Appl. Math. Inf. Sci. Lett. 5, No. 2, 63-69 (2017)

Applied Mathematics & Information Sciences Letters
An International Journal

http://dx.doi.org/10.18576/amisl/050205

Some Coding Theorems on New Generalized Fuzzy Entropy of Order α and Type β

Ashiq Hussain Bhat and M. A. K Baig*

P. G. Department of Statistics University of Kashmir, Srinagar-190006, India

Received: 7 Feb. 2016, Revised: 16 Aug. 2016, Accepted: 16 Aug. 2016
Published online: 1 May 2017

Abstract: In this communication we introduce a new generalized fuzzy information measure $H_{\beta}^{\alpha}(A)$ of order α and type β for a fuzzy set A and establishes the validity of the measure as a fuzzy entropy. Also we define a new generalized fuzzy average code-word length $L_{\beta}^{\alpha}(A)$ of order α and type β for a fuzzy set A and its relationship with generalized fuzzy information measures $H_{\beta}^{\alpha}(A)$ have been discussed. Using $L_{\beta}^{\alpha}(P)$, some coding theorems for discrete noiseless channel has been proved. The measures defined in this communication are not only new but some known measures are the particular cases of our proposed measures that already exist in the literature of fuzzy information theory.

Keywords: Fuzzy set, Membership function, Shannons entropy, Reynis entropy, Fuzzy entropy, Code-word length, Kraft inequality, Coding theorem, Holders inequality and Optimal code length.
AMS classification 94A17, 94A24

1 Introduction:

Fuzziness and uncertainty are the basic nature of human thinking and many real world objectives. Fuzziness is found in our decision, in our language and in the way of process information. The main objective of information is to remove uncertainty and fuzziness. In fact, we measure information supplied by the amount of probabilistic uncertainty removed in an experiment and the measure of uncertainty removed is also called as a measure of information, while measure of fuzziness is the measure of vagueness and ambiguity of uncertainties. The concept of entropy has been widely used in different areas, e.g. communication theory, statistical mechanics, finance pattern recognition, and neural network etc. Fuzzy set theory developed by Lotfi. A. Zadeh [29] has found wide applications in many areas of science and technology, e.g. clustering, image processing, decision making etc. because of its capability to model non-statistical imprecision or vague concepts. The importance of fuzzy sets comes from the fact that it can deal with imprecise and inexact information, many fuzzy measures have been discussed and derived by Kapur [11], Lowen [15], Nguyen and Walker [19], Parkash [24], Pal and Bezdek [22], Zadeh [29] etc. Application of fuzzy measures to engineering, fuzzy traffic control, fuzzy aircraft control, medicines, computer science and decision making etc, have already been established. The basic noiseless coding theorems see for instance; the papers, Aczel [1], Kapur [10], Khan et al [13], Renyi [25], Van Der Lubbe [28], give the lower bound for the mean code-word length of a uniquely decipherable code in terms of Shannons [26] entropy. Kapur [12] has established relationships between probability entropy and coding. But there are situations where probabilistic measures of entropy do not work, to tackle such situations, instead of taking the probability, the idea of fuzziness can be explored.

Let a universe of discourse $X = \{x_1, x_2, ..., x_n\}$ then a fuzzy subset of universe X is defined as:

$$A = \{(x_i, \mu_A(x_i)) : x_i \in X, \mu_A(x_i) \in [0,1]\}$$

Where $\mu_A(x_i) : X \to [0,1]$ is a membership function and gives the degree of belongingness of the element x_i to the set A and is defined as follows:

$$\mu_A(x_i) = \begin{cases} 0 & \text{If } x_i \notin A \text{ and there is no ambiguity,} \\ 1 & \text{If } x_i \in A \text{ and there is no ambiguity,} \\ 0.5 & \text{If } x_i \in A \text{ or } x_i \notin A \text{ and there is maximum ambiguity,} \end{cases}$$

* Corresponding author e-mail: ashiqhb14@gmail.com

In fact $\mu_A(x_i)$ associates with each $x_i \in, X$ gives a grade of membership function in the set A. When $\mu_A(x_i)$ takes values only 0 or 1, there is no uncertainty about it and a set is said to be a crisp (i.e. non-fuzzy) set. Some notions related to fuzzy sets which we shall need in our discussion Zadeh [29].

- **Containment**: If $A \subset B \Leftrightarrow \mu_A(x_i) \leq \mu_B(x_i) \forall x_i \in X$
- **Equality** : If $A = B \Leftrightarrow \mu_A(x_i) = \mu_B(x_i) \forall x_i \in X$
- **Complement**: If A^c is complement of A $\Leftrightarrow \mu_{A^c}(x_i) = (1-\mu_A(x_i)) \forall x_i \in X$
- **Union**: If $A \cup B$ is union of A and B $\Leftrightarrow \mu_{A\cup B}(x_i) = Max\{\mu_A(x_i), \mu_B(x_i)\} \forall x_i \in X$
- **Intersection**: If $A \cap B$ is intersection of A and B $\Leftrightarrow \mu_{A\cap B}(x_i) = Min\{\mu_A(x_i), \mu_B(x_i)\} \forall x_i \in X$
- **Product**: If AB is product of A and B $\Leftrightarrow \mu_{AB}(x_i) = \mu_A(x_i)\mu_B(x_i) \forall x_i \in X$
- **Sum**: If $A+B$ is sum of A and B $\Leftrightarrow \mu_{A+B}(x_i) = \mu_A(x_i) + \mu_B(x_i) - \mu_A(x_i)\mu_B(x_i) \forall x_i \in X$

2 Basic Concepts:

Let X be a discrete random variable taking values $x_1, x_2, ..., x_n$ with respective probabilities $p_1, p_2, ..., p_n$ $p_i \geq 0 \ \forall i = 1,2,3,...,n$ and $\sum_{i=1}^{n} p_i = 1$. Shannon[26] gives the following measure of information and call it entropy.

$$H(P) = -\sum_{i=1}^{n} p_i \log_D p_i \tag{1}$$

The measure (1) serves as a suitable measure of entropy. Let $p_1, p_2, p_3, , p_n$ be the probabilities of n codewords to be transmitted and let their lengths $l_1, l_2, , l_n$ satisfy Kraft [14] inequality,

$$\sum_{i=1}^{n} D^{-l_i} \leq 1 \tag{2}$$

For uniquely decipherable codes, Shannon[26] showed that for all codes satisfying (2),the lower bound of the mean codeword length,

$$L = \sum_{i=1}^{n} p_i l_i \tag{3}$$

lies between H(P) and H(P)+1.Where D is the size of code alphabet.

