Applied Mathematics & Information Sciences Letters
An International Journal

http://dx.doi.org/10.18576/amisl/050203

Some New Bounds of the Quadrature Formula of Gauss-Jacobi Type via (p,q)-Preinvex Functions

Muhammad Aslam Noor[1,*], *Muhammad Uzair Awan*[2] *and Khalida Inayat Noor*[1]

[1] Mathematics Department, COMSATS Institute of Information Technology, Park Road, Islamabad, Pakistan.
[2] Department of Mathematics, Government College University, Faisalabad, Pakistan

Received: 2 Jul. 2016, Revised: 26 Aug. 2016, Accepted: 28 Aug. 2016
Published online: 1 May 2017

Abstract: In this article, authors introduced the notion of (p,q)-preinvex functions. Some new and interesting estimates of the integral $\int_a^{a+\eta(b,a)}(a+\eta(b,a)-u)^p(u-a)^q f(u)du$ via (p,q)-preinvex functions are obtained. These estimates can be viewed as refined bounds of the quadrature formula of Gauss-Jacobi type. The ideas and technique of this paper may be starting point for further research in this dynamic and interesting field.

Keywords: Convex, preinvex, Beta functions, inequalities, quadrature.
2010 AMS Subject Classification: 26D15, 26A51

1 Introduction and Preliminaries

Theory of convexity plays a pivotal role in modern analysis through its numerous applications. This theory has received special attention by several authors over the years. Consequently the classical concepts of convexity have been extended and generalized in different directions using novel and innovative ideas, see [3,4,7,14,17,26,27]. Hanson [7] introduced the notion of differentiable invex functions, without calling them by this word. In the same year, Craven [1] introduced the term invex for calling this class of functions. Mititelu [10] defined the concept of invex set, as follows:
Let K_η be be a nonempty set in $\mathbb{R}$. Let $f : K_\eta \to \mathbb{R}$ be a continuous function and let $\eta(.,.) : \mathbb{R} \to \mathbb{R}$ be a continuous function.

Definition 1.*A set $K_\eta \in \mathbb{R}$ is said to be invex with respect to the bifunction $\eta(.,.)$, if*

$$u+t\eta(v,u) \in K_\eta, \quad \forall u,v \in K_\eta, t \in [0,1].$$

The concept of invex set K_η is sometimes referred to as η-connected set.

*Remark.*Note that $\eta(v,u) = v-u$, the invex set reduces to classical convex set. Thus, every convex set is also an invex set with respect to $\eta(v,u) = v-u$, but the converse is not necessarily true. For further details, see [11,12,13,27] and the references therein.

Preinvex functions are defined as:

Definition 2([27]). *A function $f : K_\eta \to \mathbb{R}$ is said to be preinvex function with respect to the bifunction $\eta(.,.)$, if*

$$f(u+t\eta(v,u)) \leq (1-t)f(u)+tf(v), \quad \forall u,v \in K_\eta, t \in [0,1].$$

A function f is said to be preincave if and only if $-f$ is preinvex. For $\eta(v,u) = v-u$ in Definition 2 a preinvex function reduces to a convex function in the classical sense.This shows that every convex function is a preinvex function, but the converse is not true.

Remark. In this paper function $\eta(.,.) : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is supposed to have the following property:

$$\eta(v+t_1\eta(u,v), v+t_2\eta(u,v)) = (t_1-t_2)\eta(u,v), \quad \forall t_1,t_2 \in [0,1], t_1 \leq t_2. \tag{1}$$

In this case the following consequences hold:

* Corresponding author e-mail: noormaslam@hotmail.com

1. If $t_1 = t_2 = 0$ then (1) implies that $\eta(v,v) = 0$ for all $v \in \mathbb{R}$.
2. If $t_1 = 0$ and $t_2 = t > 0$ then $\eta(v, v+t\eta(u,v)) = -t\eta(u,v)$ for all $u,v \in \mathbb{R}$. This is the first requirement of Condition C introduced in [13].
3. If $\eta(u,v) > 0$ for some $(u,v) \in \mathbb{R}$ then $\eta(v, v+t\eta(u,v)) \leq 0$ for all $t \in [0,1]$. It means that property (1) implies that function η has not constant sign on $\mathbb{R}\times\mathbb{R}$.

