Journal of Analysis & Number Theory
An International Journal

http://dx.doi.org/10.18576/jant/050204

Ascertainment of The Certain Fundamental Units in a Specific Type of Real Quadratic Fields

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Received: 7 Mar. 2017, Revised: 24 Jun. 2017, Accepted: 27 Jun. 2017
Published online: 1 Jul. 2017

Abstract: The aim of the paper is to classify the real quadratic number fields $Q(\sqrt{d})$ having specific form of continued fraction expansions of algebraic integer w_d and is to determine the general explicit parametric representation of the fundamental unit ε_d for such real quadratic number fields where $d \equiv 2,3(mod4)$ is a square free positive integer. Also, Yokoi's d-invariants n_d and m_d will be calculated in the relation to continued fraction expansion of w_d for such real quadratic fields.

Keywords: Continued Fraction Expansion, Fundamental Unit, Quadratic Field.
AMS Subject Classification: 11A55, 11R27, 11R11

1 Introduction

In Number Theory, real quadratic number fields have great importance. Many researchers have obtained their results on the real quadratic number fields ([1]-[22]). The author ([7]-[12]) considered some types of real quadratic fields and determined their fundamental unit as well as Yokoi's invariants. The purpose of this paper is to study on a particular real quadratic field and determine to classification of real quadratic fields including the continued fraction expansion which has got the partial quotients are equal to 8 in the symmetric part of period length by considering ([7]-[12]). Also, Yokoi's invariants are calculated and presented in tables.

In any $k = Q(\sqrt{d})$ real quadratic number field, integral basis element of algebraic integers ring in real quadratic fields is determined by $w_d = \sqrt{d} = [a_0; \overline{a_1, a_2, \ldots, a_{\ell(d)-1}, 2a_0}]$ in the case of $d \equiv 2,3(mod4)$ where $\ell = \ell(d)$ is the period length of continued fraction expansion. The fundamental unit ε_d of real quadratic number field is also denoted by $\varepsilon_d = (t_d + u_d\sqrt{d})/2 > 1$. Also, Yokoi's invariants are expressed by $n_d = \left[\left[\frac{t_d}{u_d^2}\right]\right]$ and $m_d = \left[\left[\frac{u_d^2}{t_d}\right]\right]$ where $[[x]]$ represents the greatest integer less than or equal to x.

2 Preliminaries

Definition 1. *$\{R_i\}$ sequence is defined by recurrence relation as follows:*

$$R_i = 8R_{i-1} + R_{i-2}$$

for $i \geq 2$ with initial values $R_0 = 0$ and $R_1 = 1$.

Definition 2. *Let $c_n = ac_{n-1} + bc_{n-2}$ recurrence relation of $\{c_n\}$ sequence where a, b are real numbers. The polynomial is called as a characteristic equation if it is written in the form:*

$$x^2 - ax - b = 0$$

For our sequence, it can be written for each element of sequence as follows:

$$R_k = \frac{1}{2\sqrt{17}}\left[(4+\sqrt{17})^k - (4-\sqrt{17})^k\right]$$

for $k \geq 1$

Remark. Let $\{R_n\}$ be the sequence defined as in Definition 1. Then, we state the following:

$$R_n = \begin{cases} 0(\text{mod } 4), & n \equiv 0(mod2); \\ 1(\text{mod } 4), & n \equiv 1(mod2). \end{cases}$$

for $n \geq 0$.

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Lemma 1. *For a square-free positive integer d congruent to 2,3 modulo 4, we put $w_d = \sqrt{d}$, $a_0 = [w_d]$, $w_R = a_0 + w_d$. Then $w_d \notin R(d)$, but $w_R \in R(d)$ holds. Moreover for the period $\ell = \ell(d)$ of w_R, we get $w_R = [\overline{2a_0, a_1, \ldots, a_{\ell-1}}]$ and $w_d = [a_0, \overline{a_1, \ldots, a_{\ell-1}, 2a_0}]$. Furthermore, let $w_R = \dfrac{(P_\ell w_R + P_{\ell-1})}{Q_\ell w_R + Q_{\ell-1}} = [2a_0, a_1, \cdots, a_{\ell-1}, w_R]$ be a modular automorphism of w_R, then the fundamental unit ε_d of $Q(\sqrt{d})$ is given by the following formula:*

$$\varepsilon_d = \frac{t_d + u_d\sqrt{d}}{2} = (a_0 + \sqrt{d})Q_{\ell(d)} + Q_{\ell(d-1)} > 1$$

and

$$t_d = 2a_0.Q_{\ell(d)} + 2Q_{\ell(d)-1},\ u_d = 2Q_{\ell(d)}.$$

where Q_i is determined by $Q_0 = 0, Q_1 = 1$ and $Q_{i+1} = a_iQ_i + Q_{i-1}, (i \geq 1)$.

