$x(0)=c \tag{9}$$

is $x(t)=Y(t)Y^{-1}(0)c$ where c is a constant n-vector. Abbreviate $Y(t)Y^{-1}(0)$ to $Y_0(t)$. Thus $x(t)=Y_0(t)c$.

Lemma 2. *Using Lemma 1, let $x(\tau)=Y_0(\tau)c$ be a solution of $\frac{dx}{d\tau}=A(t)x$. Then any solution of (6) can be written as*

$$x(\tau,c,\varepsilon)=Y_0(\tau)c \tag{10}$$

$$+\int_0^\tau Y(ws)Y^{-1}(ws)\varepsilon F(x,\sin ws,\cos ws,\varepsilon)ds. \tag{11}$$

The solution (10) satisfies the boundary conditions (2) if and only if

$$\begin{aligned}&Mc+N\Big(Y_0(\eta)c+\int_0^\eta Y(\eta)Y^{-1}(s)\varepsilon F(x,\sin ws,\cos ws,\varepsilon)ds\Big)\\ &+R\Big(Y_0(1)c+\int_0^1 Y(1)Y^{-1}(s)\varepsilon F(x,\sin ws,\cos ws,\varepsilon)ds\Big)=\ell,\end{aligned}$$

or equivalently

$$\mathscr{L}c=\varepsilon\mathscr{N}(c,\alpha,\eta,\varepsilon)+d, \tag{12}$$

where

$$\begin{aligned}\mathscr{L}&=M+NY_0(\eta)+RY_0(1),\\ \mathscr{N}(c,\alpha,\eta,\varepsilon)&=-\Big(\int_0^\eta NY(\eta)Y^{-1}(s)F(x,\sin ws,\cos ws,\varepsilon)ds\\ &\quad+\int_0^1 RY(1)Y^{-1}(s)F(x,\sin ws,\cos ws,\varepsilon)ds\Big),\\ d&=\ell,\end{aligned}$$

and $x(t,c,\varepsilon)$ is the solution of (6) given $x(0)=c$.

Thus (12) is a system of n real equations in $\varepsilon,c_1,\cdots,c_n$ where $c_1,\cdots,c_n$ are components of c. The system (12) is sometimes called the branching equations.

Remark.[32] Let $x_1,\cdots,x_n$ be a basis for $\mathbb{R}^n$ such that $x_1,\cdots,x_r$ is a basis for E_r, and $x_{r+1},\cdots,x_n$ a basis for E_{n-r}. Let P_r be the matrix projection onto Ker $\mathscr{L}=E_r$, and $P_{n-r}=I-P_r$, where I is the identity matrix. Thus P_{n-r} is a projection onto the complementary space E_{n-r} of E_r, and

$$P_r^2=P_r,\ P_{n-r}^2=P_{n-r} \text{ and } P_{n-r}P_r=P_rP_{n-r}=0.$$

Without loss of generality, we may assume

$$P_rc=(c_1,\cdots,c_r,0,\cdots,0) \text{ and } P_{n-r}c=(0,\cdots,0,c_{r+1},\cdots,c_n).$$

Let H be a nonsingular $n\times n$ matrix satisfying

$$H\mathscr{L}=P_{n-r}. \tag{13}$$

Matrix H can be computed easily (see Cronin [7]). The nature of the solutions of the branching equations depends heavily on the rank of the matrix $\mathscr{L}$.

3 Existence Results

We need to solve (12) for c when ε is sufficiently small. The problem of finding solutions to (6), (2) has become the problem of solving the branching equation (12) for c. So consider (12), which is equivalent to

$$\mathscr{L}(P_r+P_{n-r})c=\varepsilon\mathscr{N}((P_r+P_{n-r})c,\alpha,\eta,\varepsilon)+d. \quad (14)$$

Multiplying (14) by the matrix H and using (13), we have

$$P_{n-r}c=\varepsilon H\mathscr{N}((P_r+P_{n-r})c,\alpha,\eta,\varepsilon)+Hd \quad (15)$$

where

$$H\mathscr{N}((P_r+P_{n-r})c,\alpha,\eta,\varepsilon))=$$
$$-H\Big(\int_0^\eta NY(\eta)Y^{-1}(s)F(x,\sin ws,\cos ws,\varepsilon)ds$$
$$+\int_0^1 RY(1)Y^{-1}(s)F(x,\sin ws,\cos ws,\varepsilon)ds\Big),$$

and $Hd=H\ell$.

The following theorem gives a necessary condition for the existence of solutions to the BVP (6) and (2).

Theorem 1. *A necessary condition that (15) can be solved for c, with $|\varepsilon|<\varepsilon_0$, for some $\varepsilon_0>0$ is $P_rHd=0$.*

Proof The proof is very similar to that of Theorem 1 of [32] and is omitted.

Definition 1. *[32] Let E_r denote the null space of $\mathscr{L}$ and let E_{n-r} denote the complement in $\mathbb{R}^n$ of E_r. Let P_r be the matrix projection onto Ker $\mathscr{L}=E_r$, and $P_{n-r}=I-P_r$, where I is the identity matrix. Thus P_{n-r} is a projection onto the complementary space E_{n-r} of E_r. If E_{n-r} is properly contained in $\mathbb{R}^n$ then E_r is an $r-$dimensional vector space where $0<r<n$. If $c=(c_1,\cdots,c_n)$, let $c^r=(c_1,\cdots,c_r)$ and $c^{n-r}=(c_{r+1},\cdots,c_n)$, then define a continuous mapping $\Phi_\varepsilon:\mathbb{R}^r\to\mathbb{R}^r$, given by*

$$\Phi_\varepsilon(c_1,\cdots,c_r)=P_rH\mathscr{N}(c^r\oplus c^{n-r}(c^r,\varepsilon),\alpha,\eta,\varepsilon),$$

where where $c^{n-r}(c^r,\varepsilon)=c^{n-r}$ is a differentiable function of c^r and ε, $P_rH\mathscr{N}$ is interpreted as $(H\mathscr{N}_1,\cdots,H\mathscr{N}_r)$. Similarly we will sometimes identify $P_{n-r}c$ and c^{n-r}. Setting $\varepsilon=0$, we have

$$\Phi_0(c_1,\cdots,c_r)=P_rH\mathscr{N}(c^r\oplus P_{n-r}Hd,\alpha,\eta,0)$$

where $c^{n-r}(c^r,0)=P_{n-r}Hd$; note that from the context $c^{n-r}(c^r,0)=P_{n-r}Hd$ is interpreted as $c^{n-r}(c^r,0)=(Hd_{r+1},\cdots,Hd_n)$.