Theory of convexity has a strong relationship with theory of inequalities. Several inequalities have been obtained for convex functions, see [2,5,6,9,14,16,17,18,19,20,21,23,24,22,26]. One of the most interesting and extensively studied inequality in the literature for convex functions is Hermite-Hadamard's inequality. This gives an equivalent property for convexity property. This inequality is stated as:
Let $f : I = [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a convex function, then the following inequality holds:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(u)\mathrm{d}u \leq \frac{f(a)+f(b)}{2}.$$

Noor [14] extended the Hermite-Hadamard's inequality for preinvex functions as:
Let $f : K_\eta \to \mathbb{R}$ be a preinvex function such that $\eta(.,.)$ satisfies (1), then the following inequality holds:

$$f\left(\frac{2a+\eta(b,a)}{2}\right) \leq \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(u)\mathrm{d}u \leq \frac{f(a)+f(b)}{2}.$$

We now recall some known concepts which will be helpful in obtaining some of our main results.
Beta functions $\mathbb{B}(.,.)$ are defined as:

$$\mathbb{B}(u,v) = \int_0^1 t^{u-1}(1-t)^{v-1}\,\mathrm{d}t.$$

It is known that

$$\mathbb{B}(u,v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}.$$

The generalized quadrature formula of Gauss-Jacobi type has the form:

$$\int_a^b (x-a)^p(b-x)^q f(x)\mathrm{d}x = \sum_{k=0}^{m} B_{m,k} f(\gamma k) + R_m[f],$$

for some $B_{m,k}, \gamma_k$ and rest term $R_m[f]$. For more information, see [25]

2 Main Results

In this section, we define the class of (p,q)-preinvex functions and obtain some new integral inequalities for (p,q)-prinvex functions. This is the main motivation of this paper.

Definition 3. *A function $f : K_\eta \to \mathbb{R}$ is said to be (p,q)-preinvex function with respect to bifunction $\eta(.,.)$, if*

$$f(u+t\eta(v,u)) \leq t^p(1-t)^q[f(u)+f(v)], \quad \forall u,v \in K_\eta, t \in [0,1]. \tag{2}$$

Remark. Note that if $\eta(v,u) = v-u$ in (2) then we have a new definition of (p,q)-convex function.

Definition 4. *A function $f : K \to \mathbb{R}$ is said to be (p,q)-convex function, if*

$$f((1-t)u+tv) \leq t^p(1-t)^q[f(u)+f(v)], \quad \forall u,v \in K, t \in [0,1].$$

Remark. It is worth to mention here that for $p = 1 = q$ in Definition 3 and Definition 4, we recover the definitions of so-called *tgs*-preinvex functions [15] and *tgs*-convex functions [26].

Theorem 1. *Let $f : K_\eta \to \mathbb{R}$ be a (p,q)-preinvex function such that $\eta(.,.)$ satisfies (1) with $\eta(b,a) > 0$. If $f \in \mathscr{L}[a, a+\eta(b,a)]$, then*

$$2^{p+q-1} f\left(\frac{2a+\eta(b,a)}{2}\right) \leq \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(u)\mathrm{d}u \leq \mathbb{B}(p+1,q+1)[f(a)+f(b)].$$

Proof. Since $\eta(.,.)$ satisfies (1) and f is (p,q)-preinvex function, so, for $u = a+t\eta(b,a)$, $v = a+(1-t)\eta(b,a)$ and $t = \frac{1}{2}$, we have

$$f\left(\frac{2a+\eta(b,a)}{2}\right) \leq \frac{f(a+t\eta(b,a)) + f(a+(1-t)\eta(b,a))}{2^{p+q}}.$$

Integrating both sides of the above inequality with respect to t on $[0,1]$, we have

$$2^{p+q-1} f\left(\frac{2a+\eta(b,a)}{2}\right) \leq \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(u)\mathrm{d}u. \tag{3}$$

We now prove second inequality. Since it is known that f is (p,q)-preinvex function, then, we have

$$f(a+t\eta(b,a)) \leq t^p(1-t)^q[f(a)+f(b)].$$

Integrating both sides of the above inequality with respect to t on $[0,1]$, we have

$$\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(u)\mathrm{d}u \leq \mathbb{B}(p+1,q+1)[f(a)+f(b)]. \tag{4}$$

On summation of inequalities (3) and (4) the proof is complete. □

Note that when $p \to 1$ and $q \to 1$ in Theorem 1, we have the following new result for tgs-preinvex functions.

