Journal of Statistics Applications & Probability
An International Journal

http://dx.doi.org/10.18576/jsap/060210

Characterization of Lindley Distribution Based on Truncated Moments of Order Statistics

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Received: 24 Feb. 2017, Revised: 20 Mar. 2017, Accepted: 22 Mar. 2017
Published online: 1 Jul. 2017

Abstract: Characterization of a probability distribution plays an important role in statistics and probability applications. A probability distribution can be characterized through various methods. In this paper, a new theorem for characterization of the continuous distribution based on truncated moments of order statistics is introduced. Via this theorem, the characterization of Lindley distribution by the truncated moments of order statistics has been established. Finally, a simulation study is conducted to help the practitioner to check whether the available data follows the underlying distribution.

Keywords: Lindley distribution, Characterization, Survival function, Order statistics, Truncated moments.

1 Introduction

In designing a stochastic model for a particular modeling problem, an investigator will be vitally interested to know if their model fits the requirements of a specific underlying probability distribution. To this end, the investigator will rely on the characterizations of the selected distribution. Generally speaking, the problem of characterizing a distribution is an important problem in various fields and has recently attracted the attention of many researchers. Consequently, various characterization results have been reported in the literature. Laurent [10] presented characterization of distributions by truncated moments. Gupta & Gupta [8] characterized distributions by the moments of residual life. Galambos & Kotz [6] presented a various methods of characterizations of probability distributions. In recent years, there has been a great interest in the characterizations of probability distributions. For example, Su & Huang [13] studied the characterizations of distributions based on conditional expectations. Nanda [12] studied the characterizations through mean residual life and failure rates functions of absolutely continuous random variables. Afify, et al. [1] used the conditional expectation of order statistics to characterize the exponential and power function distributions. Ahsanullah et al. [4] characterized Lindley distribution by truncated moments. Ahsanullah & Hamedani [2] characterized beta of the first kind and power function distribution using order statistics. Many other papers dealing with characterization, see for instance, Hamedani [9] and Ahsanullah & Shakil [3].

In this paper, a theorem for characterization the continuous distributions based on the truncated moment of the r^{th} order statistic is introduced. The advantage of the characterization based on this theorem relates the survival function of a distribution to the solution of a first order differential equation. Furthermore, the characterization of Lindley distribution by truncated moments order statistics will be presented. The paper is organized as follows; the main statistical functions of Lindely distribution are introduced in Section 2. Section 3 includes the main characterization theorems and results for Lindley distribution based on the truncated moments of order statistics and a simulation study is used to illustrate the characterization results.

2 Lindley Distribution

Recently, one parameter Lindley distribution has attracted the researchers for its use in modelling lifetime data, and it has been observed in several papers that this distribution has performed excellently. The Lindley distribution was originally

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proposed by Lindley [11] in the context of Bayesian statistics. Ghitany et al. [7] studied many statistical properties of the Lindley distribution from the reliability and survival analysis point of view. They also showed, using a real data set, that the Lindley distribution provides a better fit than the exponential distribution.

A random variable X is said to have Lindley distribution with parameter θ if its probability density function (p.d.f.) is defined as:

$$f(x) = \left(\frac{\theta^2}{\theta+1}\right)(1+x)e^{-\theta x} \quad ;x,\theta > 0 \tag{1}$$

The corresponding cumulative distribution function (c.d.f.), hazard rate function and survival function are, respectively, given by

$$F(x) = 1 - \left(\frac{1+\theta+\theta x}{1+\theta}\right)e^{-\theta x}, \tag{2}$$

$$h(x) = \frac{\theta^2(1+x)}{1+\theta(1+x)} \tag{3}$$

and

$$S(x) = \left(\frac{1+\theta+\theta x}{1+\theta}\right)e^{-\theta x} \tag{4}$$

The mean and variance of the Lindley distribution with parameter θ are as follows,

$$\mu = \frac{\theta+2}{\theta(\theta+1)} \quad , \quad \sigma^2 = \frac{\theta^2+4\theta+2}{\theta^2(\theta+1)^2}$$

