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Inference for Constant-Stress Partially Accelerated Life Test Model with Progressive Type-II Censoring Scheme

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Abstract: In this paper, the point at issue of this paper is to deliberate point and interval estimations for the parameters of Weibull-exponential distribution (WED) using progressively Type-II censored (PRO-II-C) sample under constant stress partially accelerated life tests (CSPALT) model. The maximum likelihood, Bayes and parametric bootstrap methods are used for estimating the unknown parameters and acceleration factor. Markov chain Monte Carlo (MCMC) and Lindley's approximation are used to get the Bayes estimators. Finally, an example presented to illustrate the results.

Keywords: Weibull-exponential distribution; Constant stress partially accelerated life tests; Parametric bootstrap; Bayesian estimation; MCMC technique.

1 Introduction

Accelerated life test ALT and partial accelerated life test PALT are used to obtain failure data in a short time. The most types of stresses used to accelerate the life test are constant-stress, step-stress and progressive-stress. The main assumption in ALT is that the relationship between life and stress must be known or can be assumed so that the data obtained from accelerated conditions can be extrapolated to normal use conditions. If such a relationship is unknown or cannot be assumed, one cannot apply the ALT approach. So, PALT are often used in such cases. In CSPALT each unit run either normal use condition or accelerated (stress) use condition only until the test is terminated. These conditions are often referred to as stresses which may be in the form of temperature, pressure, vibrations, and so on.

The CSPALT studied by many authors, Abdel-Hamid [1] discussed CSPALT for Burr type-XII distribution with progressive type-II censoring, EL-Sagheer [5] studied CSPALT under progressive type-II censoring, Srivastava and Mittal [14]optimized CSPALT for the truncated logistic distribution under time constraint and Abushal and Soliman [2]estimated the Pareto parameters under progressive censoring data for CSPALT.

Also, Several authors preferred to use Lindley's approximation beside classical methods to get the Bayes estimators. Metiri [8] showed Bayes estimates of Lindley distribution under linex loss function, informative and non informative priors, Preda et al.[9] applied Bayes estimators of modified-Weibull distribution parameters using Lindley's approximation, Singh et al.[10] evaluated Bayes estimator of inverse gaussian parameters under general entropy loss function using Lindley's approximation, Singh et al.[11] computed Bayes estimator of generalized-exponential parameters under linex loss function using Lindley's approximation, Soliman et al.[13]estimated under progressive first-failure censored sampling with binomial removals by using classical and Bayesian methods and Singh et al. [12] estimated the parameter of Marshall-Olkin exponential distribution under type-I hybrid censoring scheme.

This paper focused on point and interval estimations for the parameters of WED under PRO-II-C by used non-Bayesian and Bayesian methods. Finally, this paper organized as follows: Sec. 2 the assumptions and description of WED are shown. Sec. 3 devoted to study the maximum likelihood estimations (MLEs) used to evaluate point and interval estimation for the unknown parameters under consideration, asymptotic variance-covariance matrix and parametric bootstrap confidence intervals. Sec. 4 Bayesian estimation computed by using both MCMC and Lindley's approximation methods. Sec. 5 an illustrative example is developed to explain the theoretical results. Eventually conclusion is inserted in Sec. 6.

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2 Assumptions and Model Description

2.1 Basic assumptions

1. n identical units are put on the life test .
2. The lifetimes of the units have independent and identically WED.
3. The lifetime of units tested under normal condition follows WED with probability density function (PDF), cumulative distribution function (CDF), survival function (SF) and hazard rate function (HRF) are given, respectively, by

$$f_1(t;\alpha,\gamma,\beta)=\alpha\gamma\beta\left(1-e^{-\gamma t}\right)^{\beta-1}e^{\gamma\beta t}e^{-\alpha(e^{\gamma t}-1)^{\beta}}\ ,t>0;\alpha,\gamma,\beta>0, \tag{1}$$

$$F_1(t,\alpha,\gamma,\beta)=1-e^{-\alpha(e^{\gamma t}-1)^{\beta}},t>0;\alpha,\gamma,\beta>0, \tag{2}$$

$$S_1(t,\alpha,\gamma,\beta)=e^{-\alpha(e^{\gamma t}-1)^{\beta}},t>0;\alpha,\gamma,\beta>0, \tag{3}$$

$$h_1(t,\alpha,\gamma,\beta)=\alpha\gamma\beta e^{\gamma\beta t}\left(1-e^{-\gamma t}\right)^{\beta-1},t>0;\alpha,\gamma,\beta>0, \tag{4}$$

   where α and γ are the scale parameters and β is the shape parameter.
4. The HRF of units tested under accelerated condition is given by $h_2(t)=\lambda h_1(t)$, where $\lambda>1$ is an acceleration factor. Therefore, the HRF, SF,CDF and PDF under accelerated condition with $t>0$; $\alpha,\gamma,\beta>0$ and $\lambda>1$, are given, respectively, by

$$h_2(t,\alpha,\gamma,\beta,\lambda)=\alpha\gamma\beta\lambda e^{\gamma\beta t}e^{-\alpha(e^{\gamma t}-1)^{\beta-1}}, \tag{5}$$

$$S_2(t,\alpha,\gamma,\beta,\lambda)=e^{[-\int_0^t h_2(u)du]}=e^{[-\alpha\lambda e^{\gamma\beta t}\left(1-e^{-\gamma t}\right)^{\beta}]}, \tag{6}$$

$$F_2(t,\alpha,\gamma,\beta,\lambda)=1-e^{[-\alpha\lambda e^{\gamma\beta t}\left(1-e^{-\gamma t}\right)^{\beta}]}, \tag{7}$$

$$f_2(t,\alpha,\gamma,\beta,\lambda)=\alpha\gamma\beta\lambda e^{\gamma\beta t}\left(1-e^{-\gamma t}\right)^{\beta-1}e^{[-\alpha\lambda e^{\gamma\beta t}\left(1-e^{-\gamma t}\right)^{\beta}]}. \tag{8}$$

