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Discrete Wave Equation with Infinite Differences

*Vasily E. Tarasov**

Skobeltsyn Institute of Nuclear Physics, Lomonosov Moscow State University, Moscow 119991, Russia

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Abstract: There are different approaches to derive the wave equation with some approximations. In this paper, considering the wave equation as already specified, we get an exact discrete analog of this equation. We derive an discrete equation that exactly corresponds to the continuum wave equation. The proposed discrete equations are represented as equations with $\mathscr{T}$-differences that are represented by infinite series. From a physical point of view, this discrete equation describes a lattice with long-range interactions of power-law type. From a mathematical point of view, it is a uniquely selected difference equation that exactly corresponds to continuous wave equation.

Keywords: Wave equation, infinite difference, discrete approximation, discrete version of derivative

1 Introduction

It is well-known that the wave equations can be derived by the continualization method applied to a lattice. For example, the one-dimensional system of particles and springs, where all particles have mass M and all springs have spring stiffness K, is usually described by the equations of motion in the form

$$M\frac{d^2u_n(t)}{dt^2} = K\,(u_{n+1}(t) - 2u_n(t) + u_{n-1}(t)). \quad (1)$$

In the continualization method, it is assumed that the continuous displacement $u(x,t)$ is equals to the lattice displacement $u_n(t)$ at particle n by $u_n(t) = u(nh,t)$, where the particle spacing is denoted by h. Expressing the displacements $u_{n\pm1}(t)$ in terms of the continuous displacement $u(x \pm h,t)$, the Taylor series is used in the form

$$u_{n\pm1}(t) = u(x\pm h,t) = u(x,t)\pm$$

$$\pm h\frac{\partial u(x,t)}{\partial x} + \frac{h^2}{2}\frac{\partial^2 u(x,t)}{\partial x^2} \pm \frac{h^3}{6}\frac{\partial^3 u(x,t)}{\partial x^3} + O(h^4). \quad (2)$$

Then substitution of the terms (2) into equation (1) and division by the cross-section area of the medium A and the inter-particle distance h, gives

$$\rho\,\frac{\partial^2 u(x,t)}{\partial t^2} = E\,\frac{\partial^2 u(x,t)}{\partial x^2} + O(h^2), \quad (3)$$

where $\rho = M/(Ah)$ is the mass density and $E = (Kh)/A$ is the Young's modulus. Note that all odd-order derivatives of $u(x,t)$ are cancelled. Removing all the terms $O(h^2)$, equation (3) is represented in the form

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2\,\frac{\partial^2 u(x,t)}{\partial x^2}, \quad (4)$$

where $c = \sqrt{E/\rho}$ is the velocity. We can see that continualization by Taylor series cannot give the wave equation exactly since we removed all $O(h^2)$-terms. If we consider the next term of the Taylor series (2) with h^4, we get the equations of the gradient elasticity [3,4,5] instead of the wave equation (4).

There are different methods to derive the wave equations with some approximations. In this paper, we do not discuss these methods of a derivation of the wave equation. We try to understand an inverse problem. Considering the wave equation as already given, we would like to get exact discrete analogue of this equation. In this paper we obtain exact discrete equations that correspond to the wave equation (4). The proposed discrete equations are equations with differences, which are represented by infinite series. From a physical point of view, these equations describe models of lattices with long-range interactions [6,7]. From a mathematical point of view, these questions are uniquely selected difference equations that exactly correspond to the continuous wave equation. For simplification, we will consider one dimensional wave equation only. A generalization for

* Corresponding author e-mail: tarasov-theory.sinp.msu.ru

three-dimensional case can be easily realized by the approach that is proposed in [10,11,12].