Corollary 1. *Let $f : K_\eta \to \mathbb{R}$ be a tgs-preinvex function such that $\eta(.,.)$ satisfies (1) with $\eta(b,a) > 0$. If $f \in \mathscr{L}[a, a+\eta(b,a)]$, then*

$$2f\left(\frac{2a+\eta(b,a)}{2}\right) \le \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u)\mathrm{d}u \le \frac{f(a)+f(b)}{6}.$$

Theorem 2. *Left $f,g : K_\eta \to \mathbb{R}$ be two (p,q)-preinvex functions such that $\eta(.,.)$ satisfies (1) and $\eta(b,a) > 0$. If $fg \in \mathscr{L}[a, a+\eta(b,a)]$, then*

$$2^{2(p+q)-1} f\left(\frac{2a+\eta(b,a)}{2}\right) g\left(\frac{2a+\eta(b,a)}{2}\right) - \mathbb{B}(2p+1,2q+1)[M(a,b)+N(a,b)] \le \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u)g(u)\mathrm{d}u,$$

where

$$M(a,b) = f(a)g(a) + f(b)g(b), \tag{5}$$

and

$$N(a,b) = f(a)g(b) + f(b)g(a), \tag{6}$$

respectively.

Proof. Since f and g are (p,q)-preinvex functions, so

$$\begin{aligned}
& f\left(\frac{2a+\eta(b,a)}{2}\right) g\left(\frac{2a+\eta(b,a)}{2}\right) \\
&\le \frac{1}{2^{p+q}}[f(a+t\eta(b,a)) + f(a+(1-t)\eta(b,a))] \\
&\quad \times \frac{1}{2^{p+q}}[g(a+t\eta(b,a)) + g(a+(1-t)\eta(b,a))] \\
&= \frac{1}{2^{2(p+q)}}[f(a+t\eta(b,a))g(a+t\eta(b,a)) \\
&\quad f(a+(1-t)\eta(b,a))g(a+(1-t)\eta(b,a)) \\
&\quad + f(a+(1-t)\eta(b,a))g(a+t\eta(b,a)) \\
&\quad f(a+t\eta(b,a))g(a+(1-t)\eta(b,a))] \\
&\le \frac{1}{2^{2(p+q)}}[f(a+t\eta(b,a))g(a+t\eta(b,a)) \\
&\quad f(a+(1-t)\eta(b,a))g(a+(1-t)\eta(b,a)) \\
&\quad +2t^{2p}(1-t)^{2q}[f(a)+f(b)][g(a)+g(b)]].
\end{aligned}$$

Integrating the above inequality with respect to t on $[0,1]$, we have

$$\begin{aligned}
& f\left(\frac{2a+\eta(b,a)}{2}\right) g\left(\frac{2a+\eta(b,a)}{2}\right) \\
&\le \frac{1}{2^{2(p+q)-1}}\left[\frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u)g(u)\mathrm{d}u \right. \\
&\quad \left. + \mathbb{B}(2p+1,2q+1)[M(a,b)+N(a,b)]\right].
\end{aligned}$$

□

Theorem 3. *Left $f,g : K_\eta \to \mathbb{R}$ be two (p,q)-preinvex functions such that $\eta(b,a) > 0$. If $fg \in \mathscr{L}[a, a+\eta(b,a)]$, then*

$$\frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u)g(u)\mathrm{d}u \le \mathbb{B}(2p+1,2q+1)[M(a,b)+N(a,b)],$$

where $M(a,b)$ and $N(a,b)$ are given by (5) and (6) respectively.

Proof. Since f and g are (p,q)-preinvex functions, so

$$f(a+t\eta(b,a)) \le t^p(1-t)^q[f(a)+f(b)],$$

and

$$g(a+t\eta(b,a)) \le t^p(1-t)^q[g(a)+g(b)].$$

Multiplying both sides of the above inequality and then integrating the resultant respect to t on $[0,1]$, we have

$$\int_0^1 f(a+t\eta(b,a))g(a+t\eta(b,a))\mathrm{d}t \le \int_0^1 t^{2p}(1-t)^{2q}[f(a)+f(b)][g(a)+g(b)]\mathrm{d}t.$$

This implies

$$\frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u)g(u)\mathrm{d}u \le \mathbb{B}(2p+1,2q+1)[M(a,b)+N(a,b)].$$

□

We now need an auxiliary result, which will be helpful in obtaining our next results.

