Journal of Statistics Applications & Probability
An International Journal

http://dx.doi.org/10.18576/jsap/060305

A New Method for Adding Two Parameters to a Family of Distributions with Application

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Received: 20 Mar. 2017, Revised: 2 Aug. 2017, Accepted: 8 Aug. 2017
Published online: 1 Nov. 2017

Abstract: In this paper, a new method is proposed for adding two parameters to a continuous distribution that extends the methods of [5] and [14] for adding a parameter to a family of distributions. Using the added parameters, the skewness and kurtosis of the resulting family can be fully controlled. Also simple sufficient conditions for the shape of the density and hazard rate functions of the new family are provided. Finally, the new method is applied in particular, to the one parameter Burr XII distribution to yield a three parameter extended Burr XII distribution which may serve as a competitor to such commonly used three parameters families of distributions such as generalized gamma and inverse Gaussian distributions.

Keywords: Markov-Bernoulli Geometric distribution; Maximum Likelihood Estimation; Censored Data; Kaplan-Meier Estimator; (P-P) plot; Geometric-Extreme Stability.

1 Introduction

The problem of constructing and extending classes and families of continuous distributions is one of the continue active research problems in statistics, This is due, from one side, to the limitations of the existing distributions and its lack in modeling some random phenomena, and from the other side, to meet the sustainable needs of modern and developed application fields (for a detailed review, see e.g. [15] and the references therein). Expanding families of distributions by adding one or more parameter make it richer and more flexible for modeling numerous types of data. For example, introducing a scale parameter leads to the accelerated life model, and taking powers of the survival function introduces a parameter that leads to the proportional hazards model. [5], assuming some dependency in the sequence of independent and identically distributed (IID) Bernoulli random variables (RV'S), gives an additional parameter that is interpreted as a correlation coefficient. [14] introduced another method for adding a parameter to a family of distributions. This method is used by numerous authors to extend some important distributions such as [7], [8] and others. In order to include skewness in the normal distribution, [4] introduced the skew normal family of distributions. The paper [6] pioneered the class of beta generated distributions with two added parameters. Many new families of distributions utilizing this technique are defined and studied (see e.g. [3]). A lot of developments in this area and others were recently reviewed by [12].

In this paper another general method of adding two parameters to a family of distributions is discussed. The new method represents an extension of the methods of [5] and [14]. Also, families expanded using the method introduced here have the property that the minimum (maximum) of a Marko-Bernoulli geometric number of independent random variables with common distribution in the family has a distribution again in the family. Finally the new method is applied, particularly, to a one parameter Burr XII distribution. The resulting three parameter Markov-Bernoulli extended Burr XII (MBEB XII) distribution is statistically analyzed. It is shown that the MBEB XII distribution can be represented as a mixture of one and three parameters Burr XII distributions. Further, simple sufficient conditions for the shape of the density and hazard rate functions of the MBEB XII distribution are provided. Finally, utilizing maximum likelihood estimation, the MBEB XII distribution is fitted to a set of randomly censored data.

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Starting with a survival function (SF) $\overline{F}$, the two parameters family of SF'S

$$\overline{G}(x,\ \alpha,\ \rho) = \frac{\alpha\,\overline{F}(x)\ \left[1-\ \rho\,\overline{F}(x)\right]}{1-\ [\rho+\ (1-\ \rho)\ \overline{\alpha}]\overline{F}(x)} = \frac{\alpha\,\overline{F}(x)\ \left[1-\ \rho\,\overline{F}(x)\right]}{1-\ [1-\ (1-\ \rho)\ \alpha]\ \overline{F}(x)}$$
$$= \frac{\alpha\,\overline{F}(x)\ \left[1-\ \rho\,\overline{F}(x)\right]}{F(x)\ +\ (1-\ \rho)\ \alpha\,\overline{F}(x)},\ (-\infty < x < \infty,\quad \alpha > 0,\quad 0 \le \rho < 1) \tag{1}$$

where, $\overline{\alpha} = 1-\ \alpha$, is proposed and discussed in section 2.

Note that, when $\rho = 0$, the family (1) reduces to the one parameter family introduced by [14].

Also, if (1-ρ)α = 1, i. e., ρ=$\frac{\alpha-1}{\alpha}$, then the family (1) reduces in this case to the one parameter family

$$\overline{G}(x,\ \alpha) = \overline{F}(x)\left[\alpha F(x)+\overline{F}(x)\right] \qquad (-\infty < x < \infty,\quad \alpha > 1\ ) \tag{2}$$

For α = 1 both (1) and (2) give $\overline{G}(x) = \overline{F}(x)$.