If $E_r=\mathbb{R}^n$ and $P_r=I$, then $P_{n-r}=0$. Since $P_{n-r}=0$ it follows that the matrix H is the identity matrix. Thus define a continuous mapping $\Phi_\varepsilon:\mathbb{R}^n\to\mathbb{R}^n$, given by

$$\Phi_\varepsilon(c)=\mathscr{N}(c,\alpha,\eta,\varepsilon).$$

Setting $\varepsilon=0$, we have

$$\Phi_0(c)=\mathscr{N}(c,\alpha,\eta,0).$$

Lemma 3. (see Cronin [7] p. 297) Let $P_rHd=0$. Suppose that

$$\Phi_0^1(c_1,c_2)=k_1c_1+k_2c_2+V_1(c_1,c_2)$$
$$\Phi_0^2(c_1,c_2)=k_3c_1+k_4c_2+V_2(c_1,c_2)$$

where $\det\begin{pmatrix}k_1&k_2\\k_3&k_4\end{pmatrix}\neq0$ and for $i=1,2$, the polynomial $V_i(c_1,c_2)$ consists of terms of the form $Kc_1^{q_1}c_2^{q_2}$ where $q_1+q_2\geq2$ and K is a constant. If B_k is a ball with centre at the origin and sufficiently small radius k, then

$$d(\Phi_0,B_k,0)=\text{sgn}\det\begin{pmatrix}k_1&k_2\\k_3&k_4\end{pmatrix}.$$

Proof Set $A=\begin{pmatrix}k_1&k_2\\k_3&k_4\end{pmatrix}$, and let $A=\sup_{|c|\leq1}|Ac|$. Since A is nonsingular it follows that there exist $l>0$ such that given $c=(c_1,c_2)$, we have $|Ac|\geq l|c|$. By assumption on V_i, for each $k>0$ there is a constant d_k such that

$$|V_i(c_1,c_2)|\leq d_k|c|^2,\ i=1,2,$$

for $c\in B_k$. Notice that for $c\in\partial B_k$ we have $|\Phi_0(c)|>0$. Thus we choose $k>0$ such that

$$\begin{aligned}|\Phi_0(c)|&\geq l|c|-|V(c_1,c_2)|\\&>l|c|-d_k|c|^2\\&=|c|(l-d_k|c|)\\&>0\end{aligned}$$

for $|c|<\min\{\frac{l}{d_k},1\}$. Define the homotopy

$$H(c,\lambda)=Ac+\lambda V(c_1,c_2),\ 0\leq\lambda\leq1.$$

Therefore if $c\in\partial B_k$ then $|H(c,\lambda)|>0$. Thus $0\notin H(\partial B_k,\lambda)$ for $0\leq\lambda\leq1$ and therefore

$$d(\Phi_0,B_k,0)=d(A,B_k,0)=\text{sgn}\ \det A\neq0.$$

Hence we conclude Lemma 3 with Theorem 2.

Theorem 2. (Compare with Theorem 6.12, p. 93 of Cronin [6]) If **Assumption (D1)** is satisfied, and $d(\Phi_0,B_k,0)$ is defined for B_k, a ball with centre at the origin and sufficiently small radius, then for all sufficiently small ε, the system (1) has at least one solution with the boundary conditions

$$\begin{aligned}&Mx(0,c_0,0)+Nx(\eta,c_0,0)+Rx(1,c_0,0)=Mx(0,c(\varepsilon),\varepsilon)\\&+Nx\Big(\tfrac{\eta}{1+\varepsilon\mu(\varepsilon)},c(\varepsilon),\varepsilon\Big)+Rx\Big(\tfrac{1}{1+\varepsilon\mu(\varepsilon)},c(\varepsilon),\varepsilon\Big)=\ell,\end{aligned} \quad (16)$$

where $c(0)=c_0$.

As a consequence of Theorem 3.8, p. 69 of Cronin [6], we have the following Theorem 3.

Theorem 3. *If $r=n$, a necessary condition in order that (12) has a solution c for each ε with $|\varepsilon|<\varepsilon_0$ is $\ell=0$.*

4 Application to Second-Order Equations

In this section we use results of Section 3 to find solutions to the three-point BVP

$$y'' + 16\pi^2 y = \varepsilon f(y, y', \sin 4\pi wt, \cos 4\pi wt, \varepsilon), 0 \le t \le 1, \tag{17}$$

$$2y(0) - y(1/2) - y(1) = 0,\ -y'(1/2) + y'(1) = 0, \tag{18}$$

as well as the two-point BVP

$$y'' + 4\pi^2 y = \varepsilon f(y, y', \sin 2\pi wt, \cos 2\pi wt, \varepsilon), 0 \le t \le 1, \tag{19}$$

$$y(0) = 0,\ -y'(0) + y'(1) = 0, \tag{20}$$

where $w = (1 + \varepsilon\mu(\varepsilon))^{-1}$, $\mu(\varepsilon)$ is a twice differentiable function of ε such that $\mu(0) = \mu_0$ and $f \in C(\mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \times (-\varepsilon_0, \varepsilon_0); \mathbb{R})$. We will use the following facts in solving the examples.

$$\int_0^{1/2} \sin^n 4\pi s \cos^m 4\pi s\, ds \ne 0,$$

$$\int_0^1 \sin^n 4\pi s \cos^m 4\pi s\, ds \ne 0 \tag{21}$$

if and only if both n and m are even.

$$\int_0^1 \sin^n 2\pi s \cos^m 2\pi s\, ds \ne 0 \tag{22}$$

if and only if both n and m are even.

4.1 A Three-point BVP

Consider (17), (18). Then (17) may be written as

$$\begin{aligned} \frac{dx_1}{dt} &= x_2 \\ \frac{dx_2}{dt} &= -16\pi^2 x_1 + \varepsilon f(x_1, x_2, \sin 4\pi wt, \cos 4\pi wt, \varepsilon). \end{aligned} \tag{23}$$

By the chain rule (23) becomes

$$\begin{aligned} \frac{dx_1}{d\tau} &= x_2 + \varepsilon\mu(\varepsilon) x_2 \\ \frac{dx_2}{d\tau} &= -16\pi^2 x_1 + \varepsilon\Big[-16\mu(\varepsilon)\pi^2 x_1 + \\ &\quad (1 + \varepsilon\mu(\varepsilon)) f(x_1, x_2, \sin 4\pi\tau, \cos 4\pi\tau, \varepsilon)\Big]. \end{aligned} \tag{24}$$

Writing (24), (18) into matrix form, we have

$$\begin{pmatrix} \frac{dx_1}{d\tau} \\ \frac{dx_2}{d\tau} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -16\pi^2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \varepsilon \begin{pmatrix} \mu(\varepsilon) x_2 \\ -16\mu(\varepsilon)\pi^2 x_1 (1 + \varepsilon\mu(\varepsilon)) f() \end{pmatrix} \tag{25}$$

where

$$f(\,) = f(x_1, x_2, \sin 4\pi\tau, \cos 4\pi\tau, \varepsilon).$$

and

$$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1(1/2) \\ x_2(1/2) \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1(1) \\ x_2(1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{26}$$

where $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $A = \begin{pmatrix} 0 & 1 \\ -16\pi^2 & 0 \end{pmatrix}$, $M = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$, $N = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $R = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, and

$$F(x_1, x_2, \sin 4\pi\tau, \cos 4\pi\tau, \varepsilon) = \begin{pmatrix} \mu(\varepsilon) x_2 \\ -16\mu(\varepsilon)\pi^2 x_1 + (1 + \varepsilon\mu(\varepsilon)) f(x_1, x_2, \sin 4\pi\tau, \cos 4\pi\tau, \varepsilon) \end{pmatrix}.$$