If $x_1, x_2, ..., x_n$ are members of the universe of discourse, with respective membership functions $\mu_A(x_1), \mu_A(x_2), ..., \mu_A(x_n)$,then all $\mu_A(x_1)$, $\mu_A(x_2)$, $..., \mu_A(x_n)$ lies between 0 and 1 but these are not probabilities because their sum is not unity. $\mu_A(x_i)$, gives the element x_i the degree of belongingness to the set A The function $\mu_A(x_i)$ associates with each $x_i \in R^n$ a grade of membership to the set A and is known as membership function.

Denote

$$F.S = \begin{bmatrix} x_1 & x_2 & \ldots & x_n \\ \mu_A(x_1) & \mu_A(x_2) & \ldots & \mu_A(x_n) \end{bmatrix}, 0 \leq \mu_A(x_i) \leq 1 \forall x_i \in X \tag{4}$$

We call the scheme (4) as a finite fuzzy information scheme. Every finite scheme describes a state of uncertainty, corresponding to Shannons [26] probabilistic entropy, De-Luca and Termini [7] suggested the following measure of fuzzy entropy.

$$H(A) = -\frac{1}{n}\sum_{i=1}^{n}\left[\mu_A(x_i)\log\mu_A(x_i) + (1-\mu_A(x_i))\log(1-\mu_A(x_i))\right] \tag{5}$$

The measure (5) serves as a very suitable measure of fuzzy entropy for the finite fuzzy information scheme (4)

De-Luca and Termini [7] introduced a set of four properties and these properties are widely accepted as for defining new fuzzy entropy. In fuzzy set theory, the entropy is a measure of fuzziness which expresses the amount of average ambiguity in making a decision whether an element belongs to a set or not. So, a measure of average fuzziness $H(A)$ in a fuzzy set A should have the following properties to be valid fuzzy entropy:

1. **Sharpness**: $H(A)$ is minimum if and only if A is a crisp set, i,e,.
$$\mu_A(x_i) = 0 \text{ or } 1; \forall x_i, i = 1,2,...,n$$
2. **Maximality**: $H(A)$ is maximum if and only if A is most fuzzy set, i.e.,
$$\mu_A(x_i) = \frac{1}{2}; \forall x_i, i = 1,2,...,n$$
3. **Resolution**: $H(A^*) \leq H(A)$, where A^* is sharpened version of A.
4. **Symmetry**: $H(A) = H(A^c)$, where A^c is the complement of A, i.e.,
$$\mu_{A^c}(x_i) = 1-\mu_A(x_i); \forall x_i, i = 1,2,...,n$$

Later on Bhandari and Pal [4] made a survey on information measures on fuzzy sets and gave some fuzzy information measures, analogous to Reynis [25] information measure; they have suggested the following fuzzy information measure of order α as:

$$H_\alpha(A) = \frac{1}{1-\alpha}\sum_{i=1}^{n}\log\left[\mu_A^\alpha(x_i) + (1-\mu_A(x_i))^\alpha\right]; \alpha \neq 1, \alpha \geq 0 \tag{6}$$

and analogous to Pal and Pals [23] exponential entropy, they have suggested the following fuzzy information measure:

$$H_e(A) = \frac{1}{n\sqrt{e}-1}\sum_{i=1}^{n}\log\left[\mu_A(x_i)e^{1-\mu_A(x_i)} + (1-\mu_A(x_i))e^{\mu_A(x_i)} - 1\right] \tag{7}$$

Analogous to Havrda and Charvats [8] information measure, Kapur [11] suggests the following fuzzy information measure:

$$H^\alpha(A) = \frac{1}{1-\alpha}\sum_{i=1}^{n}\left[\mu_A^\alpha(x_i) + (1-\mu_A(x_i))^\alpha - 1\right]; \alpha \neq 1, \alpha \geq 0 \tag{8}$$

Corresponding to Boekee and Lubbe [5] R-norm information measure, Hooda [9] proposed and characterize the following fuzzy information measure:

$$H_R(A) = \frac{R}{R-1}\left[\sum_{i=1}^{n} 1 - \left(\mu_A^R(x_i) + (1-\mu_A(x_i))^R\right)^{\frac{1}{R}}\right]; R \neq 1, R \geq 0 \tag{9}$$

Corresponding to Campbells [6] measure of entropy, Parkash and Sharma [20] proposed and characterize the following fuzzy information measure:

$$H^{/}_{\alpha}(A)=\frac{1}{1-\alpha}\log\left[\sum_{i=1}^{n}\left\{\mu_A^{\alpha}(x_i)+(1-\mu_A(x_i))^{\alpha}\right\}^{\alpha}\right];\alpha\neq 1,\alpha\geq 0 \quad (10)$$

Corresponding to Sharma and Taneja [27] measure of entropy of degree (α,β) Kapur [11] has taken the following measure of fuzzy entropy:

$$H^{/}_{\alpha,\beta}(A)=\frac{1}{\beta-\alpha}\sum_{i=1}^{n}\left[\left\{\mu_A^{\alpha}(x_i)+(1-\mu_A(x_i))^{\alpha}\right\}-\left\{\mu_A^{\beta}(x_i)+(1-\mu_A(x_i))^{\beta}\right\}\right] \quad (11)$$

Where, $\alpha\geq 1,\beta\leq 1$ or $\alpha\leq 1,\beta\geq 1$

In the next section we propose a new generalized fuzzy information measure of order α and type β analogous to Ashiq and Baigs [3] generalized entropy of order α and type β and show that it is a valid fuzzy information measure.