Proof. Proof is omitted in [[16], Lemma 1]

Lemma 2. *Let d be the square free positive integer congruent to 2,3 modulo 4. We will consider w_d which has got partial constant elements repeated 8s in the case of period $\ell = \ell(d)$. If we let a_0 denote the $a_0 = [[\sqrt{d}]]$ the integer part of w_d for d congruent to 2,3(mod 4), then we have continued fration expansions*

$$w_d = \sqrt{d} = [a_0; \overline{a_1, a_2, \ldots, a_{\ell(d)-1}, a_{\ell(d)}}] = [a_0; \overline{8, 8, \cdots, 8, 2a_0}]$$

for quadratic irrational numbers and $w_R = a_0 + \sqrt{d} = [\overline{2a_0, 8, \ldots, 8}]$ for reduced quadratic irrational numbers.
In the continued fraction $w_R = a_0 + \sqrt{d} = [b_1, b_2, \ldots, b_n, \ldots] = [2a_0, 8, \ldots, 8, \ldots], P_k = 2a_0R_k + R_{k-1}$ and $Q_k = R_k$ are determined in the continued fraction expansion where P_k and Q_k are two sequences defined by:

$$P_{-1} = 0, P_0 = 1, P_{j+1} = b_{j+1}.P_j + P_{j-1},$$

$$Q_{-1} = 1, Q_0 = 0, Q_{j+1} = b_{j+1}.Q_j + Q_{j-1},$$

for $j \geq 0$

Proof. It can be proved easily by considering references ([7]-[12]).

3 Main Results

Theorem 1. *Let d be a square free positive integer and ℓ be a positive integer satisfying that $\ell \geq 2$. Suppose that parameterizations of d is*

$$d = \mu^2 R_\ell^2 + \mu(8R_\ell + R_{\ell-1}) + 17$$

for $\mu \geq 1$ integer. If μ is odd positive integer, we have $d \equiv 2, 3(mod4)$ and

$$w_d = \left[\mu R_\ell + 4; \underbrace{\overline{8, 8, \ldots, 8}}_{\ell-1}, \overline{8 + 2\mu R_\ell}\right]$$

with $\ell = \ell(d)$. Moreover, we can get fundamental unit ε_d, coefficients of fundamental unit t_d, u_d as follows:

$$\varepsilon_d = (\mu R_\ell + 4)R_\ell + R_{\ell-1} + R_\ell\sqrt{d},$$

$$t_d = 2(\mu R_\ell + 4)R_\ell + 2R_{\ell-1} \text{ and } \mu_d = 2R_\ell.$$

Proof. Let $\ell \geq 2$ be the positive integer. Using Remark, we get $R_\ell \equiv 1(mod4)$ and $R_{\ell-1} \equiv 0(mod4)$ for $\ell \equiv 1(mod4)$ and $\ell \equiv 3(mod4)$. By considering μ is odd positive integer and substituting these equalities into the $d = \mu^2R_\ell^2 + \mu(8R_\ell + R_{\ell-1}) + 17$, we get $d \equiv 2(mod4)$.
Also, we have that $R_\ell \equiv 0(mod4)$ and $R_{\ell-1} \equiv 1(mod4)$ for both $\ell \equiv 0(mod4)$ and $\ell \equiv 2(mod4)$. By considering μ is odd positive integer and substituting these equalities into the $d = \mu^2R_\ell^2 + \mu(8R_\ell + R_{\ell-1}) + 17$, so $d \equiv 3(mod4)$ holds.
On substituting w_d into the w_R, we get

$$w_R = (\mu R_\ell + 4) + \left[\mu R_\ell + 4; \underbrace{\overline{8, 8, \ldots, 8}}_{\ell-1}, \overline{8 + 2\mu R_\ell}\right]$$

and we have

$$w_R = (2\mu R_\ell + 8) + \cfrac{1}{8 + \cfrac{1}{8 + \cfrac{1}{\ddots \cfrac{1}{8 + \cfrac{1}{w_R}}}}}$$

$$= (2\mu R_\ell + 8) + \frac{1}{8}_{+\ldots+} \frac{1}{w_R}$$

Using Lemma 1. and Lemma 2. about the properties of continued fraction expansion, we get

$$w_R = (2\mu R_\ell + 8) + \frac{R_{\ell-1}w_R + R_{\ell-2}}{R_\ell w_R + R_{\ell-1}}$$