The fundamental matrix

$$Y(\tau) = e^{A\tau} = \begin{pmatrix} \cos 4\pi\tau & \sin 4\pi\tau/(4\pi) \\ -4\pi \sin 4\pi\tau & \cos 4\pi\tau \end{pmatrix},$$

$$Y^{-1}(\tau) = \begin{pmatrix} \cos 4\pi\tau & -\sin 4\pi\tau/(4\pi) \\ 4\pi \sin 4\pi\tau & \cos 4\pi\tau \end{pmatrix},$$

$$Y_0(\tau) = \begin{pmatrix} \cos 4\pi\tau & \sin 4\pi\tau/(4\pi) \\ -4\pi \sin 4\pi\tau & \cos 4\pi\tau \end{pmatrix},$$

$$Y_0(1/2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and $Y_0(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then by Lemma 2, solving the problem (25), (26) is reduced to that of solving $\mathscr{L}c = \varepsilon\mathscr{N}(c, \alpha, \eta, \varepsilon) + d$ for c. Thus we find $\mathscr{L}$ and $\mathscr{N}(c, \alpha, \eta, \varepsilon)$.

$$\begin{aligned} \mathscr{L} &= M + NY_0(1/2) + RY_0(1) \\ &= \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Rank $\mathscr{L} = 0$. Let $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, be a basis for $Ker(\mathscr{L})$, and $Ker(\mathscr{L}) =$ Span (e_1, e_2). Let P_1 be the matrix projection onto Ker($\mathscr{L}$), $P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. So $P_2 = I - P_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Set $H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ so that $H\mathscr{L} = P_2$ and

$$\begin{aligned}
&\mathcal{N}(c,\alpha,\eta,\varepsilon) = \\
&-\int_0^{1/2}\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}Y^{-1}(s)\times \\
&\begin{pmatrix}\mu(\varepsilon)x_2\\ (-16\mu(\varepsilon)\pi^2x_1+(1+\varepsilon\mu(\varepsilon))f())\end{pmatrix}ds \\
&-\int_0^{1}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}Y^{-1}(s)\times \\
&\begin{pmatrix}\mu(\varepsilon)x_2\\ (-16\mu(\varepsilon)\pi^2x_1+(1+\varepsilon\mu(\varepsilon))f())\end{pmatrix}ds \\
&= (\mathcal{N}_1(c,\alpha,\eta,\varepsilon),\mathcal{N}_2(c,\alpha,\eta,\varepsilon)),
\end{aligned}$$

where

$$\begin{aligned}
&\mathcal{N}_1(c,\alpha,\eta,\varepsilon) = \int_0^{1/2}\{\mu(\varepsilon)x_2(s,c,\varepsilon)\cos 4\pi s \\
&+16\pi^2\mu(\varepsilon)\frac{\sin 4\pi s}{4\pi}x_1(s,c,\varepsilon) \\
&-(1+\varepsilon\mu(\varepsilon))\frac{\sin 4\pi s}{4\pi}\Big[h(x_1(s,c,\varepsilon),x_2(s,c,\varepsilon),\varepsilon) \\
&k(\sin 4\pi s,\cos 4\pi s,\varepsilon)\Big]\}ds \\
&+\int_0^{1}\{\mu(\varepsilon)x_2(s,c,\varepsilon)\cos 4\pi s+16\pi^2\mu(\varepsilon)\frac{\sin 4\pi s}{4\pi}x_1(s,c,\varepsilon) \\
&-(1+\varepsilon\mu(\varepsilon))\frac{\sin 4\pi s}{4\pi}\Big[h(x_1(s,c,\varepsilon),x_2(s,c,\varepsilon),\varepsilon) \\
&+k(\sin 4\pi s,\cos 4\pi s,\varepsilon)\Big]\}ds,
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{N}_2(c,\alpha,\eta,\varepsilon) = -\int_{1/2}^{1}\{4\pi\mu(\varepsilon)x_2(s,c,\varepsilon)\sin 4\pi s \\
&-16\pi^2\mu(\varepsilon)x_1(s,c,\varepsilon)\cos 4\pi s \\
&+(1+\varepsilon\mu(\varepsilon))\cos\ 4\pi s\Big[h(x_1(s,c,\varepsilon),x_2(s,c,\varepsilon),\varepsilon)+ \\
&k(\sin 4\pi s,\cos 4\pi s,\varepsilon)\Big]\}ds,
\end{aligned}$$

and f is defined in **Assumption (D1)** as

$$\begin{aligned}
&f(\,) := f(x_1,x_2,\sin 4\pi s,\cos 4\pi s,\varepsilon) \\
&= h(x_1(s,c,\varepsilon),x_2(s,c,\varepsilon),\varepsilon)+k(\sin 4\pi s,\cos 4\pi s,\varepsilon).
\end{aligned}$$

Since $d=0$, it follows that $P_1Hd=0$. Thus a necessary condition of Theorem 1 is satisfied. We compute

$$\begin{aligned}
P_1H\mathcal{N}(c,\alpha,\eta,\varepsilon) &= \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\mathcal{N}_1(c,\alpha,\eta,\varepsilon)\\ \mathcal{N}_2(c,\alpha,\eta,\varepsilon)\end{pmatrix} \\
&= \begin{pmatrix}\mathcal{N}_1(c,\alpha,\eta,\varepsilon)\\ \mathcal{N}_2(c,\alpha,\eta,\varepsilon)\end{pmatrix}.
\end{aligned}$$

Thus

$$\Phi_\varepsilon(c_1,c_2) = \begin{pmatrix}\mathcal{N}_1(c,\alpha,\eta,\varepsilon)\\ \mathcal{N}_2(c,\alpha,\eta,\varepsilon)\end{pmatrix}.$$