3 New Generalized Fuzzy Information Measure and Its Properties:

Let A be the fuzzy set defined on a discrete universe of discourse taking values $x_1,x_2,,x_n$ having the membership values $\mu_A(x_1),\mu_A(x_2),...,\mu_A(x_n)$ respectively.

We define a new generalized fuzzy information measure of order α and type β analogous to Ashiq and Baigs [3] generalized entropy of order α and type β as:

$$H^{\beta}_{\alpha}(A)=\frac{\beta}{1-\alpha}\log_D\left[\frac{1}{n}\sum_{i=1}^{n}\left(\mu_A^{\alpha\beta}(x_i)+(1-\mu_A(x_i))^{\alpha\beta}\right)\right],0<\alpha<1,0<\beta\leq 1 \quad (12)$$

In order to prove that (12) i.e., $H^{\beta}_{\alpha}(A)$ is a valid fuzzy information measure, we shall show that four properties (P1-P4) i.e. sharpness, maximality, resolution and symmetry are satisfied.

P1.**(Sharpness)**: $H^{\beta}_{\alpha}(A)$ is minimum if and only if A is a crisp set, i,e,

$$H^{\beta}_{\alpha}(A)=0 \text{ if and only if } \mu_A(x_i)=0 \text{ or } 1\forall i=1,2,...,n$$

Proof: Let $H^{\beta}_{\alpha}(A)=0$

$$\Rightarrow\frac{\beta}{1-\alpha}\log_D\left[\frac{1}{n}\sum_{i=1}^{n}\left(\mu_A^{\alpha\beta}(x_i)+(1-\mu_A(x_i))^{\alpha\beta}\right)\right]=0$$

$$\Rightarrow\log_D\left[\frac{1}{n}\sum_{i=1}^{n}\left(\mu_A^{\alpha\beta}(x_i)+(1-\mu_A(x_i))^{\alpha\beta}\right)\right]=0 \quad (13)$$

We know that $\log_D x=0$, if $x=1$. Using this result in equation (13), we get

$$\left[\frac{1}{n}\sum_{i=1}^{n}\left(\mu_A^{\alpha\beta}(x_i)+(1-\mu_A(x_i))^{\alpha\beta}\right)\right]=1$$

Or equivalently, we can write,

$$\sum_{i=1}^{n}\left(\mu_A^{\alpha\beta}(x_i)+(1-\mu_A(x_i))^{\alpha\beta}\right)=n \quad (14)$$

Now $0<\alpha<1,0<\beta\leq 1$ (14), will hold when either $\mu_A(x_i)=1$ or $\mu_A(x_i)=0,\forall i=1,2,...,n$

Next, conversely, if A is a crisp set, then either $\mu_A(x_i)=1$ or $\mu_A(x_i)=0,\forall i=1,2,...,n$

It implies

$$\left[\frac{1}{n}\sum_{i=1}^{n}\left(\mu_A^{\alpha\beta}(x_i)+(1-\mu_A(x_i))^{\alpha\beta}\right)\right]=1,\forall 0<\alpha<1,0<\beta\leq 1$$

$$\Rightarrow\log_D\left[\frac{1}{n}\sum_{i=1}^{n}\left(\mu_A^{\alpha\beta}(x_i)+(1-\mu_A(x_i))^{\alpha\beta}\right)\right]=0$$

$$\Rightarrow\frac{\beta}{1-\alpha}\log_D\left[\frac{1}{n}\sum_{i=1}^{n}\left(\mu_A^{\alpha\beta}(x_i)+(1-\mu_A(x_i))^{\alpha\beta}\right)\right]=0$$

$$\Rightarrow H^{\beta}_{\alpha}(A)=0$$

Hence, $H^{\beta}_{\alpha}(A)=0$, if and only if A is non fuzzy set or crisp set.

P2.**(Maximality)**: $H^{\beta}_{\alpha}(A)$ is maximum if and only if A is most fuzzy set.

Proof: We have

$$H^{\beta}_{\alpha}(A)=\frac{\beta}{1-\alpha}\log_D\left[\frac{1}{n}\sum_{i=1}^{n}\left(\mu_A^{\alpha\beta}(x_i)+(1-\mu_A(x_i))^{\alpha\beta}\right)\right] \quad (15)$$

Now differentiating equation (15) with respect to $\mu_A(x_i)$, we get

$$\frac{\partial H^{\beta}_{\alpha}(A)}{\partial\mu_A(x_i)}=\frac{\alpha\beta^2}{1-\alpha}\left[\frac{\sum_{i=1}^{n}\left(\mu_A^{\alpha\beta-1}(x_i)-(1-\mu_A(x_i))^{\alpha\beta-1}\right)}{\sum_{i=1}^{n}\left(\mu_A^{\beta}(x_i)+(1-\mu_A(x_i))^{\beta}\right)}\right]$$

Let $0\leq\mu_A(x_i)<0.5$, then

$$\frac{\partial H^{\beta}_{\alpha}(A)}{\partial\mu_A(x_i)}>0,\forall 0<\alpha<1,0<\beta\leq 1.$$

Hence, $H^{\beta}_{\alpha}(A)$ is an increasing function of $\mu_A(x_i)$, whenever, $0\leq\mu_A(x_i)<0.5$

