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Intuitionistic Fuzzy (ψ,η)-Contractive Mapping and Fixed Points

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Abstract: In this article, using the definition of fuzzy ψ-contractive mapping, we introduce intuitionistic fuzzy (ψ,η)-contractive mapping and extend the fixed point results to intuitionistic fuzzy metric spaces.

Keywords: intuitionistic fuzzy (ψ,η)-contractive mapping; fixed point; intuitionistic fuzzy metric space

1 Introduction

In 1988, the famous fixed point theorems of Banach and Edelstein for contraction mapping are extended to fuzzy metric spaces in the sense of Kramsoil and Michalek[12] by M.Grabiec[8]. Further, Valentine Gregori and Almanzor Sapena [9]and D.Mihet[4] extended the fixed point theorem of Banach for contraction mapping to fuzzy metric spaces in the sense of George and Veeramani[6]. Several authors have studied the kinds of Contraction mappings in fuzzy metric spaces[1][10]. D.Mihet[5] has also introduced the concept of fuzzy ψ-contractive mapping in fuzzy metric spaces. He proved fixed point theorems using ψ-contractive mapping in non-Archimedean fuzzy metric spaces in the sense of George and Veeramani. Later Shenghua Wang [17] proved that the above-fixed point theorems are also true of fuzzy metric space in the sense of Kramsoil and Michalek[12]. Continuing this, Ishak Altun and D.Mihet[11] defined the order fuzzy ψ-contractive mapping in ordered fuzzy metric spaces and proved two kinds of fixed point theorems in ordered non-Archimedean fuzzy metric spaces. But they can not prove the existence of uniqueness. Many Mathematicians has studied the concept of the intuitionistic fuzzy metric spaces [15][3]. Very recently, L.A. Ricarte and S. Romaguera[14] has introduced the existence of fixed points of ϕ-Contractions in fuzzy metric spaces with the application for an intuitionistic setting. In this article, using the definition of fuzzy ψ-contractive mapping, we introduce intuitionistic fuzzy (ψ,η)-contractive mapping and extend the fixed point results to intuitionistic fuzzy metric spaces.

Definition 1.1. [16] A binary operation $*:[0,1]\times[0,1]\rightarrow[0,1]$ is called a t-norm if the following conditions hold:
(i)$*$ is associative and commutative;
(ii)$a*1=a,\forall a\in[0,1]$;
(iii)$a*b\leq c*d$ whenever $a\leq c$ and $b\leq d,\forall a,b,c,d\in[0,1]$.
If $*$ is continuous then it is called a continuous t-norm.

Definition 1.2. [16] A binary operation $\diamond:[0,1]\times[0,1]\rightarrow[0,1]$ is called a t-conorm if the following conditions hold:
(i)$\diamond$ is associative and commutative;
(ii)$a\diamond 0=a,\forall a\in[0,1]$;
(iii)$a\diamond b\leq c\diamond d$ whenever $a\leq c$ and $b\leq d,\forall a,b,c,d\in[0,1]$.
If $\diamond$ is continuous then it is called a continuous t-conorm.

Definition 1.3. [13] Let X be an arbitrary set, $*$ be a continuous t-norm, $\diamond$ be a continuous t-conorm and M,N be fuzzy sets on $X^2\times(0,\infty)$. Consider the following conditions $\forall u,v,w\in X$ and $t>0$,
(i)$M(u,v,t)+N(u,v,t)\leq 1$;
(ii)$M(u,v,0)=0$;
(iii)$M(u,v,t)=1$ if and only if $u=v$;
(iv)$M(u,v,t)=M(v,u,t)$;
(v)$M(u,w,t+s)\geq M(u,v,t)*M(v,w,s)$;
(vi)$M(u,v,.):(0,\infty)\rightarrow[0,1]$ is left continuous;
(vii)$N(u,v,0)=1$;