2 From discrete equation with finite difference to continuum wave equation

Let us consider the linear wave equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2}. \quad (5)$$

Usually a discrete analogue of this equation is considered in the form of the equation with finite difference of second order

$$\frac{d^2 u_n(t)}{dt^2} = \frac{c^2}{h^2}\Big(u_{n+1}(t) - 2u_n(t) + u_{n-1}(t)\Big). \quad (6)$$

It is well-known [1,2] that this discrete equation is not exact analog of the wave equation (5). Let us give some details to explain a connection of equations (5) and (6). Applying the Fourier series transform $\mathscr{F}_{h,\Delta}$, which is defined by the equation

$$\hat{f}(k) = \sum_{n=-\infty}^{+\infty} f[n]\, e^{-iknh} = \mathscr{F}_{h,\Delta}\{f[n]\}, \quad (7)$$

equation (6) gives

$$\frac{d^2 \hat{u}(k,t)}{dt^2} = -\frac{2c^2}{h^2}\sum_{m=1}^{\infty} \frac{(-1)^m}{(2m)!}(kh)^{2m}\,\hat{u}(k,t). \quad (8)$$

The inverse Fourier integral transform $\mathscr{F}^{-1}$, which is defined by the equation

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk\, \hat{f}(k)\, e^{ikx} = \mathscr{F}^{-1}\{\hat{f}(k)\}, \quad (9)$$

gives

$$\frac{d^2 u(x,t)}{dt^2} = \frac{2c^2}{h^2}\sum_{m=1}^{\infty}\frac{h^{2m}}{(2m)!}\frac{\partial^{2m} u(x,t)}{\partial x^{2m}}. \quad (10)$$

Equation (10) also can be obtained (for details, see Section 8 of [2]) by using the well-known relation

$$\exp\left(h\frac{\partial}{\partial x}\right) f(x) = f(x+h).$$

Equation (10) gives the wave equation (5) only in the limit $h \to 0$, since

$$\lim_{h\to 0}\frac{2}{h^2}\sum_{m=1}^{\infty}\frac{h^{2m}}{(2m)!}\frac{\partial^{2m} u(x,t)}{\partial x^{2m}} = \frac{\partial^2 u(x,t)}{\partial x^2}. \quad (11)$$

Therefore equation (6) cannot be considered as an exact discretization of (5).

Using (8), we see that the central finite difference ${}^c\Delta^2$, which is used in equation (6), is characterized by the inequality

$$\mathscr{F}_{h,\Delta}\left({}^c\Delta^c\right) \neq (ikh)^2. \quad (12)$$

This inequality leads us (see equation (10)) to the corresponding inequality

$$\frac{1}{h^2}\mathscr{F}^{-1}\left(\mathscr{F}_{h,\Delta}\left({}^c\Delta^2\right)\right) \neq \frac{\partial^2}{\partial x^2}, \quad (13)$$

which means that this finite difference of second orders cannot give exactly the derivative of second order $\partial^2/\partial x^2$. The second-order derivative can be obtained only by the limit $h \to 0$, such that

$$\lim_{h\to 0}\frac{\mathscr{F}^{-1}\left(\mathscr{F}_{h,\Delta}\left({}^c\Delta^2\right)\right)}{h^2} = \frac{\partial^2}{\partial x^2}. \quad (14)$$

As a result, the discrete equation (6) can be considered only as approximation of the wave equation (5). Equation (6) cannot be considered as an exact analogue of the wave equation (5).

3 From continuum wave equation to discrete equation

Using the approach, which is proposed in [10,11,12], we can suggest an exact discrete analog of the wave equation (5).

Let us consider the Fourier integral transform $\mathscr{F}$, which is defined by equation

$$\hat{f}(k) = \int_{-\infty}^{+\infty} dx\, f(x)\, e^{-ikx} = \mathscr{F}\{f(x)\}. \quad (15)$$

Applying this Fourier transform to the wave equation (5), we get

$$\frac{d^2 \hat{u}(k,t)}{dt^2} = -c^2 k^2\, \hat{u}(k,t). \quad (16)$$

Using the inverse Fourier series transform

$$f[n] = \frac{h}{2\pi}\int_{-\pi/h}^{+\pi/h} dk\, \hat{f}(k)\, e^{iknh} = \mathscr{F}^{-1}_{h,\Delta}\{\hat{f}(k)\}, \quad (17)$$

we obtain

$$\frac{d^2 u_n(t)}{dt^2} = \frac{c^2}{h^2}\, {}^{\mathscr{T}}\Delta^2\, u_n(t), \quad (18)$$

where ${}^{\mathscr{T}}\Delta^2$ is the $\mathscr{T}$-difference of second order that is defined by

$${}^{\mathscr{T}}\Delta^2 u_n := -\sum_{\substack{m=-\infty \\ m\neq 0}}^{+\infty} \frac{2(-1)^m}{m^2}\, u_{n-m} - \frac{\pi^2}{3} u_n. \quad (19)$$