2.2 Model description

According to CSPALT, n units are divided into two groups: n_1 units for group 1 (normal condition) and $n_2=n-n_1$ for group 2 (accelerated condition). Pro-II-C is applied as follows: In group j, $j=1,2$, at the time of the i^{th} failure, a random number of the surviving units R_{ji} ,$i=1,2,...,m_j-1$, are randomly removed from the test. Finally, at the time of the m_j^{th} failure, the remaining surviving units $R_{jm_j}=n_j-m_j-\sum_{i=1}^{m_j-1}R_{ji}$ are removed from the test and the test is terminated. Suppose that R_{ji} and $(m_j\le n_j)$ are prefixed. Then the observed progressive censored data are

$$t^{\mathbf{R_j}}_{j1;m_j,n_j}<t^{\mathbf{R_j}}_{j2;m_j,n_j}<...<t^{\mathbf{R_j}}_{jm_j;m_j,n_j},\quad j=1,2 \tag{9}$$

3 Methods of Estimation

This section contains different estimation methods to estimate unknown parameters of WED and acceleration factor by using the MLEs method and two parametric bootstrap methods, percentile bootstrap and bootstrap-t.

3.1 Maximum likelihood estimation

Let, $t^{\mathbf{R_j}}_{j1;m_j,n_j} < t^{\mathbf{R_j}}_{j2;m_j,n_j} < \cdots < t^{\mathbf{R_j}}_{jm_j;m_j,n_j}$, for $j = 1,2$ denote two Pro-II-C samples from two populations whose CDFs and PDFs are as given in (1), (2) and (7), (8) with $\mathbf{R}_j = R_{j1}, R_{j2}, \ldots, R_{jm_j}$, $i = 1,2,\cdots,m$. The likelihood function based on the Pro-II-C is given by

$$\begin{aligned} L(\alpha,\gamma,\beta,\lambda|\underline{t}) &= \prod_{j=1}^{2} A_j \prod_{i=1}^{m_j} f_j(t_{ji},m_j,n_j)\left[S_j(t_{ji},m_j,n_j)\right]^{\mathbf{R_{ji}}} \\ &\propto \prod_{i=1}^{m_1}\left(\alpha\gamma\beta\left(1-e^{-\gamma t_{1i}}\right)^{\beta-1} e^{\gamma\beta t_{1i}} e^{-\alpha\left(e^{\gamma t_{1i}}-1\right)^{\beta}}\right)\cdot\left(e^{-\alpha\left(e^{\gamma t_{1i}}-1\right)^{\beta}}\right)^{\mathbf{R_{1i}}} \\ &\times \prod_{i=1}^{m_2}\left(\alpha\gamma\beta\lambda e^{\gamma\beta t_{2i}}\left(1-e^{-\gamma t_{2i}}\right)^{\beta-1} e^{-\alpha\lambda e^{\gamma\beta t_{2i}}\left(1-e^{-\gamma t_{2i}}\right)^{\beta}}\right)\left(e^{-\alpha\lambda e^{\gamma\beta t_{2i}}\left(1-e^{-\gamma t_{2i}}\right)^{\beta}}\right)^{\mathbf{R_{2i}}}, \end{aligned} \tag{10}$$

where $A_j = n_j(n_j-1-R_{j_1})(n_j-2-R_{j_1}-R_{j_2})\cdots(n_j-m_j-\sum_{i=1}^{m_j-1} R_{ji})$.

The log-likelihood function $\ell(\alpha,\gamma,\beta,\lambda) = \log L(\alpha,\gamma,\beta,\lambda)$ without normalized constant is given by

$$\begin{aligned} \ell(\alpha,\gamma,\beta,\lambda) &= (m_1+m_2)(\log\alpha+\log\gamma+\log\beta) + m_2\log\lambda + \gamma\beta\left(\sum_{i=1}^{m_1} t_{1i} + \sum_{i=1}^{m_2} t_{2i}\right) \\ &- \alpha\sum_{i=1}^{m_1}\left(e^{\gamma t_{1i}}-1\right)^{\beta}(\mathbf{R}_{1i}+1) - \alpha\lambda\sum_{i=1}^{m_2} e^{\gamma\beta t_{2i}}\left(1-e^{-\gamma t_{2i}}\right)^{\beta}(\mathbf{R}_{2i}+1) \\ &+ (\beta-1)\left(\sum_{i=1}^{m_1}\log(1-e^{-\gamma t_{1i}}) + \sum_{i=1}^{m_2}\log(1-e^{-\gamma t_{2i}})\right). \end{aligned} \tag{11}$$

Calculating the first order partial derivatives of log-likelihood function with respect to α, γ, β and λ, respectively, and equating each to zero, we get

$$\frac{m_1+m_2}{\alpha} - \sum_{i=1}^{m_1}\left(e^{\gamma t_{1i}}-1\right)^{\beta}(\mathbf{R}_{1i}+1) - \lambda\sum_{i=1}^{m_2} e^{\gamma\beta t_{2i}}\left(1-e^{-\gamma t_{2i}}\right)^{\beta}(\mathbf{R}_{2i}+1) = 0, \tag{12}$$

$$\begin{aligned} &\frac{m_1+m_2}{\gamma} + \beta\left(\sum_{i=1}^{m_1} t_{1i} + \sum_{i=1}^{m_2} t_{2i}\right) - \alpha\beta\sum_{i=1}^{m_1} t_{1i} e^{\gamma t_{1i}}\left(e^{\gamma t_{1i}}-1\right)^{\beta-1}(\mathbf{R}_{1i}+1) \\ &+(\beta-1)\left[\sum_{i=1}^{m_1}\frac{t_{1i}e^{-\gamma t_{1i}}}{(1-e^{-\gamma t_{1i}})} + \sum_{i=1}^{m_2}\frac{t_{2i}e^{-\gamma t_{2i}}}{(1-e^{-\gamma t_{2i}})}\right] - \alpha\lambda\beta\sum_{i=1}^{m_2} t_{2i} e^{\gamma\beta t_{2i}}\left(1-e^{-\gamma t_{2i}}\right)^{\beta-1}(\mathbf{R}_{2i}+1) = 0, \end{aligned} \tag{13}$$