2 Density and hazard rate functions of the new family

If $F(x)$ has a density $f(x)$ and hazard rate r_F then G has the density g given by

$$g(x,\alpha,\rho) = \frac{\alpha\ f(x)[1-2\rho\,\overline{F}(x)+\rho(1-(1-\rho)\alpha)\,\overline{F}^{2}(x)]}{[1-(1-(1-\rho)\alpha)\,\overline{F}(x)]^{2}}, \tag{3}$$

and hazard rate

$$r(x,\alpha,\rho) = r_F(x)\,\frac{\left[1-2\rho\overline{F}(x)+\rho\,(1-(1-\rho)\,\alpha)\overline{F}^{2}(x)\right]}{\left[1-\rho\,\overline{F}(x)\right]\left\{1-(1-(1-\rho)\,\alpha)\overline{F}(x)\right\}}. \tag{4}$$

Thus,

$$\lim_{x\to-\infty} r(x,\alpha,\rho) = \lim_{x\to-\infty}\frac{(1-\ \rho\ \alpha)}{(1-\rho)\alpha}\,r_F(x),$$

and

$$\lim_{x\to\infty} r(x,\alpha,\rho) = \lim_{x\to\infty} r_F(x).$$

It follows from (4) that

$$\frac{r_F(x)}{(1-\rho)\alpha} \le r(x,\alpha,\rho) \le r_F(x),\ (-\infty < x < \infty,\quad \alpha \ge 1,\quad 0 \le \rho < 1), \tag{5}$$

and

$$r_F(x) \le r(x,\alpha,\rho) \le \frac{r_F(x)}{(1-\rho)\,\alpha},\ (-\infty < x < \infty,\ 0 < \alpha \le 1,\quad 0 \le \rho < 1), \tag{6}$$

Moreover,

$$\overline{F}(x) \le \overline{G}(x,\alpha,\rho) \le [\overline{F}(x)]^{\frac{1}{(1-\rho)\alpha}}, \qquad (-\infty < x < \infty,\ \alpha \ge 1,\ 0 \le \rho < 1), \tag{7}$$

and

$$[\overline{F}(x)]^{\frac{1}{(1-\rho)\alpha}} \le \overline{G}(x,\alpha,\rho) \le \overline{F}(x), \qquad (-\infty < x < \infty,\ 0 < \alpha < 1,\quad 0 \le \rho < 1), \tag{8}$$

Now, from (4) we have $(\overline{F}(x) \le 1 \Longrightarrow \overline{F}^{2}(x) \le \overline{F}(x))$

$$\begin{aligned}\frac{r(x,\alpha,\rho)}{r_F(x)} &= \frac{1-2\rho\overline{F}(x)+\rho\,[1-\ (1-\ \rho)\,\alpha]\ \overline{F}^{2}(x)}{\left[1-\rho\overline{F}(x)\right]\left\{1-[1-(1-\rho)\,\alpha]\overline{F}(x)\right\}}\\ &\le \frac{1-\ \rho\,[1+\ (1-\ \rho)\,\alpha]\overline{F}(x)}{1-[1-(1-\rho)\,\alpha]\overline{F}(x)}\\ &\le \frac{1}{1-[1-(1-\rho)\,\alpha]\overline{F}(x)}\end{aligned}$$

Hence, $\frac{r(x,\alpha,\rho)}{r_F(x)}$ is increasing (decreasing) in x for $\alpha \ge \frac{1}{(1-\rho)}$ $(\alpha < \frac{1}{(1-\rho)})$, where, $0 \le \rho < 1$.

3 A MB-Extended Burr XII Distribution

Putting $\overline{F}(x) = (1+x^c)^{-1}, x > 0,\ c > 0$, in (1) we get the three parameters survival function

$$\begin{aligned}\overline{G}(x,\alpha,\rho,c) &= \frac{\alpha(1+x^c)^{-1}\left[1-\rho(1+x^c)^{-1}\right]}{1-\left[1-(1-\rho)\alpha\right](1+x^c)^{-1}} \\ &= \alpha\,\overline{H}_1(x,c) + (1-\alpha)\,\overline{H}_2(x,\alpha,\rho,c),\end{aligned} \tag{9}$$

where,

$$\overline{H}_1(x,c) = \overline{H}(x,c,1,1),$$

$$\overline{H}_2\ (x,\alpha,\rho,c) = \sum_{r=2}^{\infty} \alpha(1-\rho)\ \left[1-(1-\rho)\alpha\right]^{r-2}\ \overline{H}(x,c,r,1),$$

and

$$\overline{H}(x,c,k,a) = \left(1+\left(\frac{x}{a}\right)^c\right)^{-k},\ x>0,\ c,\ k,\ a>0$$

which is the SF of Burr XII distribution with parameters c, k and a ([10]).

It is clear from (9) that $\overline{G}(x,\alpha,\rho,c)$ is a mixture of two Burr XII distributions one of them is one parameter and the other is a countable mixture of two parameters Burr XII distributions.