Now $\tau = wt = (1+\varepsilon\mu(\varepsilon))^{-1}t$ where at $\varepsilon=0$, $wt=t$. Set $\varepsilon=0$, and $x_1(t,c,0) = c_1\cos 4\pi t + c_2\sin 4\pi t/(4\pi)$, $x_2(t,c,0) = -4\pi c_1\sin 4\pi t + c_2\cos 4\pi t$. Thus we obtain

$$\begin{aligned}
&\Phi_0^1(c_1,c_2) = \int_0^{1/2}\{\mu_0x_2(s,c,0)\cos 4\pi s \\
&+\frac{16\pi^2\mu_0\sin 4\pi s}{4\pi}x_1(s,c,0)-\frac{\sin\ 4\pi s}{4\pi}\times \\
&\Big[h(x_1(s,c,0),x_2(s,c,0))+k(\sin 4\pi s,\cos 4\pi s,0)\Big]\}ds \\
&+\int_0^{1}\{\mu_0x_2(s,c,0)+\frac{16\pi^2\mu_0\sin 4\pi s}{4\pi}x_1(s,c,0)-\frac{\sin\ 4\pi s}{4\pi}\times \\
&\Big[h(x_1(s,c,0),x_2(s,c,0))+k(\sin 4\pi s,\cos 4\pi s,0)\Big]\}ds \\
&= \int_0^{1/2}\{\mu_0c_1\sin 4\pi s\cos 4\pi s+\mu_0c_2\cos^2 4\pi s \\
&+4\pi c_1\mu_0\sin\ 4\pi s\cos 4\pi s+\mu_0c_2\sin^2 4\pi s-\frac{\sin\ 4\pi s}{4\pi}\times \\
&\Big[h(c_1\cos 4\pi s+\frac{c_2\sin 4\pi s}{4\pi}, \\
&-4\pi c_1\sin 4\pi s+c_2\cos 4\pi s)+k(\sin 4\pi s,\cos 4\pi s,0)\Big]\}ds \\
&+\int_0^{1}\{\mu_0c_1\sin 4\pi s\cos 4\pi s+\mu_0c_2\cos^2 4\pi s \\
&+4\pi c_1\mu_0\sin\ 4\pi s\cos 4\pi s+\mu_0c_2\sin^2 4\pi s \\
&-\frac{\sin\ 4\pi s}{4\pi}\Big[h(c_1\cos 4\pi s+ \\
&\frac{c_2\sin 4\pi s}{4\pi},-4\pi c_1\sin 4\pi s+c_2\cos 4\pi s) \\
&+k(\sin 4\pi s,\cos 4\pi s,0)\Big]\}ds \\
&= \frac{3\mu_0c_2}{2}+V_1(c_1,c_2),
\end{aligned}$$

$$\begin{aligned}
\Phi_0^2(c_1,c_2) = &-\int_{1/2}^{1}\{4\pi\mu_0(-4\pi c_1\sin 4\pi s+c_2\cos 4\pi s)\sin 4\pi s \\
&-16\pi^2\mu_0(c_1\cos 4\pi s+\frac{c_2\sin 4\pi s}{4\pi})\cos 4\pi s \\
&+\cos 4\pi s\Big[h(c_1\cos 4\pi s+\frac{c_2\sin 4\pi s}{4\pi}, \\
&-4\pi c_1\sin 4\pi s+c_2\cos 4\pi s) \\
&+k(\sin 4\pi s,\cos 4\pi s,0)\Big]\}ds \\
&= 8\pi^2\mu_0c_1+V_2(c_1,c_2),
\end{aligned}$$

where for $i=1,2$ the polynomial $V_i(c_1,c_2)$ consists of terms of the form $Kc_1^{q_1}c_2^{q_2}$ where $q_1+q_2\geq 2$ and K is a constant. Now we apply Lemma 3. Since

$$\det\begin{pmatrix}0 & \frac{3\mu_0}{2}\\ 8\pi^2\mu_0 & 0\end{pmatrix}\neq 0,$$

it follows that $d(\Phi_0,B_k,0)\neq 0$. Hence for ε sufficiently small, we conclude that by Theorem 2 the problem (23) has at least one solution with the boundary condition

$$\begin{pmatrix}2&0\\0&0\end{pmatrix}\begin{pmatrix}x_1(0,c_0,0)\\x_2(0,c_0,0)\end{pmatrix}+\begin{pmatrix}-1&0\\0&-1\end{pmatrix}\begin{pmatrix}x_1(1/2,c_0,0)\\x_2(1/2,c_0,0)\end{pmatrix}+ \begin{pmatrix}-1&0\\0&1\end{pmatrix}\begin{pmatrix}x_1(1,c_0,0)\\x_2(1,c_0,0)\end{pmatrix}=\begin{pmatrix}2&0\\0&0\end{pmatrix}\begin{pmatrix}x_1(0,c(\varepsilon),\varepsilon)\\x_2(0,c(\varepsilon),\varepsilon)\end{pmatrix}+ \begin{pmatrix}-1&0\\0&-1\end{pmatrix}\begin{pmatrix}x_1(\frac{1}{2(1+\varepsilon\mu(\varepsilon))},c(\varepsilon),\varepsilon)\\x_2(\frac{1}{2(1+\varepsilon\mu(\varepsilon))},c(\varepsilon),\varepsilon)\end{pmatrix} +\begin{pmatrix}-1&0\\0&1\end{pmatrix}\begin{pmatrix}x_1(\frac{1}{1+\varepsilon\mu(\varepsilon)},c(\varepsilon),\varepsilon)\\x_2(\frac{1}{1+\varepsilon\mu(\varepsilon)},c(\varepsilon),\varepsilon)\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix},$$

where $c(0)=c_0$.

Does the problem have nontrivial solutions?

Example 1: An example with local degree zero

In system (25), let $f(x_1,x_2,\sin 4\pi\tau,\cos 4\pi\tau,\varepsilon) = x_1^2\cos 4\pi\tau + x_2^2\sin 4\pi\tau + \gamma\cos 4\pi\tau$. Note also that $\sin 4\pi\tau = \sin 4\pi(1+\varepsilon\eta(\varepsilon))^{-1}t$ at $\varepsilon=0$ is given by $\sin 4\pi t$. **Assumption (D1)** is satisfied since we have

$$\begin{aligned}&h(x_1(s,c,0),x_2(s,c,0),0)+k(\sin 4\pi t,\cos 4\pi t,0)\\&=c_1^2\cos^3 4\pi t+(\frac{c_1c_2}{2\pi}+c_2^2)\cos^2 4\pi t\sin 4\pi t\\&+(\frac{c_2^2}{16\pi^2}-8\pi c_1c_2)\sin^2 4\pi t\ \cos 4\pi t\\&+16\pi^2c_1^2\sin^3 4\pi t+\gamma\cos 4\pi t.\end{aligned}$$