3 Characterization of Lindley Distribution by Truncated Moments of Order Statistics

Suppose $X_1, X_2, \ldots, X_n$ be a random sample of size n from a continuous distribution with c.d.f. $F(x)$, p.d.f. $f(x)$ and survival function $S(x) = 1 - F(x)$. Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be the corresponding order statistics. Then the p.d.f. of $X_{(r)}$ and the joint p.d.f. of $X_{(r)}$ and $X_{(r+1)}$ are given, respectively, as follows (Arnold et al. [5])

$$f_{X_{(r)}}(x) = \frac{n!}{(r-1)!(n-r)!} f(x)[F(x)]^{r-1}[1-F(x)]^{n-r} \tag{5}$$

$$f_{X_{(r)},X_{(r+1)}}(x,y) = \frac{n!}{(r-1)!(n-r-1)!} f(x)f(y)[F(x)]^{r-1}[1-F(x)]^{n-r-1} \tag{6}$$

From Equations (5) and (6) the conditional p.d.f. of $X_{(r+1)}$ given $X_{(r)} = x$ is given by

$$f_{X_{(r+1)}|X_{(r)}}(y|x) = \frac{f_{X_{(r)},X_{(r+1)}}(x,y)}{f_{X_{(r)}}(x)} = (n-r)\frac{[1-F(y)]^{n-r-1}}{[1-F(x)]^{n-r}} f(y) \tag{7}$$

In this section, Lindley distribution is to be characterized through truncated moments of order statistics given by

$$E(X_{(r+1)}^s | X_{(r)} = x) = \int_x^\infty y^s f_{X_{(r+1)}|X_{(r)}}(y|x)dy, \quad s = 1,2,3,\ldots, r = 1,2,\ldots,n-1. \tag{8}$$

3.1 The characterization Theorems

In this subsection a theorem for characterization of the continuous distributions based on the truncated moments of the order statistics has been introduced. The advantage of the characterizations using this theorem is that, the survival function, $1 - F(x)$, need not have a closed form and are given in terms of an integral whose integrand depends on the solution of a first order differential equation, which can serve as a bridge between probability and differential equation. Based on this theorem a characterization of Lindley distribution using the truncated moments of order statistics is presented.

Theorem 1. *Suppose that X is a continuous random variable defined on the interval (a,b), where $a = inf\{x, s.t.\ F(x) > 0\}$ and $b = sup\{x, s.t.\ F(x) < 1\}$, with c.d.f. $F(x)$, p.d.f $f(x)$ and survival function $S(x) = 1-F(x)$. Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ denote the order statistics of a random sample of size n from $F(x)$. If the truncated moments of the order statistics given by*

$$E(X^s_{(r+1)}|X_{(r)} = x) = w(x), \quad s = 1,2,3,\ldots, r = 1,2,\ldots,n-1.$$

where $w(x)$ is a differentiable function in (a,b), then

$$S(x) = C\exp\left[-\frac{1}{(n-r)}\int \frac{w'(x)}{(w(x)-x^s)}dx\right], \quad x \in (a,b)$$

where C is a normalization constant.

Proof. Since

$$E(X^s_{(r+1)}|X_{(r)} = x) = \frac{(n-r)}{[1-F(x)]^{n-r}}\int_x^\infty y^s[1-F(y)]^{n-r-1}f(y)dy$$

It follows that

$$\frac{(n-r)}{[1-F(x)]^{n-r}}\int_x^\infty y^s[1-F(y)]^{n-r-1}f(y)dy = w(x)$$

which implies to

$$(n-r)\int_x^\infty y^s[S(y)]^{n-r-1}f(y)dy = [S(x)]^{n-r}w(x) \tag{9}$$

Differentiating both sides of Equation (9) with respect to x gives

$$-(n-r)x^s f(x)[S(x)]^{n-r-1} = [S(x)]^{n-r}w'(x) - (n-r)f(x)[S(x)]^{n-r-1}w(x)$$