As a result we get an exact discrete analogue of the wave equation (5) in the form of the difference equation

$$\frac{d^2 u_n(t)}{dt^2} = -\frac{2c^2}{h^2} \sum_{\substack{m=-\infty \\ m\neq 0}}^{+\infty} \frac{(-1)^m}{m^2} u_{n-m}(t) - \frac{\pi^2 c^2}{3h^2} u_n(t). \tag{20}$$

To use the Fourier series transform, we assume that the function $u_n(t)$ belongs to the Hilbert space l^2 of square-summable sequences, where the norm on the l^p-space is defined by the equation

$$\|f\|_p := \left(\sum_{n=-\infty}^{+\infty} |f[n]|^p \right)^{1/p}. \tag{21}$$

The $\mathcal{T}$-difference (19) is defined by convolution of $u_m \in l^2$ and the functions

$$K_2(m) = \frac{(-1)^m}{m^2}$$

that are belong to the space l^1. Using the Young's inequality for convolutions (see [13,14] and Theorem 276 of [15]). in the form

$$\| {}^{\mathcal{T}}\Delta^2 u\|_r = \|K_2 * u\|_r \leq \|K_2\|_p \|u\|_q, \tag{22}$$

where $m \in \mathbb{Z}$, and

$$\frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}, \tag{23}$$

we get that the result of the action of operator ${}^{\mathcal{T}}\Delta^2$ also belongs to the Hilbert space l^2 of square-summable sequences, i.e.

$${}^{\mathcal{T}}\Delta^2 u_m \in l^2 \tag{24}$$

since condition (23) holds.

Note that using equation 5.1.2.3 of [8], we can get

$$\sum_{m=1}^{\infty} K_2(m) = \sum_{m=1}^{\infty} \frac{(-1)^m}{m^{2n}} = -\frac{1}{2}\zeta(2) =$$

$$= -\frac{1}{\Gamma(x)} \int_0^{\infty} \frac{x}{e^x+1} dx = -\frac{\pi^2}{12}, \tag{25}$$

where $\zeta(z)$ is the Riemann zeta function, $\Gamma(z)$ is the Gamma function. Therefore a result of the action of the $\mathcal{T}$-difference ${}^{\mathcal{T}}\Delta^2$ on a constant function converge.

An important property of the suggested difference (19) is that the Fourier series transform $\mathcal{F}_{h,\Delta}$ of this difference is represented by the equality

$$\mathcal{F}_{h,\Delta}\left({}^{\mathcal{T}}\Delta^2 \right) = (ikh)^2. \tag{26}$$

This equation leads us to the corresponding equality

$$\frac{1}{h^n} \mathcal{F}^{-1}\left(\mathcal{F}_{h,\Delta}\left({}^{\mathcal{T}}\Delta^2 \right) \right) = \frac{1}{h^2} \mathcal{F}^{-1}\left((ikh)^2 \right) = \frac{\partial^2}{\partial x^2}, \tag{27}$$

which means that the difference of second order gives the derivative $\partial^2/\partial x^2$ exactly. This $\mathcal{T}$-difference are connected with the derivative $\partial^2/\partial x^2$ not only asymptotically defined by the limit $h \to 0$. It's obvious that the limit $h \to 0$ also gives this derivative

$$\lim_{h\to 0} \frac{\mathcal{F}^{-1}\left(\mathcal{F}_{h,\Delta}\left({}^{\mathcal{T}}\Delta^n \right) \right)}{h^n} = \frac{\partial^n}{\partial x^n}. \tag{28}$$

As a result, the suggested equations with $\mathcal{T}$-difference can be considered not only as approximation of the wave equation. The suggested discrete equations (20) are exact discrete analogue of the continuous wave equation (5).