Similarly, for $0.5<\mu_A(x_i)\leq 1$ we have

$$\frac{\partial H^{\beta}_{\alpha}(A)}{\partial\mu_A(x_i)}<0,\forall 0<\alpha<1,0<\beta\leq 1.$$

Hence, $H^{\beta}_{\alpha}(A)$ is decreasing function of $\mu_A(x_i)$, whenever, $0.5<\mu_A(x_i)\leq 1$

And for $\mu_A(x_i)=0.5$

$$\frac{\partial H^{\beta}_{\alpha}(A)}{\partial\mu_A(x_i)}=0,\forall 0<\alpha<1,0<\beta\leq 1.$$

Thus $H^{\beta}_{\alpha}(A)$ is a concave function which has a global maximum at $\mu_A(x_i)=0.5$. Hence $H^{\beta}_{\alpha}(A)$ is maximum if and only if A is the most fuzzy set, i.e. $\mu_A(x_i)=0.5,\forall i=1,2,...,n$.

P3.**(Resolution)**: $H_\alpha^\beta(A) \geq H_\alpha^\beta(A^*)$, where A^* is sharped version of A.

Proof: Since $H_\alpha^\beta(A)$ is increasing function of $\mu_A(x_i)$ in the interval $[0,0.5)$ and is decreasing function of $\mu_A(x_i)$ in the interval$(0.5,1]$, therefore

$$\mu_(A^*)(x_i) \leq \mu_A(x_i) \Rightarrow H_\alpha^\beta(A^*) \leq H_\alpha^\beta(A), \text{in}\,[0,0.5),$$

and

$$\mu_(A^*)(x_i) \geq \mu_A(x_i) \Rightarrow H_\alpha^\beta(A^*) \leq H_\alpha^\beta(A), \text{in}\,(0.5,1],$$

Taking the above equations together, we get

$$H_\alpha^\beta(A) \geq H_\alpha^\beta(A^*).$$

P4.**(Symmetry)**: $H_\alpha^\beta(A) = H_\alpha^\beta(A^c)$, where A^c is the complement of A.

Proof: It may be noted that from the definition of $H_\alpha^\beta(A)$ and $\mu_{A^c}(x_i) = (1-\mu_A(x_i))$, we conclude that

$$H_\alpha^\beta(A) = H_\alpha^\beta(A^c).$$

Hence, $H_\alpha^\beta(A)$ satisfies all the properties of fuzzy entropy, therefore $H_\alpha^\beta(A)$ is a valid measure of fuzzy entropy.

It is easy to see that as $\beta = 1$ and $\alpha \to 1$ (12), reduces to (5).

In the next section new generalized fuzzy code-word mean length analogous to Ashiq and Baigs [3] new generalized code-word length of order α and type β is considered and bounds have been obtained in terms of new generalized fuzzy entropy measure $H_\alpha^\beta(A)$ analogous to Ashiq and Baigs [3] new generalized entropy of order α and type β .The main aim of these results is that it generalizes some well-known fuzzy measures already existing in the literature.

Generalized fuzzy coding theorems by considering different fuzzy information measures were investigated by several authors see for instance, the papers: M. A. K Baig and Mohd Javid Dar [16], [17] and [18], Ashiq and Baig [2], Parkash and P. K. Sharma [20] and [21], Bhandari and Pal [4], Kapur [11].

4 Bounds for New Generalized Fuzzy Information Measure:

Consider Ashiq and Baigs [3] generalized entropy of order α and type β as:

$$H_\beta^\alpha(P) = \frac{\beta}{1-\alpha}\log_D\left[\sum_{i=1}^{n} p_i^{\alpha\beta}\right] \qquad (16)$$

Where $0<\alpha<1, 0<\beta\leq 1, p_i \geq 0\ \forall i = 1,2,3,\ldots,n$ and $\sum_{i=1}^{n} p_i = 1$.

Further consider Ashiq and Baigs [3] new generalized code-word length of order α and type β corresponding to (16) and is given by

$$L_\beta^\alpha(P) = \frac{\alpha\beta}{1-\alpha}\log_D\left[\sum_{i=1}^{n} p_i^\beta D^{-l_i}\left(\frac{\alpha-1}{\alpha}\right)\right], 0<\alpha<1, 0<\beta\leq 1. \qquad (17)$$

Where D is the size of code alphabet.

Analogous to (16) and (17), we propose the following fuzzy measures respectively.

$$H_\alpha^\beta(A) = \frac{\beta}{1-\alpha}\log_D\left[\frac{1}{n}\sum_{i=1}^{n}\left(\mu_A^{\alpha\beta}(x_i) + (1-\mu_A(x_i))^{\alpha\beta}\right)\right] \qquad (18)$$

and

$$L_\alpha^\beta(A) = \frac{\beta}{1-\alpha}\log_D\left[\frac{1}{n}\sum_{i=1}^{n}\left(\mu_A^{\alpha\beta}(x_i) + (1-\mu_A(x_i))^{\alpha\beta}\right) D^{-l_i\left(\frac{\alpha-1}{\alpha}\right)}\right] \qquad (19)$$

Where,$0<\alpha<1, 0<\beta\leq 1$

Remarks for (16)

1.When $\beta = 1$ (16) reduces to Reynis entropy,i.e.,

$$H^\alpha(P) = \frac{1}{1-\alpha}\log_D\left[\sum_{i=1}^{n} p_i{}^\alpha\right]$$

2.When $\beta = 1$ and $\alpha \to 1$ (16) reduces to Shannons [26] entropy i.e.,

$$H(P) = -\sum_{i=1}^{n} p_i \log_D p_i$$

3.When $\beta = 1$, $\alpha \to 1$ and $p_i = \frac{1}{n}\ \forall i = 1,2,3,\ldots,n$ then (16) reduces to maximum entropy i.e.,

$$H(\frac{1}{n}) = \log_D n$$

Remarks for (17)