On simplification of the above equation, one obtains the following

$$(n-r)f(x)[w(x)-x^s] = w'(x)S(x) \tag{10}$$

Since $S'(x) = -f(x)$, Equation (10) can be rewritten as follows

$$S'(x) = -\frac{w'(x)}{(n-r)(w(x)-x^s)}S(x)$$

Or equivalently

$$S'(x) + \frac{w'(x)}{(n-r)(w(x)-x^s)}S(x) = 0$$

which is a first order linear differential equation with respect to unknown function $S(x)$. Thus, the general solution of this differential equation is

$$S(x) = C\exp\left[-\frac{1}{(n-r)}\int \frac{w'(x)}{(w(x)-x^s)}dx\right], \quad x \in (a,b)$$

where C is a normalization constant. This complete the proof.

Next, the characterization of Lindley distribution via truncated moments of order statistics is provided in Theorem 2 below.

Theorem 2. *Suppose X be a continuous random variable with distribution function $F(x)$ and survival function $S(x) = 1-F(x)$. Let $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ denote the order statistics of a random sample of size n from $F(x)$. Then X has the Lindley distribution with parameter $\theta > 0$ if and only if*

$$E(X_{(r+1)}|X_{(r)} = x) = x + \frac{\Gamma[n-r+1, n-r(1+\theta+\theta x)]}{\theta(n-r)^{n-r+1}(1+\theta+\theta x)^{n-r}e^{-(n-r)(1+\theta+\theta x)}}, r = 1,2,\ldots,n-1. \tag{11}$$

Proof
Necessity: from Equations (7), (8), one obtains

$$\begin{aligned}
E(X_{(r+1)}|X_{(r)}=x) &= \int_x^\infty y(n-r)\frac{[1-F(y)]^{n-r-1}}{[1-F(x)]^{n-r}}f(y)dy \\
&= x+\frac{1}{(1+\theta+\theta x)^{n-r}e^{-(n-r)\theta x}}\int_x^\infty (1+\theta+\theta y)^{n-r}e^{-(n-r)\theta y}dy \\
&= x+\frac{1}{(1+\theta+\theta x)^{n-r}e^{-(n-r)\theta x}}\left(\frac{e^{(n-r)(1+\theta)}\Gamma[n-r+1,(n-r)(1+\theta+\theta x)]}{\theta(n-r)^{n-r+1}}\right) \\
&= x+\frac{\Gamma[n-r+1,(n-r)(1+\theta+\theta x)]}{\theta(n-r)^{n-r+1}(1+\theta+\theta x)^{n-r}e^{-(n-r)(1+\theta+\theta x)}}
\end{aligned}$$

Sufficiency: for simplification, let $p=n-r$,
$w(x)=x+v(x)$
and $w'(x)=1+v'(x)$
where, $v(x)=\frac{\Gamma[p+1,\ p(1+\theta+\theta x)]}{\theta p^{p+1}(1+\theta+\theta x)^p e^{-p(1+\theta+\theta x)}}$
Then, it is seen that