$$\begin{aligned} &\frac{m_1+m_2}{\beta} + \gamma\left(\sum_{i=1}^{m_1} t_{1i} + \sum_{i=1}^{m_2} t_{2i}\right) - \alpha\sum_{i=1}^{m_1}\left(e^{\gamma t_{1i}}-1\right)^{\beta}\left(\log\left(e^{\gamma t_{1i}}-1\right)\right)(\mathbf{R}_{1i}+1) \\ &-\alpha\lambda\sum_{i=1}^{m_2} e^{\gamma\beta t_{2i}}\left(1-e^{-\gamma t_{2i}}\right)^{\beta}\left(\log\left(1-e^{-\gamma t_{2i}}\right)\right)(\mathbf{R}_{2i}+1) - \alpha\lambda\gamma\sum_{i=1}^{m_2} t_{2i} e^{\gamma\beta t_{2i}}\left(1-e^{-\gamma t_{2i}}\right)^{\beta}(\mathbf{R}_{2i}+1) \\ &+\left(\sum_{i=1}^{m_1}\log(1-e^{-\gamma t_{1i}}) + \sum_{i=1}^{m_2}\log(1-e^{-\gamma t_{2i}})\right) = 0, \end{aligned} \tag{14}$$

and

$$\frac{m_2}{\lambda} - \alpha\sum_{i=1}^{m_2} e^{\gamma\beta t_{2i}}\left(1-e^{-\gamma t_{2i}}\right)^{\beta}(\mathbf{R}_{2i}+1) = 0. \tag{15}$$

The solutions of likelihood equations (12), (13), (14) and (15) can not be obtained in a closed form. So they can be solved numerically by using Newton–Raphson iteration method.

3.1.1 Asymptotic variance–covariance matrix

The asymptotic Fisher information matrix I of maximum likelihood estimates is the 4×4 symmetric matrix of negative second order partial derivatives of the log-likelihood function with respect to α, γ, β and λ. Let $\psi_1 = \alpha,\ \psi_2 = \gamma,\ \psi_3 = \beta$ and $\psi_4 = \lambda$, then

$$I = \left(\frac{\partial^2 \ell}{\partial \psi_i \partial \psi_j} \right)_{\downarrow(\hat{\alpha},\hat{\gamma},\hat{\beta},\hat{\lambda})}, i,j = 1,2,3,4 \tag{16}$$

Therefore, the asymptotic variance-covariance matrix can be written as follows

$$V = I^{-1} = \left(\frac{\partial^2 \ell}{\partial \psi_i \partial \psi_j} \right)^{-1}_{\downarrow(\hat{\alpha},\hat{\gamma},\hat{\beta},\hat{\lambda})} = \begin{bmatrix} var(\hat{\alpha}) & Cov(\hat{\alpha}\hat{\gamma}) & Cov(\hat{\alpha}\hat{\beta}) & Cov(\hat{\alpha}\hat{\lambda}) \\ Cov(\hat{\gamma}\hat{\alpha}) & var(\hat{\gamma}) & Cov(\hat{\gamma}\hat{\beta}) & Cov(\hat{\gamma}\hat{\lambda}) \\ Cov(\hat{\beta}\hat{\alpha}) & Cov(\hat{\beta}\hat{\gamma}) & var(\hat{\beta}) & Cov(\hat{\beta}\hat{\lambda}) \\ Cov(\hat{\lambda}\hat{\alpha}) & Cov(\hat{\lambda}\hat{\gamma}) & Cov(\hat{\lambda}\hat{\beta}) & var(\hat{\lambda}) \end{bmatrix}^{-1}. \tag{17}$$

Thus, the $(1-\zeta)100\%$ approximate confidence intervals (ACIs) for the parameters α, γ, β and λ, are obtained as

$$\begin{aligned} (\hat{\alpha}_L, \hat{\alpha}_U) &= \hat{\alpha} \pm z_{\frac{\zeta}{2}} \sqrt{var(\hat{\alpha})},\ (\hat{\gamma}_L, \hat{\gamma}_U) = \hat{\gamma} \pm z_{\frac{\zeta}{2}} \sqrt{var(\hat{\gamma})}, \\ (\hat{\beta}_L, \hat{\beta}_U) &= \hat{\beta} \pm z_{\frac{\zeta}{2}} \sqrt{var(\hat{\beta})},\ (\hat{\lambda}_L, \hat{\lambda}_U) = \hat{\lambda} \pm z_{\frac{\zeta}{2}} \sqrt{var(\hat{\lambda})}, \end{aligned} \tag{18}$$

where $z_{\frac{\zeta}{2}}$ is the value of the standard normal distribution leaving an area of $\frac{\zeta}{2}$ to the right and $var(\hat{\alpha})$, $var(\hat{\gamma})$, $var(\hat{\beta})$, and $var(\hat{\lambda})$ are the elements on the main diagonal of the variance-covariance matrix.

3.2 Parametric bootstrap methods

The second method used to estimate unknown parameters of WED is the parametric bootstrap methods. This subsection present two parametric bootstrap methods, percentile bootstrap method (Boot$_p$) see Efron [4] and bootstrap-t method (Boot$_t$) see Hall [6]. The following algorithm is followed to obtain PRO-II-C bootstrap samples from WED under CSPALT for both parametric bootstrap methods:

1.Determine the values of n_j and m_j $(1 \le m_j \le n_j)$, $j = 1,2$
2.Generate two independent random samples U_{ji} of size m_j, $j = 1,2$ from Uniform (0,1) distribution $(U_{j1}, U_{j2}, ..., U_{jm_j})$.
3.Determine the values of censored R_{ji}, $i = 1,...,m_j$ and $j = 1,2$.
4.Set $V_{ji} = U_{ji}^{W_{ji}}$,where$W_{ji} = 1/(\sum_{k=m_j-i+1}^{m_j} R_{jk}), i = 1,...,m_j$ and $j = 1,2$.
5.Then, set $X_{ji} = 1 - \prod_{k=m_j-i+1}^{m_j} V_{ji}$,$i = 1,...,m_j$ is the required PRO-II-C samples generated from Uniform (0,1) distribution
6.Finally, set $t_{ji} = F^{-1}(X_{ji})$ where $F^{-1}(.)$ is the inverse CDF of WED under CSPALT. Then, $(t_{1;m_j,n_j} < t_{2;m_j,n_j} < ... < t_{m_j;m_j,n_j})$ represent the two PRO-II-C samples from WED under CSPALT.
7.Based on two PRO-II-C samples, obtain the MLEs of parameters $\hat{\alpha}$, $\hat{\gamma}$, $\hat{\beta}$ and $\hat{\lambda}$.
8.Repeat Steps 1-7 B times to get the bootstrap estimates $\hat{\alpha}^*$,$\hat{\gamma}^*$, $\hat{\beta}^*$ and $\hat{\lambda}^*$.
9.Arrange all $\hat{\alpha}^{*\prime}$s ,$\hat{\gamma}^{*\prime}$s, $\hat{\beta}^{*\prime}$s and $\hat{\lambda}^{*\prime}$s in ascending order to obtain the bootstrap sample $\left(\hat{\psi}_k^{*[1]}, \hat{\psi}_k^{*[2]}, ..., \hat{\psi}_k^{*[B]}\right)$, $k = 1,2,3,4$ and $\hat{\psi}_1^* = \hat{\alpha}^*$, $\hat{\psi}_2^* =, \hat{\gamma}^*$, $\hat{\psi}_3^* = \hat{\beta}^*$, $\hat{\psi}_4^* = \hat{\lambda}^*$.

3.2.1 Boot$_p$ confidence intervals

Let $\Phi(z) = P(\hat{\psi}_k^* \le z)$ be the CDF of $\hat{\psi}_k^*$. Define $\hat{\psi}_{kBoot_p}^* = \Phi^{-1}(z)$ for given z. The approximate percentile bootstrap $100(1-\zeta)\%$ confidence interval (Boot$_p$ CI) of $\hat{\psi}_k^*$ is given by

$$\left[\hat{\psi}_{kBoot_p}^*(\tfrac{\zeta}{2})\ ,\ \hat{\psi}_{kBoot_p}^*(1-\tfrac{\zeta}{2}) \right]. \tag{19}$$

3.2.2 Boot$_t$ confidence intervals

Consider the order statistics $\delta_k^{*[1]} < \delta_k^{*[2]} < \ldots < \delta_k^{*[B]}$, $k = 1,2,3,4$ where

$$\delta_k^{*[j]} = \frac{\sqrt{B}(\hat{\psi}_k^{*[j]} - \hat{\psi}_k)}{\sqrt{Var\left(\hat{\psi}_k^{*[j]}\right)}}, \; j = 1,2,...,B, \tag{20}$$

where $\hat{\psi}_k = \hat{\alpha}, \; \hat{\psi}_k =, \hat{\gamma}, \; \hat{\psi}_k = \hat{\beta}$ and $\hat{\psi}_k = \hat{\lambda}$ while $Var\left(\hat{\psi}_k^{*[j]}\right)$ is obtained using the inverse of the Fisher information matrix as done before in (17). Let $W(z) = P\left(\delta_k^* < z\right), k = 1,2,3,4$ be the CDF of δ_k^*. Define $\delta_k^* = W^{-1}(z)$ for a given z, define

$$\hat{\psi}^*_{kBoot_t} = \hat{\psi}_k + \frac{1}{\sqrt{B}}\sqrt{Var\left(\hat{\psi}_k^*\right)}W^{-1}(z). \tag{21}$$

Thus, the approximate bootstrap-t $100(1-\zeta)\%$ confidence interval (Boot$_t$ CI) of $\hat{\psi}_k^*$ is given by

$$\left[\hat{\psi}^*_{kBoot_t}(\tfrac{\zeta}{2}) \; , \; \hat{\psi}^*_{kBoot_t}(1-\tfrac{\zeta}{2})\right]. \tag{22}$$

4 Bayesian Estimation

In this section, two Bayesian estimation methods (MCMC and Lindley's approximation) are discussed to obtain Bayes estimators for unknown parameters of WED. The steps of Bayesian process are:

1. Specify prior distribution for the unknown parameters $\alpha, \; \gamma, \; \beta$ and λ which are independent and follow the gamma prior distributions, as follows:

$$\begin{aligned} &\pi(\alpha) \propto \alpha^{a_1-1}e^{-\alpha b_1}, \;\; \alpha > 0, \;\; \pi(\gamma) \propto \gamma^{a_2-1}\,e^{-\gamma b_2}, \;\; \gamma > 0, \\ &\pi(\beta) \propto \beta^{a_3-1}\,e^{-\beta b_3}, \;\; \beta > 0, \; \pi(\lambda) \propto \lambda^{a_4-1}\,e^{-\lambda b_4}, \;\; \lambda > 1. \end{aligned} \tag{23}$$

where $a_1, a_2, a_3, a_4, b_1, b_2, b_3$ and b_4 are the hyper parameters and they are non negative.

2. The joint prior of the parameters $\alpha, \; \gamma, \; \beta$ and λ can be written as

$$\pi(\alpha,\gamma,\beta,\lambda) \propto \alpha^{a_1-1}\,\gamma^{a_2-1}\,\beta^{a_3-1}\,\lambda^{a_4-1}\,e^{-\alpha b_1-\gamma b_2-\beta b_3-\lambda b_4}, \; \alpha > 0, \; \gamma > 0, \; \beta > 0, \lambda > 1. \tag{24}$$