Using condition (21), we obtain

$$\begin{aligned}&\Phi_0^1(c_1,c_2)\\&=\frac{3\mu_0c_2}{2}-\int_0^{1/2}\{\frac{\sin 4\pi s}{4\pi}\Big[h(x_1(s,c,0),x_2(s,c,0),0)\\&+k(\sin 4\pi s,\cos 4\pi s,0)\Big]\}ds\\&-\int_0^1\{\frac{\sin 4\pi s}{4\pi}\Big[h(x_1(s,c,0),x_2(s,c,0),0)\\&+k(\sin 4\pi s,\cos 4\pi s,0)\Big]\}ds\\&=\frac{3\mu_0c_2}{2}-\int_0^{1/2}\{\frac{1}{4\pi}(\frac{c_1c_2}{2\pi}+c_2^2)\cos^2 4\pi s\ \sin^2 4\pi s\\&+16\pi^2c_1^2\sin^4 4\pi s\}ds\\&-\int_0^1\{\frac{1}{4\pi}(\frac{c_1c_2}{2\pi}+c_2^2)\cos^2 4\pi s\ \sin^2 4\pi s\\&+16\pi^2c_1^2\sin^4 4\pi s\}ds\\&=\frac{3\mu_0c_2}{2}-3(\frac{c_1c_2}{128\pi^2}+\frac{c_2^2}{64\pi}+3\pi^2c_1^2),\end{aligned}$$

$$\begin{aligned}&\Phi_0^2(c_1,c_2)\\&=8\pi^2\mu_0c_1-\int_{1/2}^1\{\cos 4\pi s\Big[h(x_1(s,c,0),x_2(s,c,0),0)\\&+k(\sin 4\pi s,\cos 4\pi s,0)\Big]\}ds\\&=8\pi^2\mu_0c_1-\int_{1/2}^1\{c_1^2\cos^4 4\pi s+\\&(\frac{c_2^2}{16\pi^2}-8\pi c_1c_2)\sin^2 4\pi s\cos^2 4\pi s+\gamma\cos^2 4\pi s\}ds\\&=8\pi^2\mu_0c_1-(\frac{3\pi c_1^2}{4}+\frac{c_2^2}{256\pi^2}-\frac{\pi c_1c_2}{2}+\frac{\gamma}{4}).\end{aligned}$$

If $\gamma\neq 0$ and k is small enough then $\Phi_0(c)\neq 0$ for all $c\in B_k$. So $d(\Phi_0,B_k,0)=0$. Thus the branching equation has no real solutions if ε is sufficiently small. However, if $\gamma=0$, and $\mu_0\neq 0$ then the polynomial $\Phi_0^2(c_1,c_2)$ has roots in B_k and, in particular, solutions exist. Indeed we can apply Lemma 3. Since

$$\det\begin{pmatrix}0&\frac{3\mu_0}{2}\\8\pi^2\mu_0&0\end{pmatrix}\neq 0,$$

it follows that $d(\Phi_0,B_k,0)\neq 0$. Hence for $(\varepsilon,\gamma)\neq(0,0)$ but small enough, we conclude that by Theorem 2 the problem (23) has at least one solution with the boundary condition

$$\begin{pmatrix}2&0\\0&0\end{pmatrix}\begin{pmatrix}x_1(0,c_0,0)\\x_2(0,c_0,0)\end{pmatrix}+\begin{pmatrix}-1&0\\0&-1\end{pmatrix}\begin{pmatrix}x_1(1/2,c_0,0)\\x_2(1/2,c_0,0)\end{pmatrix}+ \begin{pmatrix}-1&0\\0&1\end{pmatrix}\begin{pmatrix}x_1(1,c_0,0)\\x_2(1,c_0,0)\end{pmatrix}=\begin{pmatrix}2&0\\0&0\end{pmatrix}\begin{pmatrix}x_1(0,c(\varepsilon),\varepsilon)\\x_2(0,c(\varepsilon),\varepsilon)\end{pmatrix}+ \begin{pmatrix}-1&0\\0&-1\end{pmatrix}\begin{pmatrix}x_1(\frac{1}{2(1+\varepsilon\mu(\varepsilon))},c(\varepsilon),\varepsilon)\\x_2(\frac{1}{2(1+\varepsilon\mu(\varepsilon))},c(\varepsilon),\varepsilon)\end{pmatrix} +\begin{pmatrix}-1&0\\0&1\end{pmatrix}\begin{pmatrix}x_1(\frac{1}{1+\varepsilon\mu(\varepsilon)},c(\varepsilon),\varepsilon)\\x_2(\frac{1}{1+\varepsilon\mu(\varepsilon)},c(\varepsilon),\varepsilon)\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix},$$

where $c(0)=c_0$.

Remark. If the local degree at the origin and relative to a small ball, B_k with center at the origin is nonzero, then Φ_0 has at least one root in B_k. We note that Φ_0 may have solutions even if the local degree is equal to zero.

The following example indicates the degree is defined if the radius of B_k is sufficiently small.

Example 2: An example where local degree is nonzero

In system (25), let $\varepsilon f(x_1,x_2,\sin 4\pi\tau,\cos 4\pi\tau,\varepsilon)=\varepsilon(x_2^2+\cos 4\pi\tau)$. Note that $\cos 4\pi\tau=\cos 4\pi(1+\varepsilon\eta(\varepsilon))^{-1}t$ at $\varepsilon=0$ is given by $\cos 4\pi t$. **Assumption (D1)** is satisfied since we have

$$\begin{aligned}&h(x_1(t,c,0),x_2(t,c,0),0)+k(\sin 4\pi t,\cos 4\pi t,0)\\&=16\pi^2c_1^2\sin^2 4\pi t-8\pi c_1c_2\sin\ 4\pi t\ \cos 4\pi t\\&+c_2^2\cos^2 4\pi t+\cos 4\pi t.\end{aligned}$$

Using condition (21), we obtain

$$\begin{aligned}
&\Phi_0^1(c_1,c_2)\\
&= \frac{3\mu_0 c_2}{2} - \int_0^{1/2}\{\frac{\sin 4\pi s}{4\pi}\Big[h(x_1(s,c,0),x_2(s,c,0),0)\\
&\quad +k(\sin 4\pi s,\cos 4\pi s,0)\Big]\}ds\\
&\quad -\int_0^1\{\frac{\sin 4\pi s}{4\pi}\Big[h(x_1(s,c,0),x_2(s,c,0),0)\\
&\quad +k(\sin 4\pi s,\cos 4\pi s,0)\Big]\}ds\\
&= \frac{3\mu_0 c_2}{2},
\end{aligned}$$

$$\begin{aligned}
&\Phi_0^2(c_1,c_2)\\
&= 8\pi^2\mu_0 c_1 - \int_{1/2}^1\{\cos\, 4\pi s\Big[h(x_1(s,c,0),x_2(s,c,0),0)\\
&\quad +k(\sin\, 4\pi s,\cos\, 4\pi s,0)\Big]\}ds\\
&= -\int_{1/2}^1\{\cos^2 4\pi s\}ds\\
&= -\int_{1/2}^1\{\frac{\cos 8\pi s+1}{2}\}ds\\
&= 8\pi^2\mu_0 c_1 - \frac{1}{4}.
\end{aligned}$$

The Jacobian is

$$\det\begin{pmatrix} 0 & \frac{3\mu_0}{2} \\ 8\pi^2\mu_0 & 0 \end{pmatrix} < 0.$$