$$\begin{aligned}
-\frac{1}{(n-r)}\int\frac{w'(x)}{(w(x)-x)}dx &= -\frac{1}{p}\int\frac{1+v'(x)}{v(x)}dx=-\frac{1}{p}\left[\int\frac{1}{v(x)}dx+\int\frac{v'(x)}{v(x)}dx\right] \\
&= -\frac{1}{p}\left[\int\frac{\theta p^{p+1}(1+\theta+\theta x)^p e^{-p(1+\theta+\theta x)}}{\Gamma[p+1,\ p(1+\theta+\theta x)]}dx+\ln v(x)\right] \\
&= \frac{1}{p}\left[\ln(\Gamma[p+1,\ p(1+\theta+\theta x)])-\ln\left(\frac{\Gamma[p+1,\ p(1+\theta+\theta x)]}{\theta p^{p+1}(1+\theta+\theta x)^p e^{-p(1+\theta+\theta x)}}\right)\right] \\
&= \frac{1}{p}\left[\ln\left(\Gamma[p+1,\ p(1+\theta+\theta x)]\times\frac{\theta p^{p+1}(1+\theta+\theta x)^p e^{-p(1+\theta+\theta x)}}{\Gamma[p+1,\ p(1+\theta+\theta x)]}\right)\right] \\
&= \ln\left(\theta^{\frac{1}{p}}p^{\frac{1}{p}+1}(1+\theta+\theta x)e^{-(1+\theta+\theta x)}\right)
\end{aligned}$$

Hence, using Theorem 1, one gets

$$S(x)=C\exp\left[-\frac{1}{n-r}\int\frac{w'(x)}{(w(x)-x)}dx\right]=C\left(\theta^{\frac{1}{p}}p^{\frac{1}{p}+1}(1+\theta+\theta x)e^{-(1+\theta+\theta x)}\right)$$

where C is a normalization constant. Let $C=\frac{e^{-(1+\theta)}}{(1+\theta)\theta^{\frac{1}{p}}p^{\frac{1}{p}+1}}$, thus

$$S(x)=\left(\frac{1+\theta+\theta x}{1+\theta}\right)e^{-\theta x}$$

This completes proof of the theorem.

3.2 Simulation Study

This section illustrates the practical importance of the results above through an experimental validation, using simulated data. The aim of the simulation study is to show the characterization results give an easy way for the practitioner to test whether the available data follows a particular distribution.

To validate the correctness of the theoretical results obtained in this paper for the characterization of Lindley distribution based on truncated moments of order statistics, a simulation study has been presented by generating samples $X_i, i=1,2,\ldots mn$ are randomly classified into m samples, each of containing n observations. The simulation study showed in *Table*1 is conducted with $m=30$ and $n=30,50,70$ at various choices of the order r and the parameter θ. The right most column in *Table*1 shows the absolute relative difference between the two sides of the characterizing Equation (11) is very small which validate the correctness of characterization results for Lindley distribution.

*Table*1: Verification of the characterization results

n	r	θ	X	$L.H.S.$	$R.H.S.$	$\lvert L.H.S. - R.H.S. \rvert$	$\lvert \frac{L.H.S.-R.H.S.}{R.H.S.} \rvert$
30	10	0.1	17.2876	18.2329	18.0505	0.1823	0.0100
		0.5	2.1021	2.2668	2.1021	0.1647	0.0783
		2	0.4323	0.4758	0.4323	0.0435	0.1006
		5	0.5089	0.5876	0.5089	0.0787	0.1546
		10	0.1070	0.1176	0.1070	0.0106	0.0990
50	20	0.1	17.5721	18.1582	18.0802	0.0779	0.0043
		0.5	2.3782	2.4929	2.4832	0.0097	0.0039
		2	0.4805	0.5085	0.5027	0.0058	0.0115
		5	0.4257	0.4570	0.4333	0.0237	0.0546
		10	0.1211	0.1287	0.1247	0.0040	0.0320
70	40	0.1	17.2647	17.7243	17.7757	0.0514	0.0028
		0.5	2.7322	2.8379	2.8337	0.0042	0.0014
		2	0.5724	0.5972	0.5943	0.0029	0.0048
		5	0.4032	0.4220	0.4108	0.0112	0.0272
		10	0.0581	0.0604	0.0617	0.0013	0.0210

4 Conclusion

In this paper, a new theorem for characterization the continuous distribution by truncated moments of order statistics is introduced. This theorem is used to provide the characterization of Lindley distribution based on the truncated moments of the r^{th} order statistics and a simulation study is performed to validate the correctness of the theoretical results

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