Lemma 1. *Let $f : K_\eta \to \mathbb{R}$ be a continuous function such that $f \in \mathscr{L}[a, a+\eta(b,a)]$. Then*

$$\int_a^{a+\eta(b,a)} (u-a)^\alpha (a+\eta(b,a)-u)^\beta f(u)\mathrm{d}u = \eta^{\alpha+\beta+1}(b,a) \int_0^1 t^\alpha (1-t)^\beta f(a+t\eta(b,a))\mathrm{d}t.$$

Proof. Simple calculations yield the required result. □

Theorem 4. *Let $f : K_\eta \to \mathbb{R}$ be a continuous function such that $f \in \mathscr{L}[a, a+\eta(b,a)]$. If f is (p,q)-preinvex function. Then*

$$\int_a^{a+\eta(b,a)} (u-a)^\alpha (a+\eta(b,a)-u)^\beta f(u)\mathrm{d}u$$
$$\leq \eta^{\alpha+\beta+1}(b,a)\mathbb{B}(\alpha+p+1,\beta+q+1)[f(a)+f(b)].$$

Proof. Using Lemma 1, the definition of Beta function and the fact that f is a (p,q)-preinvex function, we have

$$\int_a^{a+\eta(b,a)} (u-a)^\alpha (a+\eta(b,a)-u)^\beta f(u)\mathrm{d}u$$
$$= \eta^{\alpha+\beta+1}(b,a)\int_0^1 t^\alpha (1-t)^\beta f(a+t\eta(b,a))\mathrm{d}t$$
$$\leq \eta^{\alpha+\beta+1}(b,a)\int_0^1 t^\alpha (1-t)^\beta [t^p(1-t)^q][f(a)+f(b)]\mathrm{d}t$$
$$= \eta^{\alpha+\beta+1}(b,a)\mathbb{B}(\alpha+p+1,\beta+q+1)[f(a)+f(b)].$$

□

Theorem 5. *Let $f : K_\eta \to \mathbb{R}$ be a continuous function such that $f \in \mathscr{L}[a, a+\eta(b,a)]$. If $|f|^{\frac{r}{r-1}}$ is (p,q)-preinvex function. Then*

$$\int_a^{a+\eta(b,a)} (u-a)^\alpha (a+\eta(b,a)-u)^\beta f(u)\mathrm{d}u$$
$$\leq \eta^{\alpha+\beta+1}(b,a)\mathbb{B}(r\alpha+1,r\beta+1)$$
$$\times \left[\mathbb{B}(p+1,q+1)[|f(a)|^{\frac{r}{r-1}}+|f(b)|^{\frac{r}{r-1}}]\right]^{\frac{r-1}{r}}.$$

Proof. Using Lemma 1, Holder's inequality, the definition of Beta functions and the fact that $|f|^{\frac{r}{r-1}}$ is (p,q)-preinvex function, we have

$$\int_a^{a+\eta(b,a)} (u-a)^\alpha (a+\eta(b,a)-u)^\beta f(u)\mathrm{d}u$$
$$\leq \eta^{\alpha+\beta+1}(b,a)\left[\int_0^1 t^{r\alpha}(1-t)^{r\beta}\mathrm{d}t\right]^{\frac{1}{r}}$$
$$\times \left[\int_0^1 |f(a+t\eta(b,a))|^{\frac{r}{r-1}}\mathrm{d}t\right]^{\frac{r-1}{r}}$$
$$\leq \eta^{\alpha+\beta+1}(b,a)\mathbb{B}(r\alpha+1,r\beta+1)$$
$$\times \left[\int_0^1 \left\{t^p(1-t)^q[|f(a)|^{\frac{r}{r-1}}+|f(b)|^{\frac{r}{r-1}}]\right\}\mathrm{d}t\right]^{\frac{r-1}{r}}$$
$$\leq \eta^{\alpha+\beta+1}(b,a)\mathbb{B}(r\alpha+1,r\beta+1)$$
$$\times \left[\mathbb{B}(p+1,q+1)[|f(a)|^{\frac{r}{r-1}}+|f(b)|^{\frac{r}{r-1}}]\right]^{\frac{r-1}{r}}.$$

□

Theorem 6. *Let $f : K_\eta \to \mathbb{R}$ be a continuous function such that $f \in \mathscr{L}[a, a+\eta(b,a)]$. If $|f|^r$ is (p,q)-preinvex function. Then*

$$\int_a^{a+\eta(b,a)} (u-a)^\alpha (a+\eta(b,a)-u)^\beta f(u)\mathrm{d}u$$
$$\leq \eta^{\alpha+\beta+1}(b,a)\,[\mathbb{B}(\alpha+1,\beta+1)]^{\frac{r-1}{r}}$$
$$\times [\mathbb{B}(\alpha+p+1,\beta+q+1)[|f(a)|^r+|f(b)|^r]]^{\frac{1}{r}}.$$