3. Combine the distributions into the joint posterior distribution of $\alpha, \; \gamma, \; \beta$ and λ, denoted by $\pi^*(\alpha,\gamma,\beta,\lambda|\underline{t})$ can be written as

$$\begin{aligned} \pi^*(\alpha,\gamma,\beta,\lambda|\underline{t}) &= \frac{L(\alpha,\gamma,\beta,\lambda)\times\pi(\alpha,\gamma,\beta,\lambda)}{\int_1^\infty\int_0^\infty\int_0^\infty\int_0^\infty L(\alpha,\gamma,\beta,\lambda)\times\pi(\alpha,\gamma,\beta,\lambda)\,d\alpha d\gamma d\beta d\lambda} \\ &= K^{-1}\alpha^{m_1+m_2+a_1-1}\gamma^{m_1+m_2+a_2-1}\beta^{m_1+m_2+a_3-1}\lambda^{m_2+a_4-1}e^{-\alpha b_1-\gamma b_2-\beta b_3-\lambda b_4} \\ &\times\prod_{i=1}^{m_1}\left(\left(1-e^{-\gamma t_{1i}}\right)^{\beta-1}e^{\gamma\beta t_{1i}}e^{-\alpha\left(e^{\gamma t_{1i}}-1\right)^\beta}\right)\cdot\left(e^{-\alpha\left(e^{\gamma t_{1i}}-1\right)^\beta}\right)^{\mathbf{R}_{1i}} \\ &\times\prod_{i=1}^{m_2}\left(e^{\gamma\beta t_{2i}}\left(1-e^{-\gamma t_{2i}}\right)^{\beta-1}e^{-\alpha\lambda e^{\gamma\beta t_{2i}}\left(1-e^{-\gamma t_{2i}}\right)^\beta}\right)\left(e^{-\alpha\lambda e^{\gamma\beta t_{2i}}\left(1-e^{-\gamma t_{2i}}\right)^\beta}\right)^{\mathbf{R_{2i}}} \end{aligned} \tag{25}$$

where K^{-1} is the normalizing constant, equal to

$$\begin{aligned} K^{-1} &= \int_1^\infty\int_0^\infty\int_0^\infty\int_0^\infty \alpha^{m_1+m_2+a_1-1}\gamma^{m_1+m_2+a_2-1}\beta^{m_1+m_2+a_3-1}\lambda^{m_2+a_4-1}e^{-\alpha b_1-\gamma b_2-\beta b_3-\lambda b_4} \\ &\times\prod_{i=1}^{m_1}\left(\left(1-e^{-\gamma t_{1i}}\right)^{\beta-1}e^{\gamma\beta t_{1i}}e^{-\alpha\left(e^{\gamma t_{1i}}-1\right)^\beta}\right)\cdot\left(e^{-\alpha\left(e^{\gamma t_{1i}}-1\right)^\beta}\right)^{\mathbf{R}_{1i}} \\ &\times\prod_{i=1}^{m_2}\left(e^{\gamma\beta t_{2i}}\left(1-e^{-\gamma t_{2i}}\right)^{\beta-1}e^{-\alpha\lambda e^{\gamma\beta t_{2i}}\left(1-e^{-\gamma t_{2i}}\right)^\beta}\right)\left(e^{-\alpha\lambda e^{\gamma\beta t_{2i}}\left(1-e^{-\gamma t_{2i}}\right)^\beta}\right)^{\mathbf{R_{2i}}}. \end{aligned} \tag{26}$$

4.Evaluate the posterior mean for any function of the parameters α, γ, β and λ ,say $h(\alpha,\gamma,\beta,\lambda)$ which is the Bayes estimate under squared error loss function, i.e.

$$\hat{h}(\alpha,\gamma,\beta,\lambda)=\frac{\int_1^\infty\int_0^\infty\int_0^\infty\int_0^\infty h(\alpha,\gamma,\beta,\lambda)\times L(\alpha,\gamma,\beta,\lambda)\times\pi(\alpha,\gamma,\beta,\lambda)\,d\alpha d\gamma d\beta d\lambda}{\int_1^\infty\int_0^\infty\int_0^\infty\int_0^\infty L(\alpha,\gamma,\beta,\lambda)\times\pi(\alpha,\gamma,\beta,\lambda)\,d\alpha d\gamma d\beta d\lambda} \quad (27)$$

Therefore, this integrals given by (27) cannot be obtained in closed form, so propose to use MCMC method and Lindley's approximation method to obtain Bayes estimators under squared error loss function.

4.1 MCMC method

The MCMC method is one of the best technique for obtaining the Bayes estimates. Suppose Gibbs sampler and Metropolis -Hastings algorithm can be used to generate samples from the full conditional posterior distributions and then compute the Bayes estimates. For applying the Gibbs algorithm, the full conditional posterior densities of α, γ, β and λ are given by

$$\pi_1^*(\alpha|\gamma,\beta,\lambda,\underline{t})\propto\alpha^{m_1+m_2+a_1-1}e^{-\alpha b_1}\prod_{i=1}^{m_1}e^{-\alpha\left(e^{\gamma t_{1i}}-1\right)^{\beta}(1+\mathbf{R}_{1i})}\prod_{i=1}^{m_2}e^{-\alpha\lambda e^{\gamma\beta t_{2i}}\left(1-e^{-\gamma t_{2i}}\right)^{\beta}(1+\mathbf{R_{2i}})}, \quad (28)$$

$$\pi_2^*(\gamma|\alpha,\beta,\lambda,\underline{t})\propto\gamma^{m_1+m_2+a_2-1}e^{-\gamma b_2}\prod_{i=1}^{m_1}\left(1-e^{-\gamma t_{1i}}\right)^{\beta-1}e^{\gamma\beta t_{1i}}e^{-\alpha\left(e^{\gamma t_{1i}}-1\right)^{\beta}(1+\mathbf{R}_{1i})}$$
$$\times\prod_{i=1}^{m_2}e^{\gamma\beta t_{2i}}\left(1-e^{-\gamma t_{2i}}\right)^{\beta-1}e^{-\alpha\lambda e^{\gamma\beta t_{2i}}\left(1-e^{-\gamma t_{2i}}\right)^{\beta}(1+\mathbf{R_{2i}})}, \quad (29)$$

$$\pi_3^*(\beta|\alpha,\gamma,\lambda,\underline{t})\propto\beta^{m_1+m_2+a_3-1}e^{-\beta b_3}\prod_{i=1}^{m_1}\left(1-e^{-\gamma t_{1i}}\right)^{\beta-1}e^{\gamma\beta t_{1i}}e^{-\alpha\left(e^{\gamma t_{1i}}-1\right)^{\beta}(1+\mathbf{R}_{1i})}$$
$$\times\prod_{i=1}^{m_2}e^{\gamma\beta t_{2i}}\left(1-e^{-\gamma t_{2i}}\right)^{\beta-1}e^{-\alpha\lambda e^{\gamma\beta t_{2i}}\left(1-e^{-\gamma t_{2i}}\right)^{\beta}(1+\mathbf{R_{2i}})}, \quad (30)$$

$$\pi_4^*(\lambda|\alpha,\gamma,\beta,\underline{t})\propto\lambda^{m_2+a_4-1}e^{-\lambda b_4}\prod_{i=1}^{m_2}e^{-\alpha\lambda e^{\gamma\beta t_{2i}}\left(1-e^{-\gamma t_{2i}}\right)^{\beta}(1+\mathbf{R_{2i}})}. \quad (31)$$