Now differentiating (9) with respect to x we get

$$g(x,\alpha,\rho,c) = \alpha\ h_1(x,c) + (1-\alpha)\ h_2(x,\alpha,\rho,c), \tag{10}$$

where

$$h_1(x,c) = \frac{d}{dx}H_1(x,c),$$

and

$$h_2(x,\alpha,\rho,c) = \frac{d}{dx}H_2(x,\alpha,\rho,c).$$

From (10), after dividing by $\overline{G}(x,\alpha,\rho,c)$ and carrying some manipulations we can express

$$r_G(x,\alpha,\rho,c) = \frac{\overline{H}\left(x,c,1,(1-\rho)^{\frac{1}{c}}\right)}{\overline{H}\left(x,c,1,[(1-\rho)\alpha]^{\frac{1}{c}}\right)}\left[\ \alpha\, r_{H_1} +\ (1-\ \alpha)\ \overline{H}\left(x,c,1,[(1-\rho)\alpha]^{\frac{1}{c}}\right)\ r_{H_2}\right] \tag{11}$$

The following theorem gives simple conditions under which the pdf (10) is decreasing or unimodal.

Theorem 1.*The pdf of the MBEB XII distribution, given by* (10)*, is decreasing if* $c<1,\ \ 0<\alpha<\frac{1}{\rho}$ *, and is unimodal if* $c>1,\ \ 0<\alpha<\frac{1}{\rho}$ *.*

*Proof.*The pdf (10) can be written as

$$g(x) = \frac{\alpha c x^{c-1}\left[1-\rho(1+x^c)^{-1}-\rho(1+x^c)^{-2}(x^c+(1-\rho)\alpha)\right]}{\left[x^c+(1-\rho)\alpha\right]^2},$$

then the first derivative of g(x) is given by

$$\grave{g}(x) = -\frac{\alpha c x^{c-2}}{\left[x^c+(1-\rho)\alpha\right]^3}\Phi(x),\ \ x>0$$

where

$$\begin{aligned}\Phi(x) = &\left[(c-1)\left(\rho(1+x^c)^{-2}(x^c+(1-\rho)\alpha)-\left(1-\rho(1+x^c)^{-1}\right)\right)-2\rho c x^c(1+x^c)^{-3}(x^c+(1-\rho)\alpha)+\rho c x^c(1+x^c)^{-2}\right. \\ &\left.-\rho x^c(1+x^c)^{-2}\right]\left[x^c+(1-\rho)\alpha\right]-2cx^c\left[\left(\rho(1+x^c)^{-2}\right)(x^c+(1-\rho)\alpha)-\left(1-\rho(1+x^c)^{-1}\right)\right].\end{aligned}$$

If $\Phi(0) = \alpha(1-\rho)^2(c-1)(\rho\alpha-1) \geq 0$ and $c \leq 1,\ \alpha < \frac{1}{\rho}$, then $\Phi(x) \geq 0$ for all $x > 0$ and hence $g(x) \leq 0$ for all $x > 0$, then $g(x)$ is decreasing.

Since for $c>1,\ \alpha<\frac{1}{\rho}$, $\lim_{x\to 0} g(x) = 0$ and $\lim_{x\to\infty} g(x) = 0$, then the pdf g(x) of MB-EB XII distribution first increase and the decrease to zero and hence has a mode x_{mod} given by the equation $\Phi(x_{mod}) = 0$.

Remarks

1. For $\alpha > \frac{1}{\rho}$ the curve of the pdf will lie in the negative part of the plane and this contradicts the fact that the pdf must be non-negative and thus these choices of α and c are not suitable.
2. For $\alpha = 1$, $\rho = 0$, $g(x)$ is decreasing if $c \leq 1$ and is unimodal if $c > 1$, which is the well-known result for the Burr XII distribution ([18]).

The r^{th} moment of the MB-EB(XII) distribution

If X is a RV having the MB-EB(XII) distribution given by (9), then its r^{th} moment $E(X^r)$, $r \geq 1$, is given by:

$$\begin{aligned} E\left(X^r\right) &= r\int_0^{\infty} x^r \overline{G}\left(x\right) dx \\ &= r\alpha \int_0^{\infty} x^{r-1} \frac{\left[1-\rho\left(1+x^c\right)^{-1}\right]}{\left[x^c+\left(1-\rho\right)\alpha\right]} dx \\ &= \frac{r\alpha}{c} \sum_{\substack{u=0 \\ u \geq \frac{r}{c}}}^{\infty} \left(1-\alpha(1-\rho)\right)^u \left[\beta\left(u-\frac{r}{c}+1,\frac{r}{c}\right)-\rho\beta\left(u-\frac{r}{c}+2,\frac{r}{c}\right)\right], \end{aligned}$$

where $\beta(a,b)$ is the beta function.

Fig (1) below shows the pdf curves for the MBEBXII distribution for some selected values of the parameters c, α and ρ.