Proof. Using Lemma 1, Holder's inequality, the definition of Beta functions and the fact that $|f|^r$ is (p,q)-preinvex function, weobatin

$$\int_a^{a+\eta(b,a)} (u-a)^\alpha (a+\eta(b,a)-u)^\beta f(u)\mathrm{d}u$$
$$\leq \eta^{\alpha+\beta+1}(b,a)\left[\int_0^1 (1-t)^\alpha t^\beta \mathrm{d}t\right]^{\frac{r-1}{r}}$$
$$\times \left[\int_0^1 t^\alpha(1-t)^\beta |f(a+t\eta(b,a))|^r\mathrm{d}t\right]^{\frac{1}{r}}$$
$$\leq \eta^{\alpha+\beta+1}(b,a)\,[\mathbb{B}(\alpha+1,\beta+1)]^{\frac{r-1}{r}}$$
$$\times \left[\int_0^1 t^\alpha(1-t)^\beta t^p(1-t)^q[|f(a)|^r+|f(b)|^r]\mathrm{d}t\right]^{\frac{1}{r}}$$
$$= \eta^{\alpha+\beta+1}(b,a)\,[\mathbb{B}(\alpha+1,\beta+1)]^{\frac{r-1}{r}}$$
$$\times [\mathbb{B}(\alpha+p+1,\beta+q+1)[|f(a)|^r+|f(b)|^r]]^{\frac{1}{r}}.$$

This completes the proof. □

Note that if $p = 1 = q$ in Theorem 4, Theorem 5 and Theorem 6, we get previously known results [8]. Thus these results can be considered as significant generalizations of the results obtained in [8]

Acknowledgement

The authors are grateful to Dr. S. M. Junaid Zaidi(H.I.,S.I), Rector, COMSATS Institute of Information Technology, Pakistan for providing excellent research and academic environment.

References

[1] B.D. Craven, Duality for generalized convex fractional programs, in S. Schaible and T. Ziemba (eds.), Generalized Convexity in Optimization and Economics, Academic Press, 1981, 473–489.
[2] G. Cristescu, Improved Integral Inequalities for Products of Convex Functions, J. Inequal. Pure and Appl. Mathe., 6(2), (2005) [On line: http://www.emis.de/journals/JIPAM/article504.html?sid=504]
[3] W. W. Breckner, Stetigkeitsaussagen fiir eine Klasse verallgemeinerter convexer funktionen in topologischen linearen Raumen. Pupl. Inst. Math. 23, 13-20, (1978).
[4] G. Cristescu and L. Lupsa, Non-connected Convexities and Applications, Kluwer Academic Publishers, Dordrecht, Holland, 2002.
[5] S. S. Dragomir, C. E. M. Pearce, Selected topics on Hermite-Hadamard inequalities and applications. Victoria University, Australia, 2000.
[6] S. K. Khattri, Three proofs of the inequality $e < \left(1+\frac{1}{n}\right)^{n+0.5}$, Amer. Math. Monthly, 117(3), 273-277, (2010).
[7] M.A. Hanson, On Sufficiency of the Kuhn-Tucker Conditions, J. Math. Anal. Appl. 80(1981), 545–550.
[8] I. Ahmad, Integral inequalities under beta function and preinvex type functions, SpringerPlus, 5:521, (2016).
[9] W. Liu, New integral inequalities involving Beta function via p-convexity, Miskolc Math. Notes, 15(2), 585-591, (2014).
[10] S. Mititelu, Invex Sets, Stud. Cerc. Mat., 46(5) (1994), 529–532.
[11] S. Mititelu, Generalized invexities and global minimum properties, Balkan J. Geometry Appl., 2(1997), 1, 61–72.
[12] S. Mititelu, Generalized Convexities, Fair Partners Publishers, Bucharest, 2011.
[13] S.R. Mohan, S.K. Neogy, On invex sets and preinvex functions, J. Math. Anal. Appl. 189(1995), 901–908.
[14] M. A. Noor, Hermite-Hadamard integral inequalities for log-preinvex functions, J. Math. Anal. Approx. Theory, 2, 126-131, (2007).
[15] M. A. Noor, M. U. Awan, K. I. Noor, Some inequalities via tgs-preinvex functions in quantum analysis, preprint, (2016).
[16] M. A. Noor, G. Cristescu, M. U. Awan, Generalized fractional Hermite-Hadamard inequalities for twice differentiable s-convex functions, Filomat, 29(4), 807-815, (2015).
[17] M.A. Noor, K.I. Noor, M.U. Awan, J. Li, On Hermite-Hadamard inequalities for h-preinvex functions, Filomat, 28(7), (2014), 1463–1474.
[18] M. E. Ozdemir, E. Set, and M. Alomari, Integral inequalities via several kinds of convexity, Creat. Math. Inform., 20(1), 62-73, (2011).
[19] B.G., Pachpatte, On some inequalities for convex functions, RGMIA Research Report Collection, 6(E) (2003), [Online: http://rgmia.vu.edu.au/v6(E).html].
[20] B.G. Pachpatte, Analytic inequalities: Recent Advances, Atlantic Press, Amsterdam-Paris, 2012.
[21] E. Set, New inequalities of Ostrowski type for mappings whose derivatives are s-convex in the second sense via fractional integrals, Comput. & Math. Appl., 63(7), 1147-1154, (2012).
[22] E. Set, M. Tomar, M. Z. Sarikaya, On generalized Grss type inequalities for k-fractional integrals, Appl. Math. Comput., 269, 29-34, (2015).
[23] E. Set E, M. E. Ozdemir, S. S. Dragomir, On the Hermite-Hadamard inequality and other integral inequalities involving two functions, J. Inequal. Appl., Volume 2010, Article ID 148102, 9 pages (2010).
[24] E. Set, M. E. Ozdemir, S. S. Dragomir, On Hadamard-Type inequalities involving several kinds of convexity, J. Inequal. Appl., 2010, Article ID 286845, 12 pages (2010).
[25] D. D. Stancu, G. Coman, and P. Blaga, Analiza numerica si teoria aproximarii. Vol. II ClujNapoca: Presa Universitara Clujeana, 2002.
[26] M. Tunc, E. Gov, U. Sanal, On tgs-convex function and their inequalities, Facta universitatis (NIS) Ser. Math. Inform. 30(5), 679-691, (2015).
[27] T.Weir, B. Mond, Preinvex functions in multiobjective optimization, J. Math. Anal. Appl. 136, 29-38, (1988).