1.For $\beta = 1$ (17) reduces to code-word length corresponding to Reyni's entropy i.e.,

$$L^\alpha(P) = \frac{\alpha}{1-\alpha}\log_D\left[\sum_{i=1}^{n} p_i D^{-l_i\left(\frac{\alpha-1}{\alpha}\right)}\right]$$

2.For $\beta = 1$ and $\alpha \to 1$ (17) reduces to optimal code-word length corresponding to Shannon [26] entropy i.e.,

$$L = \sum_{i=1}^{n} p_i l_i$$

3.For $\beta = 1$ and $l_1 = l_2 = \ = l_n = 1$ then (17) reduces to 1. i.e., $L^\alpha = 1$

Now we found the bounds of (19) in terms of (18) under the condition,

$$\frac{1}{n}\sum_{i=1}^{n}\left(\mu_A^{\alpha\beta}(x_i)+(1-\mu_A(x_i))^{\alpha\beta}\right)D^{-l_i}\leq 1 \tag{20}$$

Or, we can write

$$\frac{1}{n}\sum_{i=1}^{n}\left[f\left(\mu_A(x_i),\mu_{A^c}(x_i)\right)\right]D^{l_i}\leq 1 \tag{21}$$

Where

$$f\left(\mu_A(x_i),\mu_{A^c}(x_i)\right)=\left(\mu_A^{\alpha\beta}(x_i)+(1-\mu_A(x_i))^{\alpha\beta}\right)$$

Which is generalized fuzzy Kraft [14] inequality, where D is the size of code alphabet, it is easy to see that for $\beta=1,\alpha\rightarrow 1$ the inequality (21) reduces to Kraft [14] inequality.

Theorem 4.1. For all integers $(D>1)$ the code word lengths $l_1,l_2,...,l_n$ satisfies the condition (21), then the generalized code-word length (19) satisfies the inequality

$$L_\beta^\alpha(A)\geq H_\beta^\alpha(A).\text{Where},0<\alpha<1,0<\beta\leq 1. \tag{22}$$

Where equality holds good if

$$l_i=-\log_D\left[\frac{1}{\frac{1}{n}\sum_{i=1}^{n}\left[f\left(\mu_A(x_i),\mu_{A^c}(x_i)\right)\right]}\right] \tag{23}$$

Where

$$f\left(\mu_A(x_i),\mu_{A^c}(x_i)\right)=\left(\mu_A^{\alpha\beta}(x_i)+(1-\mu_A(x_i))^{\alpha\beta}\right)$$

Proof. By Holder's Inequality we have

$$\sum_{i=1}^{n}x_iy_i\geq\left(\sum_{i=1}^{n}x_i^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n}y_i^q\right)^{\frac{1}{q}} \tag{24}$$

For all $x_i,y_i>0,i=1,2,3,...,n$ and $\frac{1}{p}+\frac{1}{q}=1,p<1(\neq 0),q<0$ or $q<1(\neq 0),p<0$. We see the equality holds iff there exists a positive constant c such that,

$$x_i^p=cy_i^q \tag{25}$$

Making the substitution

$$x_i=\left[\frac{1}{n}\left(f\left(\mu_A(x_i),\mu_{A^c}(x_i)\right)\right)\right]^{\frac{\alpha}{\alpha-1}}D^{l_i}$$

$$y_i=\left[\frac{1}{n}\left(f\left(\mu_A(x_i),\mu_{A^c}(x_i)\right)\right)\right]^{\frac{1}{1-\alpha}}$$

$$p=\frac{\alpha-1}{\alpha}\text{ and }q=1-\alpha$$

Using these values in (24), and after suitable simplification we get,

$$\frac{1}{n}\sum_{i=1}^{n}\left[f\left(\mu_A(x_i),\mu_{A^c}(x_i)\right)\right]D^{-l_i}\geq\left[\frac{1}{n}\sum_{i=1}^{n}\left\{f\left(\mu_A(x_i),\mu_{A^c}(x_i)\right)\right\}D^{-l_i\left(\frac{\alpha-1}{\alpha}\right)}\right]^{\frac{\alpha}{\alpha-1}}\left[\frac{1}{n}\sum_{i=1}^{n}\left\{f\left(\mu_A(x_i),\mu_{A^c}(x_i)\right)\right\}\right]^{\frac{1}{1-\alpha}} \tag{26}$$

Now using the inequality (21) we get,

$$\left[\frac{1}{n}\sum_{i=1}^{n}\left\{f\left(\mu_A(x_i),\mu_{A^c}(x_i)\right)\right\}D^{-l_i\left(\frac{\alpha-1}{\alpha}\right)}\right]^{\frac{\alpha}{\alpha-1}}\left[\frac{1}{n}\sum_{i=1}^{n}\left\{f\left(\mu_A(x_i),\mu_{A^c}(x_i)\right)\right\}\right]^{\frac{1}{1-\alpha}}\leq 1 \tag{27}$$

Or equation (27) can be written as

$$\left[\frac{1}{n}\sum_{i=1}^{n}\left\{f\left(\mu_A(x_i),\mu_{A^c}(x_i)\right)\right\}\right]^{\frac{1}{1-\alpha}}\leq\left[\frac{1}{n}\sum_{i=1}^{n}\left\{f\left(\mu_A(x_i),\mu_{A^c}(x_i)\right)\right\}D^{-l_i\left(\frac{\alpha-1}{\alpha}\right)}\right]^{\frac{\alpha}{1-\alpha}} \tag{28}$$