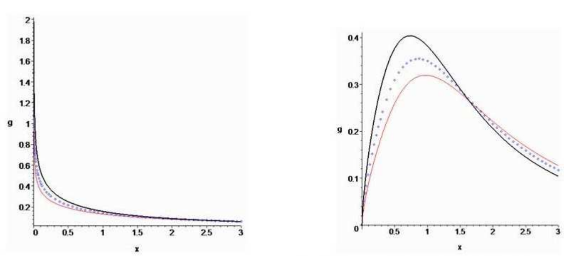

Fig. 1 (a) $\alpha = 2$ (bold), 2.5 (plain), 3 (point), $\rho = 0.1, c = 0.7$ Fig. 1 (b) $\alpha = 2$ (bold), 2.5 (plain), 3 (point), $\rho = 0.1, c = 1.7$

Fig. 1: The pdf of the MB-EB (XII)distribution for selected values of the parameters α and z.

The PDF g(x) of the MB-EB (XII) distribution for selected values of the parameters.

In fig. (1)(a) $c = 0.7$, $(c < 1)$, showing that g(x) is decreasing. In fig. (1) (b) $c = 1.7$, $(c > 1)$, showing that g(x) is increasing-decreasing.

The hazard rate function of the MB-EB(XII) distribution (9) is given by

$$h(x) = \frac{cx^{c-1}[1-\rho(1+x^c)^{-1}-\rho(1+x^c)^{-2}(x^c+\alpha(1-\rho))]}{(x^c+\alpha(1-\rho))[1-\rho(1+x^c)^{-1}]}. \tag{12}$$

Note that for all $\alpha > 0$, $0 < \rho < 1$, we have

$$h(0) = \begin{cases} \infty, & 0 < c < 1 \\ \frac{1-\rho(1-\alpha(1-\rho))}{\alpha(1-\rho)^2}, & c = 1 \\ 0, & c > 1 \end{cases}.$$

$$h(\infty) = \begin{cases} 0, & c < 1 \\ \frac{1}{\alpha(1-\rho)}, & c = 1 \\ \infty, & c > 1 \end{cases}.$$

The following theorem describes the behavior of $h(x)$.

Theorem 2. *The HRF $h(x)$ of the MBEB XII distribution given by (12), is decreasing if $c < 1,\ 0 < \alpha < \frac{1}{\rho}$, and is unimodal if $c > 1,\ 0 < \alpha < \frac{1}{\rho}$.*

Proof. The proof is similar to that of theorem 1.

Remark

For $\rho = 0,\ \alpha = 1, h(x)$ is decreasing if $c \leq 1$ and is unimodal if $c > 1$ which is the well-known result for the Burr XII distribution ([9]).

It follows from theorem 2 that the HRF of the MB-EBXII distribution have the same behavior as its pdf (Fig.**??**).

4 Maximum Likelihood Estimation (MLE) for Censored Data

Consider a data set of size n consisting of m uncensored observations $d_1,\ldots,d_m$ and n-m censored observations $e_1,\ldots,e_{n-m}$. For simplifying the notations we shall denote all the observations by $t_1,\ldots,t_n$ with censoring indicators $\delta_i = 1$ for $t_i = d_i$ and $\delta_i = 0$ for $t_i = e_i$. We have $m = \sum_{i=1}^{n} \delta_i$.

The likelihood function of MBEB (XII) distribution defined by (10)

$$Ln(\alpha,\rho,c) = \prod_{i=1}^{n} \{g(t_i)\}^{\delta_i} \{\overline{G}(t_i)\}^{1-\delta_i},$$

then the log-likelihood function is

$$\begin{aligned} ln(\alpha,\rho,c) = \sum_{i=1}^{n} \{ & \delta_i [\log c + (c-1)\,] \log(t_i) \\ & + \log\left[1-\rho(1+t_i^c\,)^{-1} - \rho(1+t_i^c\,)^{-2}\ (t_i^c + \alpha(1-\rho))\right] \\ & - (1+\delta_i)\log\ (t_i^c + \alpha(1-\rho))\ + log\alpha \\ & + (1-\delta_i)\log[1-\rho(1+t_i^c\,)^{-1}]\} \end{aligned}$$

The first derivative of $ln(\alpha,\rho,c)$ with respect to α, c and ρ, respectively, are given by

$$\frac{\partial ln}{\partial \alpha} = 0,\ \ \frac{\partial ln}{\partial c} = 0,\ \ \frac{\partial ln}{\partial \rho} = 0.$$

To test the null hypothesis $H_0 : \alpha = 1,\ \ \rho = 0$ (Burr XII distribution) we use the likelihood ratio test (LRT). Under H_0 the likelihood ratio statistic: $A = -2\left[\ln(1,\ 0, \hat{c})\ - \ln(\widehat{\alpha},\ \widehat{\rho},\ \hat{c})\,\right]$, has approximately a chi-square distribution with 2 degree of freedom. Also, for model selection, we use the Akiake information criterion (AIC) and Bayesian information criterion (BIC) defined as:

$$AIC = \log likelihood - 2k,$$

$$BIC = loglikelihood - \frac{k}{2}\log(n)\ ,$$

Where k is the number of parameters in the model and n is the sample size. For more details about the AIC and BIC, see [1] and [17], respectively.