and by using Definition 1 into the above equality, we obtain,

$$w_R^2 - (2\mu R_\ell + 8)w_R - (1 + 2\mu R_{\ell-1}) = 0$$

This requires that $w_R = (\mu R_\ell + 4) + \sqrt{d}$ since $w_R > 0$. Also, using Lemma 2, we get

$$w_d = \sqrt{d} = \left[\mu R_\ell + 4; \underbrace{\overline{8, 8, \ldots, 8}}_{\ell-1}, \overline{2\mu R_\ell + 8}\right]$$

and $\ell = \ell(d)$. This completes the first part of theorem.
Now, to determine ε_d, t_d and u_d using Lemma, we get

$$Q_1 = 1 = R_1, Q_2 = a_1.Q_1 + Q_0 \Rightarrow Q_2 = 8 = R_2$$

$$Q_3 = a_2Q_2 + Q_1 = 8R_2 + R_1 = 65 = R_3, Q_4 = 528 = R_4, \cdots$$

So, this implies that $Q_i = R_i$ by using mathematical induction $\forall\ i \geq 0$. If we substitute these values of the sequence into the

$\varepsilon_d = \frac{t_d + u_d\sqrt{d}}{2} = (a_0 + \sqrt{d})Q_{\ell(d)} + Q_{\ell(d)-1} > 1$ and rearranging, we will get

$$\varepsilon_d = (\mu R_\ell + 4)R_\ell + R_{\ell-1} + R_\ell\sqrt{d},$$

$$t_d = 2(\mu R_\ell + 4)R_\ell + 2R_{\ell-1} \text{ and } u_d = 2R_\ell.$$

Corollary 1.*If d is a square free positive integer and ℓ is a positive integer satisfying that $\ell \geq 2$ as well as parametrization of d as follows:*

$$d = R_\ell^2 + 2(4R_\ell + R_{\ell-1}) + 17$$

then, we have $d \equiv 2,3(mod4)$ and

$$w_d = \left[R_\ell + 4; \underbrace{\overline{8,8,\ldots,8}}_{\ell-1}, \overline{2\mu R_\ell + 8}\right]$$

with $\ell = \ell(d)$. Also, fundamental unit ε_d, coefficients of fundamental unit t_d, u_d and Yokoi invariant as follows:

$$\varepsilon_d = (R_\ell + 4)R_\ell + R_{\ell-1} + R_\ell\sqrt{d},$$

$$t_d = 2(R_\ell + 4)R_\ell + 2R_{\ell-1} \text{ and } u_d = 2R_\ell$$

Also, we have value of Yokoi's d-invariant $m_d = 1$.

*Proof.*This corollary is obtained by using Theorem 1 with taking $\mu = 1$. So, we should determine value of the Yokoi's invariant m_d. We know that $m_d = \left[\left[\frac{u_d^2}{t_d}\right]\right]$ from H. Yokoi's references. If we substitute t_d and u_d into the m_d, then we get

$$m_d = \left[\left[\frac{u_d^2}{t_d}\right]\right] = \left[\left[\frac{4R_\ell^2}{2R_\ell^2 + 8R_\ell + 2R_{\ell-1}}\right]\right] = 1,$$

since R_l increasing sequence and $1,984 < \frac{4R_\ell^2}{2R_\ell^2 + 8R_\ell + 2R_{\ell-1}} < 2$ for $\ell \geq 2$. Therefore we obtain $m_d = 1$ for $\ell \geq 2$ owing to definition of m_d. Besides, Table 1 is given as numerical illustrates. (In this table, $\ell(d) = 2,3,6,7,8$ are ruled out since d is not a square free positive integer in these periods).

Corollary 2.*If d is a square free positive integer and ℓ is a positive integer satisfying that $\ell \geq 2$ as well as parametrization of d is*

$$d = 9R_\ell^2 + 24R_\ell + 6R_{\ell-1} + 17$$

then, we have $d \equiv 2,3(mod4)$ and

$$w_d = \left[3R_\ell + 4; \underbrace{\overline{8,8,\ldots,8}}_{\ell-1}, \overline{8 + 6R_\ell}\right]$$

with $\ell = \ell(d)$. Additionally, we obtain fundamental unit ε_d, coefficients of fundamental unit t_d, u_d as follows:

$$\varepsilon_d = (3R_\ell + 4)R_\ell + R_{\ell-1} + R_\ell\sqrt{d},$$

$$t_d = 2(3R_\ell + 4)R_\ell + 2R_{\ell-1} \text{ and } u_d = 2R_\ell$$

Also, we have Yokoi's d-invariant value $n_d = 1$

*Proof.*This corollary is obtained by using theorem 1 for $\mu = 3$. In the same manner we obtain $n_d = 1$ for $\ell \geq 2$ owing to definition of n_d. Besides, following Table 2 gives an example for this corollary. (In this table, we also rule out $\ell(d) = 3$ since d is not a square free positive integer in this period).