Thus the local degree is -1, since the Jacobian is negative except the origin. Hence for ε sufficiently small, we conclude that by Theorem 2 the problem (23) has at least one solution with the boundary condition

$$\begin{aligned}
&\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1(0,c_0,0) \\ x_2(0,c_0,0) \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x_1(1/2,c_0,0) \\ x_2(1/2,c_0,0) \end{pmatrix} +\\
&\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1(1,c_0,0) \\ x_2(1,c_0,0) \end{pmatrix}\\
&= \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1(0,c(\varepsilon),\varepsilon) \\ x_2(0,c(\varepsilon),\varepsilon) \end{pmatrix} +\\
&\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x_1(\frac{1}{2(1+\varepsilon\mu(\varepsilon))},c(\varepsilon),\varepsilon) \\ x_2(\frac{1}{2(1+\varepsilon\mu(\varepsilon))},c(\varepsilon),\varepsilon) \end{pmatrix}\\
&+\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1(\frac{1}{1+\varepsilon\mu(\varepsilon)},c(\varepsilon),\varepsilon) \\ x_2(\frac{1}{1+\varepsilon\mu(\varepsilon)},c(\varepsilon),\varepsilon) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\end{aligned}$$

where $c(0)=c_0$. Thus there exists at least one solution to the BVP even if all the terms in V_1 or V_2 vanish, because the terms $\frac{3\mu_0}{2}c_2$ and $8\pi^2\mu_0 c_1$ are nonzero.
Next we show that in the following case the local degree is non zero.

4.2 A Two-point BVP

Consider the BVP (19), (20). Then (19) may be written as

$$\begin{aligned}
\frac{dx_1}{dt} &= x_2\\
\frac{dx_2}{dt} &= -4\pi^2 x_1 + \varepsilon f(x_1,x_2,\sin 2\pi wt,\cos 2\pi wt,\varepsilon). \qquad (27)
\end{aligned}$$

By the chain rule (27) becomes

$$\begin{aligned}
\frac{dx_1}{d\tau} &= x_2 + \varepsilon\mu(\varepsilon)x_2\\
\frac{dx_2}{d\tau} &= -16\pi^2 x_1 + \varepsilon\Big[-4\mu(\varepsilon)\pi^2 x_1 +\\
&\qquad (1+\varepsilon\mu(\varepsilon))f(x_1,x_2,\sin 2\pi\tau,\cos 2\pi\tau,\varepsilon)\Big]. \qquad (28)
\end{aligned}$$

Writing (28), (20) into matrix form, we have

$$\begin{aligned}
&\begin{pmatrix} \frac{dx_1}{d\tau} \\ \frac{dx_2}{d\tau} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4\pi^2 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\\
&+\varepsilon\begin{pmatrix} \mu(\varepsilon)x_2 \\ -4\mu(\varepsilon)\pi^2 x_1 + (1+\varepsilon\mu(\varepsilon))f(x_1,x_2,\sin 2\pi\tau,\cos 2\pi\tau,\varepsilon) \end{pmatrix}, \qquad (29)
\end{aligned}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1(1) \\ x_2(1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad (30)$$

where $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $A = \begin{pmatrix} 0 & 1 \\ -4\pi^2 & 0 \end{pmatrix}$, $M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $R = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$,

$F(x_1,x_2,\sin 2\pi\tau,\cos 2\pi\tau,\varepsilon) = \begin{pmatrix} \mu(\varepsilon)x_2 \\ -4\mu(\varepsilon)\pi^2 x_1 + (1+\varepsilon\mu(\varepsilon))f(x_1,x_2,\sin 2\pi\tau,\cos 2\pi\tau,\varepsilon) \end{pmatrix}$.

The fundamental matrix $Y(\tau) = e^{A\tau} = \begin{pmatrix} \cos 2\pi\tau & \sin 2\pi\tau/(2\pi) \\ -2\pi\sin 2\pi\tau & \cos 2\pi\tau \end{pmatrix}$,

$Y^{-1}(t) = \begin{pmatrix} \cos 2\pi t & -\sin 2\pi t/(2\pi) \\ 2\pi\sin 2\pi t & \cos 2\pi t \end{pmatrix}$,

$Y_0(\tau) = \begin{pmatrix} \cos 2\pi\tau & \sin 2\pi\tau/(2\pi) \\ -2\pi\sin 2\pi\tau & \cos 2\pi\tau \end{pmatrix}$ and $Y_0(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then by Lemma 2, solving the problem (29), (30) is reduced to that of solving $\mathscr{L}c = \varepsilon\mathscr{N}(c,\alpha,\eta,\varepsilon) + d$ for c. Thus we find $\mathscr{L}$ and $\mathscr{N}(c,\alpha,\eta,\varepsilon)$.

$$\begin{aligned}
\mathscr{L} &= M + RY_0(1)\\
&= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\\
&= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.
\end{aligned}$$

Rank $\mathscr{L} = 1$. Let E_2 denote the null space of $\mathscr{L}$. Thus $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, be a basis for $Ker(\mathscr{L})$, and $Ker(\mathscr{L})$=Span (e_2).

Let P_2 be the matrix projection onto Ker($\mathscr{L}$), $P_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. So $P_1 = I - P_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Set $H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ so that $H\mathscr{L} = P_1$, and

$$\mathscr{N}(c,\alpha,\eta,\varepsilon) =$$

$$-\int_0^1 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos 2\pi s & -\sin 2\pi s/(2\pi) \\ 2\pi \sin 2\pi s & \cos 2\pi s \end{pmatrix} \times$$
$$\begin{pmatrix} \mu(\varepsilon)x_2 \\ -4\pi^2 x_1(s,c,\varepsilon)\mu(\varepsilon) + (1+\varepsilon\mu(\varepsilon))f(x_1,x_2,\tau_1,\tau_2,\varepsilon) \end{pmatrix} ds$$
$$= \begin{pmatrix} 0 \\ \mathscr{N}_2(c,\alpha,\eta,\varepsilon) \end{pmatrix},$$

where

$$\mathscr{N}_2(c,\alpha,\eta,\varepsilon) = -\int_0^1 \{2\pi\mu(\varepsilon)x_2(s,c,\varepsilon)\sin 2\pi s$$
$$-4\pi^2\mu(\varepsilon)\cos 2\pi s x_1(s,c,\varepsilon)$$
$$+\cos 2\pi s(1+\varepsilon(\mu(\varepsilon))f(x_1,x_2,\tau_1,\tau_2,\varepsilon)\}ds$$

and $f(x_1,x_2,\tau_1,\tau_2,\varepsilon) = f(x_1,x_2,\sin 2\pi ws,\cos 2\pi ws,\varepsilon)$.