4 Solution of difference equation

Difference wave equation (20)can be solved, by using the method of separation of variables. Let us represent $u_n(t)$ in the form

$$u_n(t) = u[n]\, T(t). \tag{29}$$

Substitution of (29) into equations (20), gives equation for $u[n]$ in the form

$$\frac{c^2}{h^2} {}^{T}\Delta^2 u[n] + \omega^2 u[n] = 0, \tag{30}$$

Equations for $T(t)$ are the same for equations (5) and (20). To solve (30), we assume that solution is proportional to $\exp(\lambda n)$ with a constant λ. Substitute $u[n] = \exp(\lambda n)$ into difference equation (30), and use the relation

$${}^{T}\Delta^1 \exp(\lambda n) = \lambda \exp(\lambda n), \tag{31}$$

which is proved [12], we get a solution of difference equation (30) in the form

$$u[n] = C_1 e^{\lambda_1 \cdot hn} + C_2 e^{\lambda_2 \cdot hn}, \tag{32}$$

where $\lambda_{1,2}$ are solution of equation $\lambda^2 + \omega^2 = 0$, i.e. $\lambda_1 = +i\omega$ and $\lambda_1 = -i\omega$. It is easy to see that solutions (32) is connected with the solution $u(x,t)$ of wave equation (5) by the relation $u_n(t) = hu(hn,t)$ for all $n \in \mathbb{Z}$ and $h > 0$, up to renormalization constants $C_{1,2}$.

Difference equation (30) can be considered as an exact discretization of differential equation (5). The exact discretization means that the difference equation has the same general solution as the associated differential equation. Here we use the following criterion of exact discretization of differential equations. A discretization of differential equation is exact if and only if the associated difference equation has solutions $u_n(t)$ is exactly equal to the solutions $u(x,t)$ of associated differential equation for $x = hn$, where $n \in \mathbb{Z}$ and $h > 0$.

It should be noted that discretization of the wave equation by standard finite differences (6) cannot be considered as an exact discretization since

$${}^{c}\Delta^1 \exp(\lambda n) \neq \lambda \exp(\lambda n) \tag{33}$$

in contrast with (31). Therefore (32) is not the solution of the standard difference equation (6).

5 Conclusion

We propose exact discrete equations that corresponds to the wave equation exactly. From a mathematical point of view, these discrete equations are selected equations with nonstandard differences that exactly correspond to the continuous wave equation. Physically these equations describe microstructural models of chain with long-range interactions. The main advantage of the suggested discrete equations are the connection with continuous wave equation without any approximation. For simplification, we consider one-dimensional wave equation only. A generalization for three-dimensional case can be easily realized by the approach proposed in [10,11,12]. Computer simulation of suggested exact discrete analogue of the continuous wave equation can be realized similar to modeling of chain systems with long-range interactions. We assume that the suggested equations with $\mathscr{T}$-differences can be important in application since its allow us to reflect characteristic properties of complex systems and continua at the micro-scale and nano-scale, where long-range interactions play a crucial role in determining the properties (see [16,17,18,19] and references therein).

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Vasily Tarasov received his Ph.D. in Theoretical Physics from the Lomonosov Moscow State University in 1995. He was a Research Associate at the Skobeltsyn Institute of Nuclear Physics, Moscow State University for six years; a Senior Research Associate at the Skobeltsyn Institute of Nuclear Physics, Lomonosov Moscow State University, and then became a a Leading Researcher at the Skobeltsyn Institute of Nuclear Physics, Lomonosov Moscow State University. Vasily is an Associate Professor at the Applied Mathematics and Physics Department of Moscow Aviation Institute. He is a member of Organizing Committee of International Workshop on High Energy Physics and Quantum Field Theory. Vasily E. Tarasov was Senior Visitor at Courant Institute of Mathematical Sciences, New York University in 2005-2009. V.E. Tarasov is an Associate Editor of ”Communications in Nonlinear Science and Numerical Simulations” (Elsevier); ”Journal of Applied Nonlinear Dynamics” (LH Scientific Publishing). V.E. Tarasov is a member of Editorial Boards of the journals: ”Fractional Calculus and Applied Analysis” (De Gruyter)”, ”Fractional Differential Calculus” (Ele-Math); ”International Journal of Applied and Computational Mathematics” (Springer); ”Discontinuity, Nonlinearity, and Complexity” (LH Scientific Publishing); ”Physics International” (Science Publications). He has published about 160 scientific works, among which 8 books, 4 chapters in books and about 140 papers in refereed journals.