Theorem 2.*Let d be the square free positive integer and ℓ be a positive integer holding that $\ell = 0(mod2)$ and $\ell \geq 1$. We assume that parametrization of d is*

$$d = \frac{\mu^2 R_\ell^2}{4} + (4R_\ell + R_{\ell-1})\mu + 17.$$

for $\mu > 0$ positive integer. If $\mu \equiv 1(mod4)$ positive integer then $d \equiv 2(mod4)$ and

$$w_d = \left[\frac{\mu R_\ell}{2} + 4; \underbrace{\overline{8,8,\ldots,8}}_{\ell-1}, \overline{\mu R_\ell + 8}\right]$$

holds for $\ell = \ell(d)$. Moreover, the following equalities also hold:

$$\varepsilon_d = \left(\frac{\mu R_\ell^2}{2} + 4R_\ell + R_{\ell-1}\right) + R_\ell\sqrt{d}$$

$$t_d = \mu R_\ell^2 + 8R_\ell + 2R_{\ell-1} \text{ and } u_d = 2R_\ell$$

for ε_d, t_d and u_d.

*Proof.*Let $\ell \equiv 0(mod2)$ and $\ell > 1$ hold. If $\ell \equiv 0(mod2)$ holds then we have $R_\ell \equiv 0(mod4), R_{\ell-1} \equiv 1(mod4)$. Considering $\mu \equiv 1(mod4)$ positive integer and substituting these equivalent and equations into the parameterization of d then we get $d \equiv 2(mod4)$.
Using Lemma 2, we put

$$w_R = \frac{\mu R_\ell}{2} + 4 + \left[\frac{\mu R_\ell}{2} + 4; \underbrace{\overline{8,8,\ldots,8}}_{\ell-1}, \overline{\mu R_\ell + 8}\right],$$

we get

$$w_R = (\mu R_\ell + 8) + \cfrac{1}{8 + \cfrac{1}{8 + \cfrac{1}{\ddots \cfrac{1}{8 + \cfrac{1}{w_R}}}}}$$

$$= (\mu R_\ell + 8) + \frac{1}{8} {}_{+\cdots+} \frac{1}{8} + \frac{1}{w_R}$$

Now by using Lemma 1 and Lemma 2 about the properties of continued fraction expansion, we get

$$w_R = (\mu R_\ell + 8) + \frac{R_{\ell-1}w_R + R_{\ell-2}}{R_\ell w_R + R_{\ell-1}}$$

by using induction and property of continued fraction expansion and the Definition 1 into the above inequality, we obtain

$$w_R^2 - (\mu R_\ell + 8)w_R - (1 + \mu R_{\ell-1}) = 0.$$

Table 1: ???

d	$\ell(d)$	m_d	w_d	ε_d
283155	4	1	$[532,\overline{8,8,8,1064}]$	$280961+528\sqrt{283155}$
18430906	5	1	$[4293,\overline{8,8,8,8,8586}]$	$18413205+4289\sqrt{18430906}$
348729821225306	9	1	$[18674309,\overline{8,8,\ldots,37348618}]$	$348729818926393+18674305\sqrt{348729821225306}$
23010874291891347	10	1	$[151693356,\overline{8,8,\ldots,8,303386712}]$	$23010873666443617+151693352\sqrt{23010874291891347}$
1518368901199652330	11	1	$[1232221125,\overline{8,8,\ldots,2464442250}]$	$1518368806119074477+1232221121\sqrt{1518368901199652330}$

Table 2: ???

d	$\ell(d)$	n_d	w_d	ε_d
791	2	1	$[28,\overline{8,56}]$	$225+8\sqrt{791}$
2522135	4	1	$[1588,\overline{8,8,8,8,3176}]$	$838529+528\sqrt{2522135}$
165665810	5	1	$[12871,\overline{8,8,\ldots,8,209048}]$	$55204247+4289\sqrt{165665810}$
10925292311	6	1	$[104524,\overline{8,8,\ldots,8,209048}]$	$3641620449+34840\sqrt{10925292311}$
720853848002	7	1	$[849031,\overline{8,8,\ldots,8,1698062}]$	$240283449119+283009\sqrt{720853848002}$
47565024325655	8	1	$[6896740,\overline{8,8,\ldots,8,13793480}]$	$15854998629889+2298912\sqrt{47565024325655}$
3138567467074034	9	1	$[56022919,\overline{8,8,\ldots,8,112045838}]$	$1046189078695207+18674305\sqrt{3138567467074034}$
207097861121649431	10	1	$[455080060,\overline{8,8,\ldots,8,910160120}]$	$69032619748435425+151693352\sqrt{207097861121649431}$