Taking logarithms to both sides with base D to equation (28) we get

$$\frac{1}{1-\alpha}\log\left[\frac{1}{n}\sum_{i=1}^{n}\left\{f\left(\mu_A(x_i),\mu_{A^c}(x_i)\right)\right\}\right]\leq\frac{\alpha}{1-\alpha}\log\left[\frac{1}{n}\sum_{i=1}^{n}\left\{f\left(\mu_A(x_i),\mu_{A^c}(x_i)\right)\right\}D^{-l_i\left(\frac{\alpha-1}{\alpha}\right)}\right] \tag{29}$$

As $0<\beta\leq 1$,multiply equation (29) both sides by $\beta>0$, we get

$$\frac{\beta}{1-\alpha}\log\left[\frac{1}{n}\sum_{i=1}^{n}\left\{f\left(\mu_A(x_i),\mu_{A^c}(x_i)\right)\right\}\right]\leq\frac{\alpha\beta}{1-\alpha}\log\left[\frac{1}{n}\sum_{i=1}^{n}\left\{f\left(\mu_A(x_i),\mu_{A^c}(x_i)\right)\right\}D^{-l_i\left(\frac{\alpha-1}{\alpha}\right)}\right] \tag{30}$$

Taking

$f\left(\mu_A(x_i),\mu_{A^c}(x_i)\right)=\left(\mu_A^{\alpha\beta}(x_i)+(1-\mu_A(x_i))^{\alpha\beta}\right)$, we get

$$\frac{\beta}{1-\alpha}\log\left[\frac{1}{n}\sum_{i=1}^{n}\left(\mu_A^{\alpha\beta}(x_i)+(1-\mu_A(x_i))^{\alpha\beta}\right)\right]\leq\frac{\alpha\beta}{1-\alpha}\log\left[\frac{1}{n}\sum_{i=1}^{n}\left(\mu_A^{\alpha\beta}(x_i)+(1-\mu_A(x_i))^{\alpha\beta}\right)D^{-l_i\left(\frac{\alpha-1}{\alpha}\right)}\right] \tag{31}$$

Or equivalently, we can write

$L_\beta^\alpha(A)\geq H_\beta^\alpha(A),0<\alpha<1,0<\beta\leq 1$. Hence the result.

From equation (23) we have

$$l_i=-\log_D\left[\frac{1}{\frac{1}{n}\sum_{i=1}^{n}\left[f\left(\mu_A(x_i),\mu_{A^c}(x_i)\right)\right]}\right]$$

Or equivalently, we can write

$$D^{-l_i}=\frac{1}{\frac{1}{n}\sum_{i=1}^{n}\left[f\left(\mu_A(x_i),\mu_{A^c}(x_i)\right)\right]} \tag{32}$$

Raising both sides to the power $\frac{\alpha-1}{\alpha}$ to equation (32) and after simplification, we get

$$D^{-l_i\left(\frac{\alpha-1}{\alpha}\right)}=\left[\frac{1}{n}\sum_{i=1}^{n}\left[f\left(\mu_A(x_i),\mu_{A^c}(x_i)\right)\right]\right]^{\frac{1-\alpha}{\alpha}} \tag{33}$$

Multiply equation (33) both sides by $\frac{1}{n}\left\{f\left(\mu_A(x_i),\mu_{A^c}(x_i)\right)\right\}$ and then summing over $i=1,2,...,n$, both sides to the resultant expression and after simplification, we get

$$\left[\frac{1}{n}\sum_{i=1}^{n}\left\{f\left(\mu_A(x_i),\mu_{A^c}(x_i)\right)\right\}D^{-l_i\left(\frac{\alpha-1}{\alpha}\right)}\right]=\left[\frac{1}{n}\sum_{i=1}^{n}\left\{f\left(\mu_A(x_i),\mu_{A^c}(x_i)\right)\right\}\right]^{\frac{1}{\alpha}} \tag{34}$$

Taking logarithms both sides with base D to equation (34), then multiply both sides by $\frac{\alpha\beta}{1-\alpha}$, we get

$$\frac{\alpha\beta}{1-\alpha}\log\left[\frac{1}{n}\sum_{i=1}^{n}\{f(\mu_A(x_i),\mu_{A^c}(x_i))\}D^{-l_i\left(\frac{\alpha-1}{\alpha}\right)}\right]=\frac{\beta}{1-\alpha}\log\left[\frac{1}{n}\sum_{i=1}^{n}\{f(\mu_A(x_i),\mu_{A^c}(x_i))\}\right] \quad (35)$$

Or equivalently, we can write

$$L_\beta^\alpha(A)=H_\beta^\alpha(A).\text{Hence the result}$$

Where

$$f(\mu_A(x_i),\mu_{A^c}(x_i))=\left(\mu_A^{\alpha\beta}(x_i)+(1-\mu_A(x_i))^{\alpha\beta}\right)$$

Theorem 4.2. For every code with lengths $l_1,l_2,\ldots,l_n$ satisfies the condition (21), $L_\beta^\alpha(P)$ can be made to satisfy the inequality,

$$L_\beta^\alpha(A)<H_\beta^\alpha(P)+\beta,\ \text{where } 0<\alpha<1, 0<\beta\le 1. \quad (36)$$

Proof. From the theorem 4.1 we have

$$L_\beta^\alpha(A)=H_\beta^\alpha(A)$$

Holds if and only if

$$D^{-l_i}=\frac{1}{\frac{1}{n}\sum_{i=1}^{n}[f(\mu_A(x_i),\mu_{A^c}(x_i))]}, 0<\alpha<1, 0<\beta\le 1$$

Or equivalently we can write

$$l_i=\log_D\left[\frac{1}{n}\sum_{i=1}^{n}[f(\mu_A(x_i),\mu_{A^c}(x_i))]\right]$$