The algorithm of Gibbs sampling as suggested by Tierney [16] is as follows:

1.Start with an $\left(\alpha^{(0)}=\hat{\alpha},\ \gamma^{(0)}=\hat{\gamma},\ \beta^{(0)}=\hat{\beta}\text{ and }\lambda^{(0)}=\hat{\lambda}\right)$
2.Set $k=1$
3.Generate $\alpha^{(k)}$ from

$$\text{gamma distribution}\left[m_1+m_2+a_1,b_1+\sum_{i=1}^{m_1}\left(e^{\gamma t_{1i}}-1\right)^{\beta}(1+\mathbf{R}_{1i})+\sum_{i=1}^{m_2}\lambda e^{\gamma\beta t_{2i}}\left(1-e^{-\gamma t_{2i}}\right)^{\beta}(1+\mathbf{R_{2i}})\right]$$

4.Generate $\lambda^{(k)}$ from

$$\text{gamma distribution}\left[m_2+a_4,b_4+\sum_{i=1}^{m_2}\alpha e^{\gamma\beta t_{2i}}\left(1-e^{-\gamma t_{2i}}\right)^{\beta}(1+\mathbf{R_{2i}})\right]$$

5.Using the Metropolis-Hastings method, generate $\gamma^{(k)}$ and $\beta^{(k)}$ from normal distribution as follows

$$N(\gamma^{(k-1)},var(\gamma))\text{ and }N(\beta^{(k-1)},var(\beta))$$

6.Repeat Steps 3-5 N times. for all $k=1,2,3,...,N$
7.Obtain the Bayes MCMC point estimates of α, γ, β and λ as

$$\alpha_{MCMC}=\tfrac{1}{N-M}\textstyle\sum_{k=M+1}^{N}\alpha^{(k)},\ \ \gamma_{MCMC}=\tfrac{1}{N-M}\sum_{k=M+1}^{N}\gamma^{(k)},$$

$$\beta_{MCMC}=\tfrac{1}{N-M}\textstyle\sum_{k=M+1}^{N}\beta^{(k)},\ \ \lambda_{MCMC}=\tfrac{1}{N-M}\sum_{k=M+1}^{N}\lambda^{(k)}.$$

where M is the burn-in period.

8.Compute 100 $(1-\zeta)\%$ credible interval (CRI) of ψ_l as

$$\left[\psi_{l\left((N-M)\left(\frac{\zeta}{2}\right)\right)}, \psi_{l\left((N-M)\left(1-\frac{\zeta}{2}\right)\right)}\right],$$

where $\psi_1=\alpha,\ \psi_2=\gamma,\ \psi_3=\beta$ and $\psi_4=\lambda,\ l=1,2,3,4.$

4.2 Lindley's approximation method

Consider Lindley's approximation method for obtaining the Bayes estimator of α, γ, β and λ, which approaches the ratio of the integrals in the posterior expectation to a simplified form. According to Lindley's approximation in [7] and if n is sufficiently large, then the posterior expectation in (27) can be written as follows:

$$I(t)=E[u(\alpha,\gamma,\beta,\lambda)]=\frac{\int_{(\alpha,\gamma,\beta,\lambda)}u(\alpha,\gamma,\beta,\lambda)\times e^{L(\alpha,\gamma,\beta,\lambda)+G(\alpha,\gamma,\beta,\lambda)}d(\alpha,\gamma,\beta,\lambda)}{\int_{(\alpha,\gamma,\beta,\lambda)}e^{L(\alpha,\gamma,\beta,\lambda)+G(\alpha,\gamma,\beta,\lambda)}d(\alpha,\gamma,\beta,\lambda)} \tag{32}$$

where $u(\alpha,\gamma,\beta,\lambda)$ is a function of α,γ,β and λ
$L(\alpha,\gamma,\beta,\lambda)$ is Log-likelihood function
$G(\alpha,\gamma,\beta,\lambda)$ is Log of joint prior density
Then the ratio of the integral in equation (32) can be approximated as:

$$\begin{aligned}I(t)&=u(\hat{\alpha},\hat{\gamma},\hat{\beta},\hat{\lambda})+\frac{1}{2}\sum_{i=1}^{4}\sum_{j=1}^{4}(\hat{u}_{ij}+2\hat{u}_i\hat{g}_j)\hat{\sigma}_{ij}+\frac{1}{2}\sum_{i=1}^{4}\sum_{j=1}^{4}\sum_{k=1}^{4}\sum_{l=1}^{4}\hat{L}_{ijk}\hat{\sigma}_{ij}(\hat{\sigma}_{kl}\hat{u}_l)\\&=u(\hat{\alpha},\hat{\gamma},\hat{\beta},\hat{\lambda})+(\hat{u}_1\hat{a}_1+\hat{u}_2\hat{a}_2+\hat{u}_3\hat{a}_3+\hat{u}_4\hat{a}_4+\hat{a}_5+\hat{a}_6)\\&\quad+\frac{1}{2}[A(\hat{u}_1\hat{\sigma}_{11}+\hat{u}_2\hat{\sigma}_{12}+\hat{u}_3\hat{\sigma}_{13}+\hat{u}_4\hat{\sigma}_{14})+A_2(\hat{u}_1\sigma_{21}+\hat{u}_2\sigma_{22}+\hat{u}_3\sigma_{23}+\hat{u}_4\sigma_{24})\\&\quad+A_3(\hat{u}_1\hat{\sigma}_{31}+\hat{u}_2\hat{\sigma}_{32}+\hat{u}_3\hat{\sigma}_{33}+\hat{u}_4\hat{\sigma}_{34})+A_4(\hat{u}_1\hat{\sigma}_{41}+\hat{u}_2\hat{\sigma}_{42}+\hat{u}_3\hat{\sigma}_{43}+\hat{u}_4\hat{\sigma}_{44})]\end{aligned} \tag{33}$$