The model with higher AIC (BIC) is the one that better fits the data.

Application

The following data represent the ordered remission times (in months) of a random sample of 137 bladder cancer patient ([13] P. 231).

Table (4.1) remission times (months) of 137 bladder cancer patients.

0.08	2.09	3.48	4.87	6.94	8.66	13.11	23.63
0.20	2.23	3.52	4.98	6.97	9.02	13.29	24.80+
0.40	2.26	3.57	5.06	7.09	9.22	13.80	25.74
0.50	2.46	3.64	5.09	7.26	9.47	14.24	25.82
0.51	2.54	3.70	5.17	7.28	9.74	14.76	26.31
0.81	2.62	3.82	5.32	7.32	10.06	14.77	32.15
0.87+	2.64	3.88	5.32	7.39	10.34	14.83	34.26
0.90	2.69	4.18	5.34	7.59	10.66	15.96	36.66
1.05	2.69	4.23	5.41	7.62	10.75	16.62	43.01
1.19	2.75	4.26	5.41	7.63	10.86+	17.12	46.12
1.26	2.83	4.33	5.49	7.66	11.25	17.14	79.05
1.35	2.87	4.33+	5.62	7.87	11.64	14.36	
1.40	3.02	4.34	5.71	7.93	11.79	18.10	
1.46	3.02+	4.40	5.85	8.26	11.98	19.13	
1.76	3.25	4.50	6.25	8.37	12.02	19.36+	
2.02	3.31	4.51	6.54	8.53	12.03	20.28	
2.02	3.36	4.65+	6.76	8.60+	12.07	21.73	
2.07	3.36	4.70+	6.93	8.65	12.63	22.69	
+ censored data.							

The data set has minimum (maximum) uncensored remission time at 0.08 (79.05) months. Also the data set contains nine censored remission time at 0.87, 3.02, 4.33, 4.65, 4.70, 8.60, 10.86, 19.36, 24.80 months.

The following table gives a comparison between the MLE, log-likelihood, AIC and BIC for the fitted MBEB(XII) and Burr XII distributions to the bladders cancer data. The following table shows high values of both AIC and BIC, which favor selecting the MBEB (XII) distribution.

Table (4.2) rA comparison between the MLEs, Log-likelihood, AIC, BIC, for the fitted MBEB(XII) and Burr(XII) distributions of the remission times from 137 cancer patients.

Model	Parameter	MLE	Log-likelihood	AIC	BIC
MBEB(XII)	ρ α c	0.5435 2.4272 0.7588	-451.625	-457.625	-449.005
Burr(XII)	c	0.7495	-519.762	-521.762	-522.222

The results in the above table show that the fitted MBEB (XII) distribution should be selected based on either the BIC or AIC procedure.

For the given data, under H0, $l_n(1,0,\widehat{c}) = -519.762$, thus

$$X_L = -2[-519.762 + 451.625)] = 136.286,$$

$$\chi^2_{2.0.05} = 5.991,$$

therefore, we cannot accept the null hypothesis, i.e. the LRT rejects the assumption that the Burr(XII) model is suitable for the given data.

Let $t_{(i)}$s be the ordered survival times and $\delta_{(i)}$s be their corresponding censoring indicators. [11] estimator (KME), also known as the product-limit estimator, of a survival function is defined as

$$\overline{G}_n(t) = \prod_{\substack{t:\, t_i \leq t \\ i = 1,\ldots,n}} \left\{1 - \frac{\delta_{(i)}}{n-i+1}\right\},\ t > 0.$$

The following figures show the probability-probability (P-P) plot of the KME versus the fitted Burr (XII) and MBEB (XII) survival functions for the given data.