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(viii)$N(u,v,t)=0$ if and only if $u=v$;
(ix)$N(u,v,t)=N(v,u,t)$;
(x)$N(u,w,t+s)\leq N(u,v,t)\diamond N(v,w,s)$;
(xi)$N(u,v,.):(0,\infty)\to[0,1]$ is left continuous.
If M satisfies conditions (ii)-(vi), then the pair $(M,*)$ is called fuzzy metric on X. In this case, the triple $(X,M,*)$ is called a fuzzy metric space. If N satisfies conditions (vii)-(xi), then the pair $(N,\diamond)$ is called dual fuzzy metric on X. Then the triple $(X,N,\diamond)$ is called a dual fuzzy metric space.
If $(M,*)$ is a fuzzy metric on X and $(N,\diamond)$ is a dual fuzzy metric on X satisfying condition (i), then the 4-tuple $(M,N,*,\diamond)$ is called an intuitionistic fuzzy metric on X. In this case, the 5-tuple $(X,M,N,*,\diamond)$ is called an intuitionistic fuzzy metric space.
Example 1.4. [2] Let (X,d) be a metric space. Denote $a*b=ab$ and $a\diamond b=\min\{1,a+b\},\forall a,b\in[0,1]$ and let M_d and N_d be fuzzy sets on $X\times X\times(0,+\infty)$ defined as follows:$M_d(u,v,t)=\frac{t}{t+d(u,v)}$ and $N_d(u,v,t)=\frac{d(u,v)}{t+d(u,v)},\forall t>0$, then $(X,M_d,N_d,*,\diamond)$ is an intuitionistic fuzzy metric space.
Definition 1.5. [8] Let $(X,M,N,*,\diamond)$ be an intuitionistic fuzzy metric space. A sequence $\{u_n\}$ in X is called
(a)convergent to a point $u\in X$ if and only if $\lim_{n\to+\infty}M(u_n,u,t)=1$, and $\lim_{n\to+\infty}N(u_n,u,t)=0,\forall t>0$,
(b)Cauchy if $\lim_{n\to\infty}M(u_n,u_{n+p},t)=1$, and $\lim_{n\to+\infty}N(u_n,u_{n+p},t)=0,\forall t>0$ and $p>0$.
Definition 1.6. An intuitionistic fuzzy metric space $(X,M,N,*,\diamond)$ is said to be complete if every Cauchy sequence in X is convergent.