Muhammad Aslam Noor earned his PhD degree from Brunel University, London, UK (1975) in the field of Applied Mathematics (Numerical Analysis and Optimization). His field of interest and specialization is versatile in nature. It covers many areas of Mathematical and Engineering sciences such as Variational Inequalities, Operations Research and Numerical Analysis. He has been awarded by the President of Pakistan: President's Award for pride of performance on August 14, 2008 and Sitara-i-Imtiaz, August 14, 2016, in recognition of his contributions in the field of Mathematical Sciences. He was awarded HEC Best Research paper award in 2009. He has supervised successfully several Ph.D and MS/M.Phil students. He is currently member of the Editorial Board of several reputed international journals of Mathematics and Engineering sciences. He has more than 850 research papers to his credit which were published in leading world class journals. He is highly cited researcher in Mathematical Science (THomson Reuters, 2015, 2016).

Muhammad Uzair Awan has earned his PhD degree from COMSATS Institute of Information Technology, Islamabad, Pakistan. He is Assistant Professor at Department of Mathematics GC University Faisalabad, Pakistan.His field of interest is Convex Analysis and Numerical Optimization. He has published several research papers in reputed international journals.

Khalida Inayat Noor is a leading world-known figure in mathematics and is presently employed as an eminent Professor at CIIT, Islamabad. She obtained her PhD from Wales University (UK). She has a vast experience of teaching and research at university levels in various countries including Iran, Pakistan, Saudi Arabia, Canada and United Arab Emirates. She was awarded HEC best research paper award in 2009 and CIIT Medal for innovation in 2009. She has been awarded by the President of Pakistan: Presidents Award for pride of performance on August 14, 2010 for her outstanding contributions in mathematical sciences and other fields. Her field of interest and specialization is Complex analysis, Geometric function theory, Functional and Convex analysis. She introduced a new technique, now called as Noor Integral Operator which proved to be an innovation in the field of geometric function theory and has brought new dimensions in the realm of research in this area. She has been personally instrumental in establishing PhD/MS programs at CIIT. Dr. Khalida Inayat Noor has supervised successfully several Ph.D and MS/M.Phil students. She has been an invited speaker of number of conferences and has published more than 450 research articles in reputed international journals of mathematical and engineering sciences. She is member of editorial boards of several international journals of mathematical and engineering sciences.