We choose the code-word lengths $l_i, i=1,2,\ldots,n$ in such a way that they satisfy the inequality,

$$\log_D\left[\frac{1}{n}\sum_{i=1}^{n}[f(\mu_A(x_i),\mu_{A^c}(x_i))]\right]\le l_i<\log_D\left[\frac{1}{n}\sum_{i=1}^{n}[f(\mu_A(x_i),\mu_{A^c}(x_i))]\right]+1 \quad (37)$$

Consider the interval

$$\delta_i=\left[\log_D\left[\frac{1}{n}\sum_{i=1}^{n}[f(\mu_A(x_i),\mu_{A^c}(x_i))]\right],\log_D\left[\frac{1}{n}\sum_{i=1}^{n}[f(\mu_A(x_i),\mu_{A^c}(x_i))]\right]+1\right]$$

of length unity.In every δ_i, there lies exactly one positive integer l_i, such that,

$$0<\log_D\left[\frac{1}{n}\sum_{i=1}^{n}[f(\mu_A(x_i),\mu_{A^c}(x_i))]\right]\le l_i<\log_D\left[\frac{1}{n}\sum_{i=1}^{n}[f(\mu_A(x_i),\mu_{A^c}(x_i))]\right]+1 \quad (38)$$

We will first show that the sequence $l_1,l_2,\ldots,l_n$, thus defined satisfies the inequality (21), which is generalized fuzzy Kraft [14] inequality.

From the left inequality of (38), we have

$$\log_D\left[\frac{1}{n}\sum_{i=1}^{n}[f(\mu_A(x_i),\mu_{A^c}(x_i))]\right]\le l_i$$

Or equivalently, we can write

$$D^{-l_i}\le\frac{1}{\frac{1}{n}\sum_{i=1}^{n}[f(\mu_A(x_i),\mu_{A^c}(x_i))]} \quad (39)$$

Multiply equation (39) both sides by $\frac{1}{n}\{f(\mu_A(x_i),\mu_{A^c}(x_i))\}$ and then summing over $i=1,2,\ldots,n$, on both sides to the result that we obtain, we get the required result (21),which is generalized fuzzy Kraft [14] inequality.

Now the last inequality of (38) gives

$$l_i<\log_D\left[\frac{1}{n}\sum_{i=1}^{n}(f(\mu_A(x_i),\mu_{A^c}(x_i)))\right]+1$$

Or equivalently, we can write

$$D^{l_i}<\left[\frac{1}{n}\sum_{i=1}^{n}(f(\mu_A(x_i),\mu_{A^c}(x_i)))\right]D \quad (40)$$

As $0<\alpha<1$, then $(1-\alpha)>0$, and $\left(\frac{1-\alpha}{\alpha}\right)>0$, raising both sides to the power $\left(\frac{1-\alpha}{\alpha}\right)>0$, to equation (40),and after suitable simplification, we get

$$D^{-l_i\left(\frac{\alpha-1}{\alpha}\right)}<\left[\frac{1}{n}\sum_{i=1}^{n}(f(\mu_A(x_i),\mu_{A^c}(x_i)))\right]^{\left(\frac{1-\alpha}{\alpha}\right)}D^{\frac{1-\alpha}{\alpha}} \quad (41)$$

Multiply equation (41) both sides by $\frac{1}{n}\{f(\mu_A(x_i),\mu_{A^c}(x_i))\}$ and then summing over $i=1,2,\ldots,n$, on both sides to the resulted expression, and after making suitable operations we get,

$$\left[\frac{1}{n}\sum_{i=1}^{n}\{f(\mu_A(x_i),\mu_{A^c}(x_i))\}D^{-l_i\left(\frac{\alpha-1}{\alpha}\right)}\right]<\left[\frac{1}{n}\sum_{i=1}^{n}\{f(\mu_A(x_i),\mu_{A^c}(x_i))\}\right]^{\frac{1}{\alpha}}D^{\frac{1-\alpha}{\alpha}} \quad (42)$$

Taking logarithms to both sides with base D to equation (42), we get,

$$\log_D\left[\frac{1}{n}\sum_{i=1}^{n}\{f(\mu_A(x_i),\mu_{A^c}(x_i))\}D^{-l_i\left(\frac{\alpha-1}{\alpha}\right)}\right]<\frac{1}{\alpha}\log_D\left[\frac{1}{n}\sum_{i=1}^{n}\{f(\mu_A(x_i),\mu_{A^c}(x_i))\}\right]+\frac{1-\alpha}{\alpha} \quad (43)$$

As $0<\alpha<1, 0<\beta\le 1$ then$(1-\alpha)>0$ and $\left(\frac{\alpha\beta}{1-\alpha}\right)>0$, multiply both sides equation (43) by $\left(\frac{\alpha\beta}{1-\alpha}\right)>0$, we get

$$\frac{\alpha\beta}{1-\alpha}\log_D\left[\frac{1}{n}\sum_{i=1}^{n}\{f(\mu_A(x_i),\mu_{A^c}(x_i))\}D^{-l_i\left(\frac{\alpha-1}{\alpha}\right)}\right]<\frac{\beta}{1-\alpha}\frac{1}{n}\log_D\left[\frac{1}{n}\sum_{i=1}^{n}\{f(\mu_A(x_i),\mu_{A^c}(x_i))\}\right]+\beta \quad (44)$$

Or equivalently, we can write

$$L_\beta^\alpha(A)<H_\beta^\alpha(A)+\beta.\text{Hence the result for } 0<\alpha<1, 0<\beta\le 1.$$

Thus from above two coding theorems we have shown that

$$H_\beta^\alpha(A)\le L_\beta^\alpha(A)<H_\beta^\alpha(P)+\beta, 0<\alpha<1, 0<\beta\le 1$$

5 Conclusion:

In this paper we define a new generalized fuzzy entropy measure of order α and type β. and show that this is a valid measure of fuzzy entropy. This measure also generalizes some well-known fuzzy measures already existing in the literature. Also generalized fuzzy average codeword length is considered and bounds have been obtained in terms of new generalized fuzzy entropy measure of order α and type β.