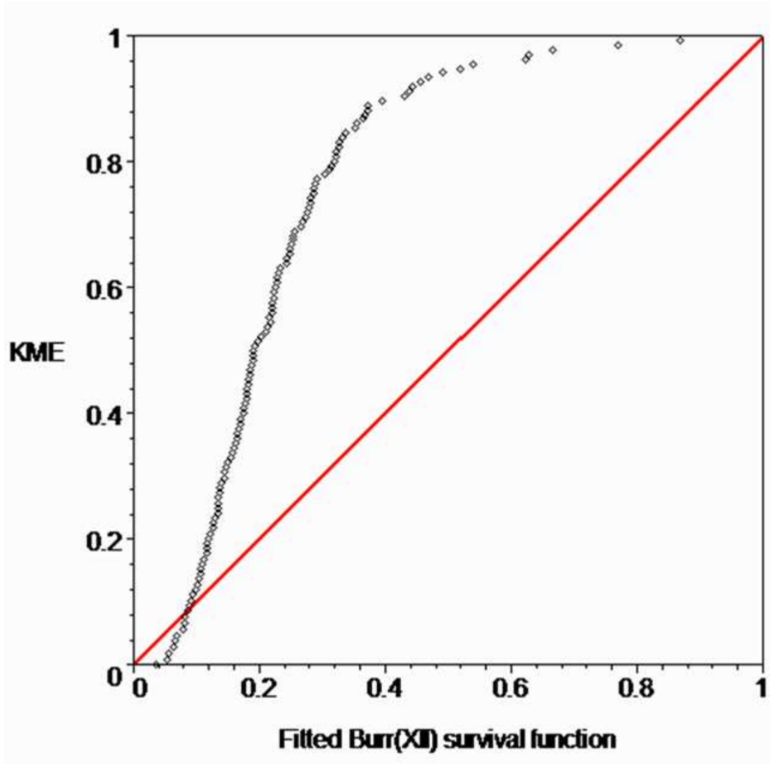

Fig. 2: p-p plot of KME versus fitted Burr (XII) survival function

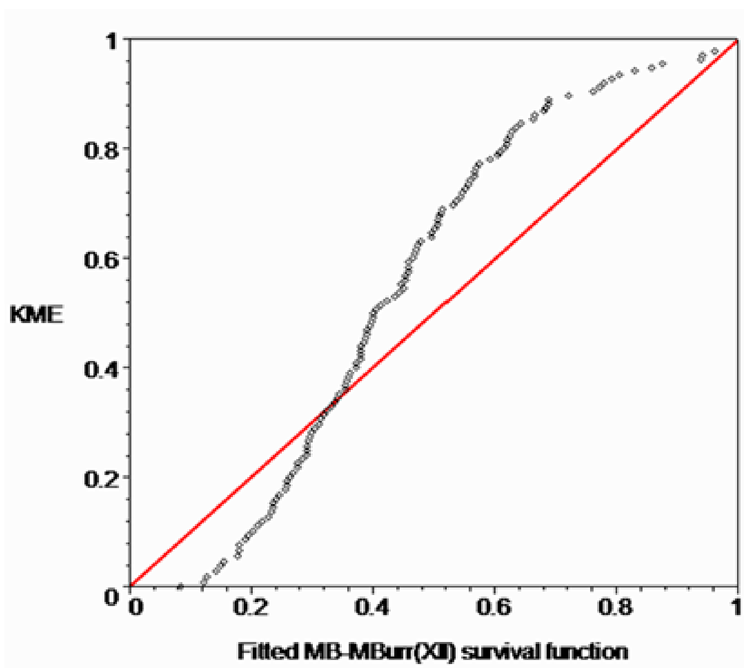

Fig. 3: p-p plot of KME versus fitted MBEB XII survival function.

We note that the depicted points for the fitted MBEB (XII) survival function are near to the 45° line, indicating good fit as comparing with the fitted Burr(XII) survival function.

Since $\widehat{\alpha}= 2.4272,\ \widehat{\rho}= 0.5435,\ \widehat{\mathrm{c}}= 0.7588$, then the estimated hazard rate function $\mathrm{h}(\mathrm{x})$ is as shown in the following figure.