Table 3: ???

d	$\ell(d)$	m_d	w_d	ε_d
66	2	1	$[8;\overline{8,16}]$	$65+8\sqrt{66}$
71890	4	3	$[268;\overline{8,8,8,536}]$	$141569+528\sqrt{66}$
303600066	6	3	$[17424;\overline{8,8,8,8,8,34848}]$	$607056449+34840\sqrt{303600066}$
25047334025145016018	12	3	$[5004731164;\overline{8,8,\ldots,8,10009462328}]$	$50094668009019961601+10009462320\sqrt{25047334025145016018}$

This requires that $w_R = \dfrac{\mu R_\ell}{2}+4+\sqrt{d}$ since $w_R > 0$. Considering Lemma 2, we get

$$w_d=\sqrt{d}=\left[\frac{\mu R_\ell}{2}+4;\underbrace{\overline{8,8,\cdots,8}}_{\ell-1},\overline{\mu R_\ell+8}\right]$$

and $\ell=\ell(d)$. This completes the first part of theorem. Now we should dettermine ε_d, t_d and u_d using Lemma 1, we have

$$Q_1=1=R_1, Q_2=a_1.Q_1+Q_0 \Rightarrow Q_2=8=R_2,$$

$$Q_3=a_2Q_2+Q_1=8R_2+R_1=65=R_3, Q_4=528=R_4,\cdots$$

This implies that $Q_i=R_i$ by using mathematical induction $\forall i\geq 0$. On substituting these values of sequence into the $\varepsilon_d=\dfrac{t_d+u_d\sqrt{d}}{2}=(a_0+\sqrt{d})Q_{\ell(d)}+Q_{\ell(d)-1}>1$ and rearranged, we get

$$\varepsilon_d=(\frac{\mu R_\ell^2}{2}+4R_\ell+R_{\ell-1})+R_\ell\sqrt{d}$$

$$t_d=R_\ell^2+8R_\ell+2R_{\ell-1} \text{ and } \mu_d=2R_\ell$$

for ε_d, t_d and u_d.

Corollary 3. *Let d be the square free positive integer and ℓ be a positive integer holding that $\ell\equiv 0(mod 2)$ and $\ell>1$. We assume that parameterizations of d is*

$$d=\frac{R_\ell^2}{4}+4R_\ell+R_{\ell-1}+17$$

Then we get $d\equiv 2(mod 4)$ and

$$w_d=\left[\frac{R_\ell+8}{2};\underbrace{\overline{8,8,\ldots,8}}_{\ell-1},\overline{R_\ell+8}\right]$$

and $\ell=\ell(d)$. Moreover, we have following equalities:

$$\varepsilon_d=\left(\frac{R_\ell^2}{2}+4R_\ell+R_{\ell-1}\right)+R_\ell\sqrt{d}$$

$$t_d=R_\ell^2+8R_\ell+2R_{\ell-1} \text{ and } u_d=2R_\ell$$

and

$$m_d=\begin{cases}1, \text{ if } \ell=2;\\ 3, \text{ if } \ell\geq 2.\end{cases}$$

Proof. We obtain this corollary by taking $\mu=1$ in Theorem 2. In a similar way, we get,

$$4>4.\left(1+\frac{8}{R_\ell}+\frac{2R_{\ell-1}}{R_\ell^2}\right)^{-1}>3,938$$

for $l\geq 4$. Therefore, we obtain

$$m_d=\left[\left[\frac{4R_\ell^2}{R_\ell^2+8R_\ell+2R_{\ell-1}}\right]\right]=3$$

for $\ell \geq 4$. Table 3 shows some numerical examples for Corollary 3. (In this table we rule out $\ell(d) = 8, 10$ since d is not a square free positive integer in these periods).

4 conclusion

In this paper, we introduced the notion of real quadratic field structures such as continued fraction expansions, fundamental unit and Yokoi invariants in the terms of special sequence. We established a practical method so as to rapidly determine continued fraction of w_d, fundamental unit ε_d and Yokoi invariants n_d, m_d for classified such real quadratic number fields.

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