References

[1] Aczel J. On Shannons inequality optimal coding and characterization of Shannons and Renyis entropies. Institute Novonal De Alta Mathematics Symposia Maths **15**,153-179 (1975).

[2] Ashiq H. B and M. A. K. Baig. Coding Theorems on Generalized Useful Fuzzy Inaccuracy Measure. International Journal of Modern Mathematical Sciences **14**, 54-62 (2016).

[3] Ashiq H. B and M. A. K Baig. Some coding theorems on generalized Reynis entropy of order α and type β. International Journal of Applied Mathematics and Information Sciences Letters **5**, 13-19 (2016).

[4] Bhandari D. and N. R. Pal. Some new information measures for fuzzy sets. Information Science **67**, 204-228 (1993).

[5] Boekee D. E and Lubbe J. C. A. The R-norm information measures. Information and Control **45**, 136-155 (1980).

[6] Campbell L. L. A coding theorem and Renyis entropy. Information and control **8**, 423-429 (1965).

[7] De Luca A and Termini S. A Definition of Non-probabilistic Entropy in the Setting of fuzzy set theory. Information and Control **20**, 301-312 (1972).

[8] Harvda J. H and Charvat F. Quantification method of classification process the concept of structural α entropy. Kybernetika **3**, 30-35 (1967).

[9] Hooda D. S. On generalized measures of fuzzy entropy. Mathematica Slovaca **54**, 315-325 (2004).

[10] Kapur J. N. A generalization of Campbells noiseless coding theorem. Journal of Bihar Mathematical Society **10**, 1-10 (1986).

[11] Kapur J. N. Measures of Fuzzy Information. Mathematical Science Trust Society, New Delhi, (1997).

[12] Kapur J. N. Entropy and Coding. Mathematical Science Trust Society, New Delhi, (1998).

[13] Khan A. B., Autar R. and Ahmad H. Noiseless Coding Theorems for Generalized Non-additive Entropy. Tam Kong Journal of Mathematics **1211**, 15-20 (1981).

[14] Kraft L. J. A device for quantizing grouping and coding amplitude modulates pulses, M.S Thesis, Department of Electrical Engineering, MIT, Cambridge, (1949).

[15] Lowen R. Fuzzy Set TheoryBasic Concepts, Techniques and Bibliography. Kluwer Academic Publication. Applied Intelligence **31**, 283-291 (1996).

[16] M. A. K Baig and Mohd Javid Dar. Some Coding theorems on Fuzzy entropy Function Depending upon Parameter R and V. IOSR Journal of Mathematics **9**, 119-123 (2014).

[17] M. A. K Baig and Mohd Javid Dar. Fuzzy coding theorems on generalized fuzzy cost measure. Asian Journal of Fuzzy and Applied Mathematics **2**, 28-34 (2014).

[18] M. A. K Baig and Mohd Javid Dar. Some new generalization of fuzzy average codeword length and their Bounds American Journal of Applied Mathematics and Statistics **2**, 73-76 (2014).

[19] Nguyen H. T and Walker E. A. A first course in fuzzy logic. C.R.C Press Inc. (1997).

[20] Om Parkash and P. K. Sharma. A new class of fuzzy coding theorems. Caribb. J. Math. Computer Sciences **12**, 1-10 (2002).

[21] Om Parkash and P. K. Sharma. Noiseless coding theorems corresponding to fuzzy entropies. Southeast Asian Bulletin of Mathematics **27**, 1073-1080 (2004).

[22] Pal and Bezdek. Measuring Fuzzy Uncertainty. IEEE Trans. of fuzzy systems **2**, 107-118 (1994).

[23] Pal N. R and Pal S. K. Object background segmentation using new definition of entropy. Proc. Inst. Elec. Eng. **136**, 284-295 (1989).

[24] Parkash O. A new parametric measure of fuzzy entropy. Information Processing and Management of Uncertainty **2**, 1732-1737 (1998).

[25] Renyi A. On measure of entropy and information. Proceeding Fourth Berkely Symposium on Mathematical Statistics and probability University of California Press **1**, 547-561 (1961).

[26] Shannon C. E. A mathematical theory of communication. Bell System Technical Journal **27**, 379-423, 623-659 (1948).

[27] Sharma B. D and Taneja I. J. Entropies of type α,β and other generalized measures of information theory. Mathematika **22**, 205-215 (1975).

[28] Van Der Lubbe J. C. A. On certain coding theorems for the information of order α and type β. In. Trans., 8th Prague conf. Inf. Theory, D. Reidell, Dordrecth 253-256 (1978).

[29] Zadeh L. A. Fuzzy sets. Information and control **8**, 338-353 (1965).

Ashiq Hussain Bhat is a Ph.D Scholar in Department Of Statistics University Of Kashmir Srinagar (India).He received the B.A degree from Govt. Degree College Bemina Srinagar in 2010 and received the Masters in Statistics from Department Of Statistics University Of Kashmir Srinagar in (June 2013). From March 2014 he is a research scholar of Ph.D in the Department Of Statistics University Of Kashmir Srinagar (India). His research interest includes Information Theory, Fuzzy Information Measures and Coding Theory.

M. A. K. Baig completed his Graduation, Masters and Doctorial degree from A.M.U Aligarh (India). He was gold Medalist both in Graduation and Post-Graduation, currently he is working in University of Kashmir Srinagar (India), as Sr. Assistant Professor. He has supervised Six (6) Ph.D and Fifteen (15) M.Phil Scholars. He served as Head of the Statistics Department University of Kashmir Srinagar (India). His research interest includes Information Theory, Fuzzy Measures, Informatic Reliability. He has published more than 40 research papers in peer reviewed Journals.