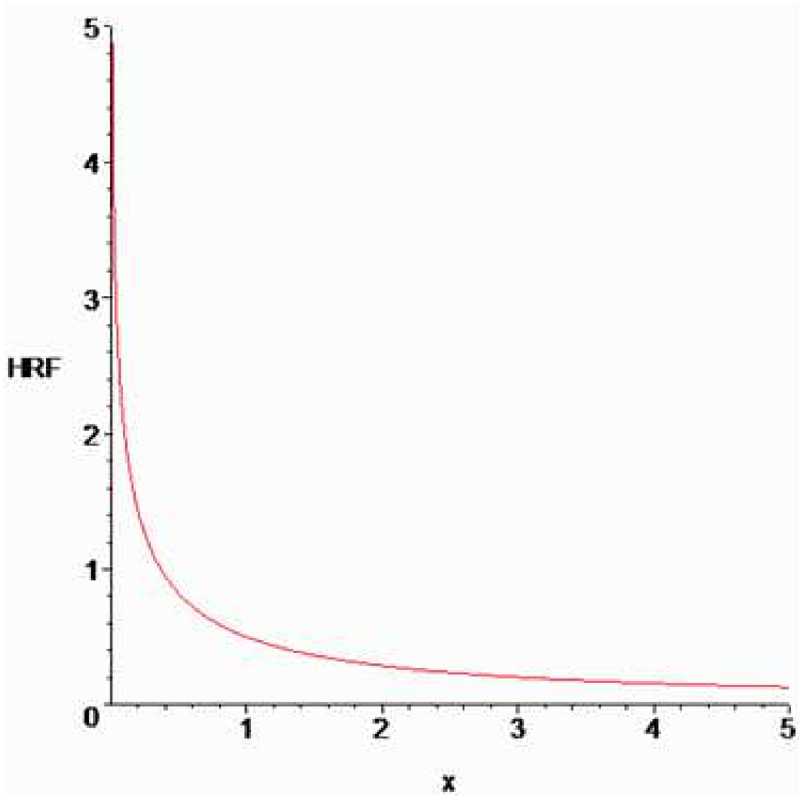

Fig. 4: The estimated hazard rate function of MBEB XII distribution based on the remission times of bladder cancer data.

5 MB Geometric-Extreme Stability

We first give the definition of MBG distribution.

Definition 1. *If $\{Z_i\}, i = 1, 2, \ldots$ is a two states Markov chain with states $E_i; i = 0, 1$, with the following one step transition probabilities matrix*

$$Z_i \begin{matrix} \\ 0 \\ 1 \end{matrix} \overset{\displaystyle Z_{i+1}}{\begin{matrix} \quad 0 & \quad\quad 1 \\ \left[\begin{matrix} 1-(1-)\,p & (1-)\,p \\ (1-)\,(1-p) & +(1-)\,p \end{matrix}\right] \end{matrix}}, \tag{13}$$

and initial distribution:

$P(Z_1 = 1) = p = 1 - P(Z_1 = 0)$, *where* $p\ [0,\ 1]$ *and* $[0,\ 1]$ *, and if W is the rv representing the number of transitions necessary for the system (13) to be in state E_1 for the first time, then the probability mass function (pmf) of W is given by:*

$$P(W = n) = \begin{cases} p, & n = 1 \\ (1-p)\,a^{-1}\left(1 - a^{-1}\right)^{n-2}, & n \geq 2, \end{cases} \tag{14}$$

where,

$$a = 1/[(1-\ \rho)\ p] = E(W) + \ \rho/(1-\ \rho).$$

The distribution (14) is called the Markov - Bernoulli geometric (MBG) distribution [2].

If $X_1, X_2, \ldots$ is a sequence of iid rv's with distribution in the family (1) and if N has a MBG distribution on $\{1, 2, \ldots\}$, then min($X_1, X_2, \ldots, X_N$) and max($X_1, X_2, \ldots, X_N$) have distributions in the family. To explain why this property is of interest, we recall that extreme value distributions are limiting distributions for extrema, and as such they are sometimes useful approximations. In applications a rv of interest may be the extreme of only a finite, possibly a random number N of rv's. When N has a MBG distribution, the rv has a nice stability property, not unlike that of extreme value distributions.

Assume that N is independent of the X_i,s with a MBG(p, ρ) distribution given by (14), and let U = min($X_1, X_2, \ldots, X_N$), V = max($X_1, X_2, \ldots, X_N$).

Definition 2.*If F∈ F implies that the distribution of U (V) is in F, then F is said to be minimum- MBG stable (maximum MBG stable). If F is both MBG-minimum and MBG-maximum stable, then F is said to be MBG-extreme stable.*

The term 'max-MBG stable' extends, in some sense, the term 'max-geometric stable' which has been used by [14], for families of distributions, has been used, also, by [16] to describe a related but more restricted concept (they apply the term not to families of distributions but to individual distributions), in their sense, a distribution is 'max-geometric stable' if the location- scale parameter family generated by the distribution is max stable in Marshall and Olkin's sense. The three ideas coincide for families F that are parameterized by location and scale. The logistic distribution is an example for this.

*Example 1.*The family of logistic distributions logistic (θ, λ), with SF of the form

$$\overline{F}(x) = \frac{1}{1+\theta e^{\lambda x}} \qquad (-\infty < x < \infty, \quad \theta, \lambda > 0).$$

Is a MBG-extreme stable family. It is interesting to note that on substituting this family in (1) then the SF of the resulting four parameters MBE logistic family can be written as:

$$\overline{G}(x,\alpha,\rho,\theta,\quad\lambda) = \frac{\alpha\rho}{1-(1-\rho)\alpha}\;\frac{1}{1+\theta e^{\lambda x}} + \frac{1-\alpha}{1-(1-\rho)\alpha}\;\frac{1}{1+\frac{\theta}{(1-\rho)\alpha}e^{\lambda x}},$$

Which is the SF of a mixture of the two logistic distributions:

logistic (θ, λ), and logistic ($\frac{\theta}{(1-\rho)\alpha}$, λ).