2 Main results

Definition 2.1. Let Ψ be the class of all mappings $\psi:[0,1]\to[0,1]$ such that
(i)ψ is nondecreasing and $\lim_{n\to\infty}\psi^n(s)=1,\forall s\in(0,1]$;
(ii)$\psi(s)>s,\forall s\in(0,1)$;
(iii)$\psi(1)=1$;
Example 2.2. Define $\psi:[0,1]\to[0,1]$ by $\psi(s)=\frac{2s}{s+1},\forall s\in[0,1]$.
$\psi^2(s)=\frac{4s}{3s+1},\psi^3(s)=\frac{8s}{7s+1},\ldots,\psi^n(s)=\frac{2^ns}{(2^n-1)s+1},\forall s\in[0,1]$.
$\lim_{n\to\infty}\psi^n(s)=\lim_{n\to\infty}\frac{2^ns}{(2^n-1)s+1}=1,\forall s\in(0,1)$.
Clearly, $\psi(s)>s,\forall s\in(0,1)$ and $\psi(1)=1$.
Definition 2.3. Let Ψ be the class of all mappings $\eta:[0,1]\to[0,1]$ such that
(i)η is nondecreasing and $\lim_{n\to\infty}\eta^n(r)=0,\forall r\in[0,1)$;
(ii)$\eta(r)<r,\forall r\in(0,1)$;
(iii)$\eta(0)=0$;
Example 2.4. Define $\eta:[0,1]\to[0,1]$ by $\eta(r)=\frac{r}{2-r}\forall r\in[0,1]$.
$\eta^2(r)=\frac{r}{4-3r},\eta^3(r)=\frac{r}{8-7r},\ldots,\eta^n(r)=\frac{r}{2^n(1-r)+r},\forall r\in[0,1]$.
$\lim_{n\to\infty}\eta^n(r)=\lim_{n\to\infty}\frac{r}{2^n(1-r)+r}=0,\forall r\in[0,1)$.
Clearly, $\eta(r)<r,\forall r\in(0,1)$ and $\eta(0)=0$.
Definition 2.5. Let $(X,M,N,*,\diamond)$ be an intuitionistic fuzzy metric space and $\psi,\eta\in\Psi$. A mapping $T:X\to X$ is called an intuitionistic fuzzy (ψ,η)-contractive mapping if $M(T(u),T(v),t)\geq\psi(M(u,v,t))$ and $N(T(u),T(v),t)\leq\eta(N(u,v,t)),\forall u,v\in X$ and $t>0$.
Proposition 2.6. An intuitionistic fuzzy (ψ,η)-contractive mapping is continuous.
Proof. Let $T:X\to X$ be an intuitionistic fuzzy (ψ,η)-contractive mapping and $\{u_n\}$ be a sequence convergent to $u\in X$. That is $\lim_{n\to\infty}M(u_n,u,t)=1$ and $\lim_{n\to\infty}N(u_n,u,t)=0$.
Now, let us prove $\lim_{n\to\infty}M(T(u_n),T(u),t)=1$ and $\lim_{n\to\infty}N(T(u_n),T(u),t)=0$.
Since T is the intuitionistic fuzzy (ψ,η)-contractive mapping,
$\lim_{n\to\infty}M(T(u_n),T(u),t)\geq\lim_{n\to\infty}\psi(M(u_n,u,t))=\psi(\lim_{n\to\infty}M(u_n,u,t))=\psi(1)=1$.
$\lim_{n\to\infty}N(T(u_n),T(u),t)\leq\lim_{n\to\infty}\eta(N(u_n,u,t))=\eta(\lim_{n\to\infty}N(u_n,u,t))=\eta(0)=0$.
That is, $\lim_{n\to\infty}M(T(u_n),T(u),t)=1$ and $\lim_{n\to\infty}N(T(u_n),T(u),t)=0$.
Therefore, intuitionistic fuzzy (ψ,η)-contractive mapping is continuous.
Theorem 2.7. Every intuitionistic fuzzy (ψ,η)-contractive mapping on a complete intuitionistic fuzzy metric space has a unique fixed point.
Proof. Let $T:X\to X$ be an intuitionistic fuzzy (ψ,η)-contractive mapping. Let $u_0\in X$ and define a sequence u_n in $X,\forall n\in N$ as follows:

$$u_{n+1}=T(u_n).$$

Then $\forall t>0$,

$$\begin{aligned}M(u_n,u_{n+1},t)&=M(T(u_{n-1}),T(u_n),t)\\&\geq\psi(M(u_{n-1},u_n,t))\\&=\psi(M(T(u_{n-2}),T(u_{n-1}),t))\\&\geq\psi^2(M(u_{n-2},u_{n-1},t))\\&\ldots\\&\geq\psi^n(M(u_0,u_1,t)).\end{aligned}$$

By taking limit as $n\to\infty$ and by our assumption

$$\lim_{n\to\infty}M(u_n,u_{n+1},t)=1.$$

$$\begin{aligned}M(u_{n+1},u_{n+2},t)&=M(T(u_n),T(u_{n+1}),t)\\&\geq\psi(M(u_n,u_{n+1},t))\\&=\psi(M(T(u_{n-1}),T(u_n),t))\\&\geq\psi^2(M(u_{n-1},u_n,t))\\&\ldots\\&\geq\psi^n(M(u_1,u_2,t)).\end{aligned}$$

By taking limit as $n\to\infty$, and by our assumption $\lim_{n\to\infty}M(u_{n+1},u_{n+2},t)=1$.