Since $d = 0$, it follows that $P_2Hd = 0$. Hence the condition of Theorem 1 is satisfied. In order to study Φ_0, we must first obtain $x(t,c,0)$, that is the solution of $x' = A(t)x$. By Lemma 1, $x' = A(t)x$ has a solution $x(t)$ with $x(0) = c = (0,c_2)^T$, where $x_1(0) = 0 = c_1$. Thus (29), (30) has a solution if $\varepsilon = 0$ namely

$$x_1(t,c,0) = \frac{c_2 \sin 2\pi t}{2\pi},$$
$$x_2(t,c,0) = c_2\cos 2\pi t.$$

We compute

$$P_2H\mathscr{N}(c,\alpha,\eta,\varepsilon) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ \mathscr{N}_2(c,\alpha,\eta,\varepsilon) \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ \mathscr{N}_2(c,\alpha,\eta,\varepsilon) \end{pmatrix}.$$

Thus $\Phi_\varepsilon(c_2) = \mathscr{N}_2(c^2,\alpha,\eta,\varepsilon)$, where $c^2 = P_2 = \binom{0}{c_2}$ and $c^1 = P_1c = \binom{c_1}{0}$. Setting $\varepsilon = 0$, we have $\Phi_0(c_2) = \mathscr{N}_2(c^2,\alpha,\eta,0)$, where $c^1(c^2,0) = P_2Hd = 0$. Using condition (22), we obtain

$$\Phi_0(c_2) = -\int_0^1 \{2\pi\mu_0(c_2\cos 2\pi s)\sin 2\pi s \tag{31}$$
$$-\frac{4\pi^2 c_2\mu_0\cos 2\pi s\sin 2\pi s}{2\pi}$$
$$+\cos 2\pi s\Big[h(\frac{c_2\sin 2\pi s}{2\pi}, c_2\cos 4\pi s) \tag{32}$$
$$+k(\sin 2\pi s,\cos 2\pi s,0)\Big]\}ds$$
$$= V_1(c_2) + K_1. \tag{33}$$

Example 3

In system (29), let

$$\varepsilon f(t,x_1,x_2,\sin\tau,\cos\tau,\varepsilon) = \varepsilon(x_2^3 + \cos 2\pi\tau).$$

Assumption (D1) is satisfied since we have

$$h(x_1(t,c,0),x_2(t,c,0),0) + k(\sin wt,\cos wt,0) = c_2^3\cos^3 2\pi t + \cos 2\pi t.$$

Using condition (22),

$$\Phi_0(c_2) = -\int_0^1 \{\cos 2\pi s(c_2^3\cos^3 2\pi s + \cos 2\pi s)\}ds$$
$$= -\int_0^1 \{c_2^3(\frac{\cos 8\pi s}{2} + 2\cos 4\pi s + \frac{3}{2}) + \frac{\cos 4\pi s + 1}{2}\}ds$$
$$= \frac{-3\pi c_2^3}{4} + \frac{1}{2}.$$

We apply the Intermediate Value Theorem when $n = 1$. Since $\Phi_0(0) = 1/2 > 0$ and $\Phi_0(1) = \frac{(-3\pi+2)}{4} < 0$, it follows then that $\Phi_0(0)\Phi_0(1) < 1$. Thus

$$d(\Phi_0,(0,1),0) = -1.$$

Hence for ε sufficiently small, we conclude that by Theorem 2 the problem (27) has at least one solution with the boundary condition

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x_1(0,c_0,0) \\ x_2(0,c_0,0) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1(1,c_0,0) \\ x_2(1,c_0,0) \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x_1(0,c(\varepsilon),\varepsilon) \\ x_2(0,c(\varepsilon),\varepsilon) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1(\frac{1}{1+\varepsilon\mu(\varepsilon)},c(\varepsilon),\varepsilon) \\ x_2(\frac{1}{1+\varepsilon\mu(\varepsilon)},c(\varepsilon),\varepsilon) \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where $c(0) = c_0$.

5 Conclusions

In this paper, the entrainment of frequency problem for a perturbed system of first order ordinary differential equations has been established by adapting the approach of Cronin [6,7]. It is shown that the problem $\frac{dx}{d\tau} = A(t)x + \varepsilon F(x,\sin\tau,\cos\tau,\varepsilon)$ with three-point and two-point boundary conditions has a solution $\overline{x}(t)$ for ε sufficiently small, and this solution is close to the solution of the problem $\frac{dx}{d\tau} = A(t)x$, the system when $\varepsilon = 0$. This is called the resonance or entrainment of frequency problem. The applications proposed in this paper for a 2-dimensional system of first-order equations can be extended to n-dimensional systems.

References

[1] Ravi P. Agarwal, On Multipoint Boundary Value Problem for Discrete Equations, *J. Math. Anal. Appl.*, **96** (1985), no. 2, 520-534.

[2] Ravi P. Agarwal and Donal O'Regan, Multipoint Boundary Value Problems for General Discrete Systems: The Degenerate case, *Commun. Appl. Anal.*, **1** (1997), no. 3, 269-288.

[3] Ravi P. Agarwal and M. Sambandham, Multipoint Boundary Value Problems for General Discrete Systems, *Dynam. Systems Appl.*, **6** (1997), no. 4, 469-492.

[4] Zhangbing Bai, Weiguo Li and Weigao Ge, Existence and Multiplicity of Solutions for Four-Point Boundary Value Problems at Resonance, *Nonlinear Anal.*, **60** (2005), no. 6, 1151-1162.

[5] E. A. Coddington and N. Levinson, *Perturbations of Linear Systems with Constants Coefficients Possesing Periodis Solutions*, Contributions to the Theory of Nonlinear Oscillations, Vo II, pp. 19-35, Princeton University Press, Princeton, 1950.

[6] Jane Cronin, *Fixed Points and Topological Degree in Nonlinear Analysis*, Math. Surveys, **11**, Amer. Math. Soc., Providence, Rhode Island, 1964.

[7] Jane Cronin, *Differential Equations: Introduction and Qualitative Theory*, Second Edition, Marcel Dekker, Inc., New York, 1994.

[8] Jane Cronin, Topological Degree of Some Mappings, *Proc. Amer. Math. Soc.,* **5** (1954), 175-178.

[9] Jane Cronin, *Analyzing Multiscale Phenomena Using Singular Perturbation Methods*, American Mathematical Society Short Course, January 5-6, Baltimore, Maryland, 1998.

[10] Jane Cronin, Entrainment of frequency in singularly perturbed systems, *Methods and Applications of Analysis*, Vol 3, (1996), 370-400.

[11] J. Cronin, Poincare's Perturbation Method and Topological Degree, *Contributions to The Theory of Nonlinear Oscillations*, Vol 5, pp. 37-54, Ann. of Math. Studies, **45**, Princeton, University Press, 1960.

[12] P. Drabek and M. Langerova, On a seond order periodic problem at resonance with impulses, Journal of Mathematical Analysis and Applications, 428 (2015) 1330 –1353.

[13] D. L. Etheridge and J. Rodriguez, Scalar Discrete Nonlinear Two-point Boundary Value Problems, *J. Difference Equ. Appl.,* **4** (1998), 127-144.

[14] W. Feng and J. R. L. Webb, Solvability of a m-Point Boudary Value Problems with Nonlinear Growth, *J. Math. Anal. Appl.,* **212** (1997), 467-480.

[15] W. Feng and J. R. L. Webb, Solvability of three point boundary value problems at resonance, Proceedings of the Second World Congress of Nonlinear Analysts, Part 6 (Athens, 1996), *Nonlinear Anal.* **30** (1997), 3227–3238.