Theorem 3.*The parametric family of distributions of the form (1) is MBG extreme stable.*

*Proof.*Spouse that $X_1, X_2, \ldots, X_M$ is a sequence of independent random variables with a common distribution G given by (1), M is independent of the X,'s with a MBG distribution given by (14), and if

$$\mathrm{U_N} = \min(\mathrm{X1,X2,...,X_N}),\ \ \mathrm{V_N} = \max(\mathrm{X1,X2,...,X_N}),$$

then

$$\overline{\mathrm{G}}(\mathrm{x}) = \Pr(\mathrm{U_N} > x)$$

$$= \Pr(\mathrm{X_1>x_1, X_2>x_2, \ldots, X_N>x_N}) = \sum_{m=1}^{\infty} \overline{\mathrm{F}}^{\mathrm{m}}(\mathrm{x}) \Pr(\mathrm{M}=\mathrm{m})$$

$$= \mathrm{p}\overline{\mathrm{F}}(\mathrm{x}) + (1-\mathrm{p})\,\mathrm{a}^{-1}\overline{\mathrm{F}}^2(\mathrm{x}) \sum_{m=2}^{\infty} \overline{\mathrm{F}}^{\mathrm{m}-2}(\mathrm{x})(1-\mathrm{a}^{-1})^{\mathrm{m}-2}$$

$$= \mathrm{p}\overline{\mathrm{F}}(\mathrm{x}) + (1-\mathrm{p})\,\mathrm{a}^{-1}\overline{\mathrm{F}}^2(\mathrm{x}) \frac{1}{[1-(1-\mathrm{a}^{-1})\overline{\mathrm{F}}(\mathrm{x})]}$$

$$= \frac{\overline{\mathrm{F}}(\mathrm{x})\,[p - \mathrm{p}\overline{\mathrm{F}}(\mathrm{x}) + \mathrm{a}^{-1}\overline{\mathrm{F}}(\mathrm{x})]}{[1-(1-\mathrm{a}^{-1})\overline{\mathrm{F}}(\mathrm{x})]}$$

where

$$\mathrm{a}^{-1} = (1-\rho)\,\mathrm{p}$$

Hence,

$$\overline{\mathrm{G}}(\mathrm{x}) = \frac{\mathrm{p}\overline{\mathrm{F}}(\mathrm{x})\,[1-\rho\overline{\mathrm{F}}(\mathrm{x})]}{[1-(1-\mathrm{a}^{-1})\overline{\mathrm{F}}(\mathrm{x})]},$$

Which is (1) with α = p, hence U_N= min(X1, X2, ..., X $_N$) where $N \sim MBG$ distribution belongs to the family (1). Hence, (1) is a minimum MBG stable family.

Also,

$$H(x) = \Pr(V_N < x) = \Pr(X_1 < x_1, X_2 < x_2, \ldots, X_N < x_N)$$

$$= \sum_{m=1}^{\infty} F^m(x) \Pr(M = m)$$

$$= pF(x) + (1-p)\, a^{-1} F^2(x) \sum_{m=2}^{\infty} F^{m-2}(x)(1-a^{-1})^{m-2}$$

$$= pF(x) + (1-p)\, a^{-1} F^2(x) \frac{1}{[1-(1-a^{-1})F(x)]} = \frac{pF(x)[1-\rho F(x)]}{[1-(1-a^{-1})F(x)]}$$

So that, $\overline{H}(x) = \frac{\overline{F}(x)(1-\rho p F(x))}{a^{-1}+(1-a^{-1})\overline{F}(x)}$, $-\infty < x < \infty$.

Which is (1) with $= \frac{(1-\rho p)}{(1-\rho)p}$, and $\frac{\rho p}{1-\rho p}$ instead of ρ, where $a^{-1} = (1-\rho)p$.

Hence, V_N= max(X1, X2, ..., X $_N$) where $N \sim MBG$ distribution belongs to the family (1). Therefore, (1) is a maximum MBG stable family.

6 Conclusion and Summary

In this paper, a new method is proposed for adding two parameters to a continuous distribution that extends the methods of [5] and [14] for adding a parameter to a family of distributions. The added parameters provide additional flexibility for fitting diverse shapes of data. Some of the properties of the new family are derived and some of its subfamilies are defined. A detailed study is provided for the particular case when the extended distribution is the one parameter Burr (XII) distribution. The derived properties include: probability density function, its shape, hazard rate function, moments, and maximum likelihood estimation.

We note that for MBEB(XII) distribution, the Bayesian information criterion (BIC) and Akaike information criterion (AIC) are higher than the corresponding (BIC) and (AIC) of the Burr(XII) distribution. Also the fitted MBEB (XII) survival function indicates strong linear relationship between the empirical and fitted survival functions compared with the fitted Burr (XII) survival functions. All these results lead us to select the MBEB (XII) distribution as the best distribution for the given data.

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