where $\hat{a}_i=\hat{g}_1\hat{\sigma}_{i1}+\hat{g}_2\hat{\sigma}_{i2}+\hat{g}_3\hat{\sigma}_{i3}+\hat{g}_4\hat{\sigma}_{i4}, i=1,2,3,4$ $\hat{a}_5=2(\hat{u}_{12}\hat{\sigma}_{12}+\hat{u}_{13}\hat{\sigma}_{13}+\hat{u}_{14}\hat{\sigma}_{14}+\hat{u}_{23}\hat{\sigma}_{23}+\hat{u}_{24}\hat{\sigma}_{24}+\hat{u}_{34}\hat{\sigma}_{34})$ $\hat{a}_6=\frac{1}{2}(\hat{u}_{11}\hat{\sigma}_{11}+\hat{u}_{22}\hat{\sigma}_{22}+\hat{u}_{33}\hat{\sigma}_{33}+\hat{u}_{44}\hat{\sigma}_{44})$ $\hat{A}_i=\hat{L}_{11i}\hat{\sigma}_{11}+\hat{L}_{22i}\hat{\sigma}_{22}+\hat{L}_{33i}\hat{\sigma}_{33}+\hat{L}_{44i}\hat{\sigma}_{44}+2(\hat{L}_{12i}\hat{\sigma}_{12}+\hat{L}_{13i}\hat{\sigma}_{13}+\hat{L}_{14i}\hat{\sigma}_{14}+\hat{L}_{23i}\hat{\sigma}_{23}+\hat{L}_{24i}\hat{\sigma}_{24}+\hat{L}_{34i}\hat{\sigma}_{34})$ Put $\theta_1=\hat{\alpha},\theta_2=\hat{\gamma},\theta_3=\hat{\beta},\theta_4=\hat{\lambda},\ \hat{g}_i=\frac{\partial G(\alpha,\gamma,\beta,\lambda)}{\partial\theta_i}, i=1,2,3,4$

$$\begin{aligned}G(\alpha,\gamma,\beta,\lambda)&=\log\pi(\alpha,\gamma,\beta,\lambda)\\&=(a_1-1)\log\alpha+(a_2-1)\log\gamma+(a_3-1)\log\beta+(a_4-1)\log\lambda-(\alpha b_1+\gamma b_2+\beta b_3+\lambda b_4)\end{aligned} \tag{34}$$

$$\hat{u}_i=\frac{\partial u(\theta_1,\theta_2,\theta_3,\theta_4)}{\partial\theta_i},\ \hat{u}_{ij}=\frac{\partial^2 u(\theta_1,\theta_2,\theta_3,\theta_4)}{\partial\theta_i\partial\theta_j},\ i,j=1,2,3,4$$

$$\hat{L}_{ij}=\frac{\partial^2 L(\theta_1,\theta_2,\theta_3,\theta_4)}{\partial\theta_i\partial\theta_j}, i,j=1,2,3,4$$

$$\hat{L}_{ijk}=\frac{\partial^3 L(\theta_1,\theta_2,\theta_3,\theta_4)}{\partial\theta_i\partial\theta_j\partial\theta_k}, i,j,k=1,2,3,4$$

$$\hat{\sigma}_{ij}=\begin{cases}\frac{-1}{\hat{L}_{ij}}, i=j\\ 0, i\neq j\end{cases}$$

Now the values of the Bayes estimates of various parameters can be obtained by used Lindley Approximation under symmetric and asymmetric loss function. For more details see Zamani [15] and Soliman [13].

4.2.1 Symmetric Bayes estimation

1.Squared error loss function (SE) The Bayes estimators of the parameters α,γ,β and λ under squared error loss function are

$$\hat{\alpha}_{SE}=E(\alpha|t),\hat{\gamma}_{SE}=E(\gamma|t),\hat{\beta}_{SE}=E(\beta|t),\hat{\lambda}_{SE}=E(\lambda|t). \tag{35}$$

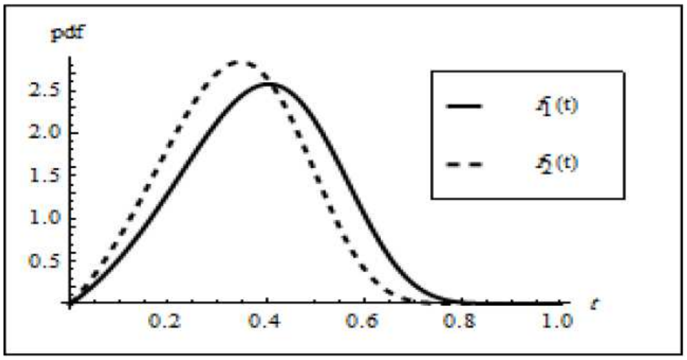

Fig. 1: PDFs under normal and accelerated conditions

4.2.2 Asymmetric Bayes estimation

1.Linex loss function The Bayes estimators of the parameters α, γ, β and λ under Linex loss function are

$$\hat{\alpha}_{Linex} = -\frac{1}{c}\log E(e^{-c\alpha}|t), \hat{\gamma}_{Linex} = -\frac{1}{c}\log E(e^{-c\gamma}|t),$$
$$\hat{\beta}_{Linex} = -\frac{1}{c}\log E(e^{-c\beta}|t), \hat{\lambda}_{Linex} = -\frac{1}{c}\log E(e^{-c\lambda}|t). \quad (36)$$

2.General entropy loss function

The Bayes estimators of the parameters α, γ, β and λ under General Entropy (GE) loss function are

$$\hat{\alpha}_{GE} = (E(\alpha^{-c}|t))^{-1/c}, \hat{\gamma}_{GE} = (E(\gamma^{-c}|t))^{-1/c},$$
$$\hat{\beta}_{GE} = (E(\beta^{-c}|t))^{-1/c}, \hat{\lambda}_{GE} = (E(\lambda^{-c}|t))^{-1/c}. \quad (37)$$