Now,
$M(u_n,u_{n+p},t)$
$\geq M(u_n,u_{n+1},\frac{t}{p})*\ldots*M(u_{n+p-1},u_{n+p},\frac{t}{p})$.
By taking limit $n\to\infty$, we have,

$$\begin{aligned}\lim_{n\to\infty}M(u_n,u_{n+p},t) &\geq \lim_{n\to\infty}M(u_n,u_{n+1},\frac{t}{p})*\ldots* \\ \lim_{n\to\infty}M(u_{n+p-1},u_{n+p},\frac{t}{p}) \\ &\geq 1*\ldots*1 \\ &= 1.\end{aligned}$$

That is,
$$\lim_{n\to\infty}M(u_n,u_{n+p},t)=1.$$

Again, $\forall t>0$,

$$\begin{aligned}N(u_n,u_{n+1},t) &= N(T(u_{n-1}),T(u_n),t) \\ &\leq \eta(N(u_{n-1},u_n,t)) \\ &= \eta(N(T(u_{n-2}),T(u_{n-1}),t)) \\ &\leq \eta^2(N(u_{n-2},u_{n-1},t)) \\ &\ldots \\ &\leq \eta^n(N(u_0,u_1,t)).\end{aligned}$$

By taking limit as $n\to\infty$ and by our assumption

$$\lim_{n\to\infty}N(u_n,u_{n+1},t)=0.$$

Similarly, we can prove,

$$\lim_{n\to\infty}N(u_{n+1},u_{n+2},t)=0.$$

Now,

$$N(u_n,u_{n+p},t)\leq N(u_n,u_{n+1},\frac{t}{p})\diamond\ldots\diamond N(u_{n+p-1},u_{n+p},\frac{t}{p}).$$

By taking limit as $n\to\infty$,

$$\begin{aligned}\lim_{n\to\infty}N(u_n,u_{n+p},t) &\leq \lim_{n\to\infty}N(u_n,u_{n+1},\frac{t}{p})\diamond\ldots\diamond \\ \lim_{n\to\infty}N(u_{n+p-1},u_{n+p},\frac{t}{p}) \\ &\leq 0\diamond\cdots\diamond 0 \\ &= 0.\end{aligned}$$

That is,
$$\lim_{n\to\infty}N(u_n,u_{n+p},t)=0.$$

Hence,$\{u_n\}$ is a Cauchy sequence in X.

Since $(X,M,N,*,\diamond)$ is a complete fuzzy metric space, there exists $u\in X$ such that $\lim_{n\to\infty}M(u_n,u,t)=1$ and $\lim_{n\to\infty}N(u_n,u,t)=0$. for each $t>0$.

Since T is continuous,

$$T(u)=T(\lim_{n\to\infty}u_n)=\lim_{n\to\infty}T(u_n)=\lim_{n\to\infty}u_{n+1}=u.$$

That is $T(u)=u$.
Uniqueness:
Assume $v=T(v)$ for some $v\in X$. Then for $t>0$, we have,

$$\begin{aligned}M(u,v,t) &= M(T(u),T(v),t) \\ &\geq \psi(M(u,v,t)) \\ &\ldots \\ &\geq \psi^n(M(u,v,t)).\end{aligned}$$

Taking limit as $n\to\infty$ and by our assumption,
$M(u,v,t)\geq\lim_{n\to\infty}\psi^n(M(u,v,t))=1$.
That is, $M(u,v,t)=1$.
Again, for $t>0$,

$$\begin{aligned}N(u,v,t) &= N(T(u),T(v),t) \\ &\leq \eta(N(u,v,t)) \\ &\ldots \\ &\geq \eta^n(N(u,v,t)).\end{aligned}$$

By taking limit as $n\to\infty$ and by our assumption,
$N(u,v,t)\leq\lim_{n\to\infty}\eta^n(N(u,v,t))=0$.
That is, $N(u,v,t)=0$.
Therefore, $u=v$.
Hence T has a unique fixed point in X.