[16] C. P. Gupta, Existence Theorems for A Second Order M-Point Boundary Value Problem at Resonance, *Internat J. Math. Sci.*, **18** (1995), 705-710.

[17] C. P. Gupta, Solvability of Multi-Point Boundary Value Problem at Resonance, *Results Math.,* **28** (1995), 270-276.

[18] C. P. Gupta, On a third-order three-point boundary value problem at resonance. *Differential Integral Equations* **2** (1989), 1–12.

[19] Gennaro Infante and Miroslawa Zima, Positive solutions of multi-point boundary value problems at resonance. *Nonlinear Anal.* **69** (2008), no. 8, 2458-2465.

[20] Philip Korman, Global solution curves for boundary value problems with linear part at resonance, *Nonlinear Anal.* **71** (2009), no. 7-8, 2456-2467.

[21] Nickolai Kosmatov, Multi-point boundary value problems on an unbounded domain at resonance, *Nonlinear Anal.* **68** (2008), no. 8, 2158-2171.

[22] N. G. Lloyd, *Degree theory,* Cambridge Univ. Press, New York, 1978.

[23] B. Liu and J. S. Yu, Solvability of Multi-Point Boundary Value Problems at Resonance (I), *Ind. J. Pure Appl. Math.,* **34** (2002), 475-494.

[24] B. Liu and J. S. Yu, Solvability of Multi-Point Boundary Value Problems at Resonance (II), *Appl. Math. Comput.,* **136** (2003), 353-373.

[25] B. Liu and J. S. Yu, Solvability of Multi-Point Boundary Value Problems at Resonance (III), *Appl. Math. Comput.,* **129** (2002), 119-143.

[26] B. Liu and J. S. Yu, Solvability of Multi-Point Boundary Value Problems at Resonance (IV), *Appl. Math. Comput.,* **143** (2003), 275-299.

[27] Ruyun Ma, Multiplicity Results for an m-point Boundary Value Problem at Resonance, *Indian J. Math.,* **47** (2005), no. 1, 15-31.

[28] J. Mawhin, Resonance and nonlinearity: a survey. *Ukrain. Mat. Zh. 59 (2007), no. 2, 190-205; translation in Ukrainian Math. J.* **59** (2007), no. 2, 197-214.

[29] J. Mawhin, Degré topologique et solutions périodiques des systèmes différentiels non linéaires. (French) *Bull. Soc. Roy. Sci. Liège* **38** (1969), 308–398.

[30] J. Mawhin, Landesman-Lazer's Type Problems for Nonlinear Equations, *Confer. Sem. Mat. Univ. Bari,* No. **147** (1977), 1-22.

[31] J. Mawhin, Topological Degree and Boundary Value Problems for Nonlinear Differential Equations, in: P. M. Fitzpatrick, M. Martelli, J. Mawhin, R. Nussbaum (Eds.), *Topological Methods for Ordinary Differential Equations, Lecture Notes in Mathematics,* Springer, New York, **1537** (1997), 74-142.

[32] M. Mohamed, H. B. Thompson and M. Jusoh, First-Order Three-Point Boundary Value Problems at Resonance, Journal of Computational and Applied Mathematics 235 (2011), 4796–4801.

[33] M. Mohamed, H. B. Thompson and M. Jusoh, First-Order three-point BVPs at Resonance (II), Electronic Journal of Qualitative Theory of Differential Equations, Vol 2011, Issue 68, 1–21.

[34] M. Mohamed, H. B. Thompson and M. Jusoh, First-Order Three-Point Boundary Value Problems at Resonance Part III, Journal of Applied Mathematics, Vol 2012, Article ID 357651,13 pages. doi:10.1155/2012/357651.

[35] L. C. Piccinini, G. Stampacchia and G. Vidossich, *Ordinary Differential Equations in* R^n, Problems and Methods, Translated by A. LoBello, New York, Springer-Verlag, 1984.

[36] Bogdan Przeradzki and Robert Stanczy, Solvability of A Multi-Point Boundary Value Problem at Resonance, *J. Math. Anal. Appl.,* **264** (2001), 253-261.

[37] J. Rodriguez, Resonance in Nonlinear Discrete Systems with Nonlinear Constraints , *Proceedings of 24th Conference on Decision and Control,* December 1985, Ft. Lauderdale, FL, IEEE.

[38] J. Rodriguez, On Perturbed Discrete Boundary Value Problems, *J. Difference Equ. Appl.,* **8** (2002), no. 5, 447-466.

[39] J. Rodriguez, Nonlinear multipoint boundary value problems at resonance. *Int. J. Pure Appl. Math.* **54** (2009), no 2, 215-226.

[40] Felix Sadyrbaev, Multiplicity of solutions for second order two-point boundary value problems with asymptotically

asymmetric nonlinearities at resonance, *Georgian Math. J.* **14** (2007), no. 2, 351-360.
[41] M. Urabe, An Existence Theorem for Multipoint Boundary Value Problems, *Funkcial. Ekvacioj,* **9** (1966), 43-60.
[42] M. Urabe, The Degenerate Case of Boundary Value Problems Associated with Weakly Nonlinear Differential Systems, *Publ. Res. Inst. Math. Sci. Kyoto Univ.,* **4** (1968), 545-584.
[43] J. R. L. Webb, M. Zima, Multiple positive solutions of resonant and non-resonant nonlocal boundary value problems, *Nonlinear Anal.* **71** (2009), no. 3-4, 1369-1378.
[44] Aijun Yang, Bo Sun and Weigao Ge, Existence of positive solutions for self-adjoint boundary-value problems with integral boundary conditions at resonance, *Electron. J. Differential Equations* **2011**, No. 11, 8 pp.
[45] Miroslawa Zima, Positive solutions for first-order boundary value problems at resonance, *Commun. Appl. Anal.* **13** (2009), no. 4, 671-679.
[46] Zengji Du, Fanchao Meng, Solutions to a second-order multi-point boundary value problem at resonance, *Acta Math. Sci. Ser. B Engl. Ed.* **30** (2010) no. 5 1567-1576.
[47] Zengji Du, Xiaojie Lin and Weigao Ge, Some Higher-Order Multi-Point Boundary Value Problem at Resonance, *J. Comput. Appl. Math.,* **177** (2005), 55-65.
[48] Zhang, X. C., Huang, Q. A and Wei, H. M., Existence and approximation of solutions of solution for a second-order three-point boundary value problem, Boundary Value Problems (2016), 169, DOI 10.1186/513661-016-0678-4.

Mesliza Mohamed is currently attached to Universiti Teknologi MARA Cawangan Pahang Malaysia. She holds a PhD from the University of Queensland, Australia. Her areas of interest are nonlinear analysis with applications to differential, difference and integral equations.

Bevan Thompson received his PhD from the University of Queensland where he is an Honorary Associate Professor. He is the author of around 130 research papers in professional journals spanning topological algebra, differential equations, nonlinear functional analysis, variational inequalities, stochastic modeling and uncertainty quantification.