5 Numerical Example

In this section for illustrative purposes, we present a simulation example to check the estimation procedures. In this example, by using the algorithm described in Balakrishnan and Sandhu [3], we generate two samples from WE(α, γ, β) distribution with parameters $\alpha = 0.5$, $\gamma = 2$, $\beta = 2$ and $\lambda = 1.5$, using progressive censoring scheme (CSs) $n_1 = n_2 = 50$, $m_1 = 20$, $m_2 = 30$, R_1=(5, 0, 0, 5, 0, 0, 3, 0, 0, 5, 2, 2, 2, 1, 1, 1, 1, 1, 1, 0) and R_2= (3, 0, 0, 0, 2, 0, 0, 0, 2, 0, 0, 0, 2, 0, 0, 3, 0, 1, 0, 1, 0, 1, 0, 0, 2, 0, 1, 0, 2, 0). The following two progressively censored data sets were observed:
Data Set 1: 0.1124, 0.1868, 0.2994, 0.3077, 0.3107, 0.3419,0.3755, 0.3775, 0.3799,0.3827, 0.3970, 0.4417,0.4788, 0.5074, 0.5749,0.6107, 0.6546, 0.7651,0.7696, 0.8122.
Data Set 2: 0.0471, 0.1578, 0.1881, 0.2148, 0.2516, 0.2866, 0.2994,0.3075, 0.3110, 0.3455, 0.3536, 0.3587, 0.3637, 0.3879, 0.3940, 0.4008, 0.4232, 0.4238, 0.4616, 0.4668, 0.4935,0.5092, 0.5716, 0.5922, 0.6111, 0.6445, 0.6500, 0.6564,0.6637, 0.7307.

Figure 1 plots the PDFs under normal and accelerated conditions. Newton–Raphson iteration method used to obtain the MLEs of WED parameters.Denote the estimates using MLEs, the bootstrap-p,bootstrap-t, Bayes estimate according to MCMC, Bayes estimate according to SE loss function, LINEX loss function, and GE loss function, respectively by MLE, $Boot_p$, $Boot_t$, $MCMC$, $Lindley_{SE}$, $Lindley_{Linex}$ and $Lindley_{GE}$.

Table 1. Different point estimates for $(\alpha, \gamma, \beta, \lambda) = (0.5, 2, 2, 1.5)$

Parameters	α	γ	β	λ
MLE	0.1901	1.9071	1.8308	1.6789
$Boot_p$	0.2237	1.9486	1.9353	1.2973
$Boot_t$	0.1495	1.6402	1.8345	1.6383
$MCMC$	0.2119	1.8448	1.8799	1.4522
$Lindley_{SE}$	0.1896	1.8057	1.8375	1.5245
$Lindley_{Linex}$ (c=0.5)	0.1894	1.8042	1.8330	1.5085
$Lindley_{GE}$ (c=0.5)	0.1868	1.8032	1.8306	1.4985
$Lindley_{Linex}$ (c=-0.5)	0.1898	1.8072	1.8424	1.5435
$Lindley_{GE}$ (c=-0.5)	0.1886	1.8048	1.8351	1.5147

Table 2. 95% confidence intervals for α and γ

Method	α	Length	γ	Length
ACI	[-0.8909,1.271]	2.16189	[-1.9884,5.6501]	7.63848
$Boot_p$ CI	[0.0433,0.8435]	0.80018	[1.0487,2.9498]	1.90117
$Boot_t$ CI	[0.0286,0.1858]	0.15721	[1.5453,1.7598]	0.21453
CRI	[0.1374,0.3041]	0.16670	[1.8301,1.8603]	0.03019

Table 3. 95% confidence intervals for β and λ

Method	β	Length	λ	Length
ACI	[0.4181,3.3961]	2.97805	[0.717,2.6409]	1.92387
$Boot_p$ CI	[1.4402,2.6574]	1.21722	[1.0084,1.8777]	0.8693
$Boot_t$ CI	[1.7762,1.8677]	0.09159	[1.6142,1.6562]	0.04198
CRI	[1.8631,1.8967]	0.03357	[0.8386,2.3268]	1.48819

Table 4. 90% confidence intervals for α and γ

Method	α	Length	γ	Length
ACI	[-0.7171,1.0972]	1.81432	[-1.3744,5.036]	6.41042
$Boot_p$ CI	[0.0452,0.6866]	0.64141	[1.1301,2.8601]	1.7300
$Boot_t$ CI	[0.0466,0.1844]	0.13776	[1.5661,1.7443]	0.17812
CRI	[0.1428,0.2734]	0.13062	[1.8371,1.8912]	0.05410

Table 5. 90% confidence intervals for β and λ

Method	β	Length	λ	Length
ACI	[0.6575,3.1568]	2.49926	[0.8717,2.4862]	1.61456
$Boot_p$ CI	[1.4731,2.5297]	1.0566	[1.0227,1.7637]	0.7410
$Boot_t$ CI	[1.7888,1.863]	0.07416	[1.6187,1.6546]	0.03594
CRI	[1.8798,1.9113]	0.03145	[0.9154,2.1195]	1.20408

6 Conclusion

Based on PRO-II-C samples, this paper is related to full Bayes and non-Bayes procedures for the analysis of the CSPALT using the WED failure model. The classical Bayes estimates cannot be obtained in explicit form. One can clearly see the scope of MCMC-based Bayesian solutions which make every inferential development routinely available. In this paper, we have considered the maximum likelihood and Bayes estimates for the parameters of WED using PRO-II-C scheme. This paper also studied the construction of confidence intervals for the parameters and acceleration factor by using the parametric bootstrap methods. It is well known that when all parameters are unknown, the Bayes estimates cannot be obtained in explicit form. We used the MCMC and Lindley's techniques to compute the approximate Bayes estimates and the corresponding credible intervals. A numerical example using the simulated data set is presented to illustrate how the MCMC, Lindley's and parametric bootstrap methods work based progressive censored data.

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