Example 2.8. Let $X=[0,\infty)$ with the metric d defined by $d(u,v)=|u-v|$, define $M(u,v,t)=\frac{t}{t+d(u,v)}$, and $N(u,v,t)=\frac{d(u,v)}{t+d(u,v)},\forall u,v\in X$ and $t>0$. Note that, $(X,M,N,*,\diamond)$ where $a*b=ab$ and $a\diamond b=\min\{1,a+b\}$ is a complete intuitionistic fuzzy metric space.

A map $T:X\to X$ is defined by $T(u)=\frac{8-u}{3}$ and $T(v)=\frac{8-v}{3}$.

Define the map $\psi:[0,1]\to[0,1]$ by $\psi(s)=\frac{2s}{s+1}$ for each $s\in[0,1]$ and $\psi\in\Psi$.

$$\begin{aligned}& M(T(u),T(v),t)\geq\psi(M(u,v,t)) \\ \text{if}\quad & M(\frac{8-u}{3},\frac{8-v}{3},t)\geq\frac{2M(u,v,t)}{M(u,v,t)+1} \\ \text{That is if}\quad & \frac{t}{t+d(\frac{8-u}{3},\frac{8-v}{3})}\geq\frac{\frac{2t}{t+d(u,v)}}{\frac{t}{t+d(u,v)}+1} \\ \text{That is if}\quad & \frac{t}{t+\left|\frac{8-u}{3}-\frac{8-v}{3}\right|}\geq\frac{\frac{2t}{t+|u-v|}}{\frac{t}{t+|u-v|}+1} \\ \text{That is if}\quad & \frac{t}{t+\frac{|u-v|}{3}}\geq\frac{t}{t+\frac{|u-v|}{2}} \\ \text{That is if}\quad & t+\frac{|u-v|}{2}\geq t+\frac{|u-v|}{3} \\ \text{That is if}\quad & 3\geq 2.\end{aligned}$$

Again define the map $\eta:[0,1]\to[0,1]$ by $\eta(r)=\frac{r}{2-r}$ for each $r\in[0,1]$ and $\eta\in\Psi$.

$$\begin{aligned}
&N(T(u),T(v),t)\le\eta(N(u,v,t))\\
&\text{if}\quad N(\frac{8-u}{3},\frac{8-v}{3},t)\le\frac{N(u,v,t)}{2-N(u,v,t)}\\
&\text{That is if}\quad \frac{d(\frac{8-u}{3},\frac{8-v}{3})}{t+d(\frac{8-u}{3},\frac{8-v}{3})}\le\frac{\frac{d(u,v)}{t+d(u,v)}}{2-\frac{d(u,v)}{t+d(u,v)}}\\
&\text{That is if}\quad \frac{\left|\frac{8-u}{3}-\frac{8-v}{3}\right|}{t+\left|\frac{8-u}{3}-\frac{8-v}{3}\right|}\le\frac{\frac{|u-v|}{t+|u-v|}}{2-\frac{|u-v|}{t+|u-v|}}\\
&\text{That is if}\quad \frac{\frac{|u-v|}{3}}{t+\frac{|u-v|}{3}}\le\frac{|u-v|}{2t+|u-v|}\\
&\text{That is if}\quad 2t+|u-v|\le 3t+|u-v|\\
&\text{That is if}\quad 2\le 3.
\end{aligned}$$

Therefore T is the intuitionistic fuzzy (ψ,η)-contractive mapping.
Then 2 is the unique fixed point.
Hence every fintuitionistic fuzzy (ψ,η)-contractive mapping on a complete fuzzy metric space has a unique fixed point.

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