Progress in Fractional Differentiation and Applications
An International Journal

http://dx.doi.org/10.18576/pfda/030307

Technical Note on a New Definition of Fractional Derivative

Vincenzo Ciancio[1,*] *and Bruno Felice Filippo Flora*[2]

[1] Department of Mathematical and Computer Sciences, Physical Sciences and Earth Sciences, University of Messina, Salita Sperone 31, 98166 I-Messina, Italy
[2] Engineering Office, G. Matteotti 39, 89044 I-Locri, Italy

Received: 2 Jan. 2017, Revised: 20 Feb. 2017, Accepted: 28 Feb. 2017
Published online: 1 Jul. 2017

Abstract: In this letter, we have discovered a connection between the new fractional time derivative of Caputo-Fabrizio and his associated ordinary derivative. We will prove this connection, giving an evident physical meaning.

Keywords: Fractional derivatives, fractional calculus,physical meaning.

1 Introduction

The fractional calculus has used in numerous applications of different scientific fields. Historically, various definitions of fractional derivative had produced from Leibniz, Abel, Riemann, Liouville, and so on. In the last century, fractional derivative has used for characterize non-linear behavior of materials subjected to fatigue or to analyze chaotic phenomena as Brownian motion. Recently, in [1], Caputo and Fabrizio have introduced a "New Fractionary Time Derivative" (NFD_t) for characterize behavior of viscoelastic materials, when order of derivation, α, is between 0.6 and 1. In this letter, it has given an evident physical meaning of the NFD_t, by means of the connection between ordinary derivative, respect to time, of the response in output from a physical system and the fractional time derivative of the input function to system itself.

2 The Connection Between the NFD_t and its Associated Ordinary Derivative

Let be $f(t) \in H^1(a,b)$, with $b > a$ and $\alpha \in [0,1]$. Let $M(\alpha)$ be finite and continue function in [0,1], such that: $M(0) = M(1) = 1$.

Caputo - Fabrizio defined the NFD_t as the following operator [1]:

$$\mathfrak{D}_t^{(\alpha)}[f(t)] = \frac{M(\alpha)}{1-\alpha}\int_a^t \frac{d}{d\tau}[f(\tau)]\, exp\left[-\left(\frac{\alpha}{1-\alpha}\right)(t-\tau)\right] d\tau, \tag{1}$$

Dirac's Delta function, $\delta(t-\tau)$, has defined, in the sense of the theory of distributions, shifted of -t respect to the time axis origin, the following limit:

$$\delta(t-\tau) = \lim_{\alpha \longrightarrow 1}\left(\frac{\alpha}{1-\alpha}\right) exp\left[-\left(\frac{\alpha}{1-\alpha}\right)(t-\tau)\right]. \tag{2}$$

Theorem 1. *Let $g(\tau)$ be given:*

$$g(\tau) = \int_a^\tau f(\mu)\, exp\left[-\left(\frac{\gamma}{1-\gamma}\right)(\tau-\mu)\right] d\mu \tag{3}$$

* Corresponding author e-mail: ciancio@unime.it

with $\gamma \in (0,1)$*; We obtain:*

$$\frac{d}{dt}[g(t)] = \mathfrak{D}_t^{(1)}[g(t)] = \left[\frac{1-\gamma}{M(\gamma)}\right] \mathfrak{D}_t^{(\gamma)}[f(t)]. \tag{4}$$

By means of (1), we conclude:

$$\mathfrak{D}_t^{(1)}[g(t)] = \lim_{\alpha \longrightarrow 1} \left\{ \mathfrak{D}_t^{(\alpha)}[g(t)] \right\} =$$

$$= \lim_{\alpha \longrightarrow 1} \left\{ \frac{M(\alpha)}{1-\alpha} \int_a^t \frac{d}{d\tau}[g(\tau)]\, exp\left[-\left(\frac{\alpha}{1-\alpha}\right)(t-\tau)\right] d\tau \right\} =$$

$$= \lim_{\alpha \longrightarrow 1} \left\{ \frac{M(\alpha)}{1-\alpha} \int_a^t \frac{d}{d\tau} \left\{ \int_a^\tau f(\mu)\, exp\left[-\left(\frac{\gamma}{1-\gamma}\right)(\tau-\mu)\right] d\mu \right\} exp\left[-\left(\frac{\alpha}{1-\alpha}\right)(t-\tau)\right] d\tau \right\}.$$

Putting:

$$\alpha\, \frac{M(\alpha)}{1-\alpha} = \frac{\gamma}{1-\gamma}$$

and inverting the integral operation with that of limit, whereas M(1)=1, we obtain:

$$\mathfrak{D}_t^{(1)}[g(t)] = \int_a^t \frac{d}{d\tau} \left\{ \int_a^\tau f(\mu) \lim_{\alpha \longrightarrow 1} \left\{ \left(\alpha \frac{M(\alpha)}{1-\alpha}\right) exp\left[-\left(\alpha \frac{M(\alpha)}{1-\alpha}\right) (\tau-\mu)\right] \right\} d\mu \right\} \lim_{\alpha \longrightarrow 1} \left\{ exp\left[-\frac{\gamma(t-\tau)}{M(\alpha)(1-\gamma)}\right] \right\} d\tau.$$

By means of the (2), we report that:

$$\mathfrak{D}_t^{(1)}[g(t)] = \int_a^t \frac{d}{d\tau} \left\{ \int_a^\tau f(\mu)\delta(\tau-\mu)\, d\mu \right\} exp\left[-\left(\frac{\gamma}{1-\gamma}\right)(t-\tau)\right] d\tau,$$

i.e.:

$$\mathfrak{D}_t^{(1)}[g(t)] = \int_a^t \frac{d}{d\tau}[f(\tau)]\, exp\left[-\left(\frac{\gamma}{1-\gamma}\right)(t-\tau)\right] d\tau.$$

Reapplying the (1) we get:

$$\mathfrak{D}_t^{(1)}[g(t)] = \left[\frac{1-\gamma}{M(\gamma)}\right] \mathfrak{D}_t^{(\gamma)}[f(t)].$$

3 Physical Meaning of the NFD_t

With reference at the theory of circuits and signals [2], the (3) represents the output signal, $g(t)$, from a linear system. Its impulse response, $h(t)$, has given from following expression:

$$h(t) = exp\left(-\frac{t}{R_1C_1}\right) \mathscr{H}(t-a), \tag{5}$$

where $\mathscr{H}(t)$ is Heaveside function. This system is known as a passive analog low pass filter.
From Figure 1, putting:

$$R_1C_1 = \left(\frac{1-\gamma}{\gamma}\right) T$$

that it represents the time constant of the low pass filter, we obtain the following parameters:

$$C_2 = \left(\frac{1}{1-\gamma}\right) C_1 = \left(\frac{1}{\gamma}\right)\left(\frac{T}{R_1}\right),$$

$$\gamma = \frac{\frac{T}{R_1C_1}}{1+\frac{T}{R_1C_1}},$$

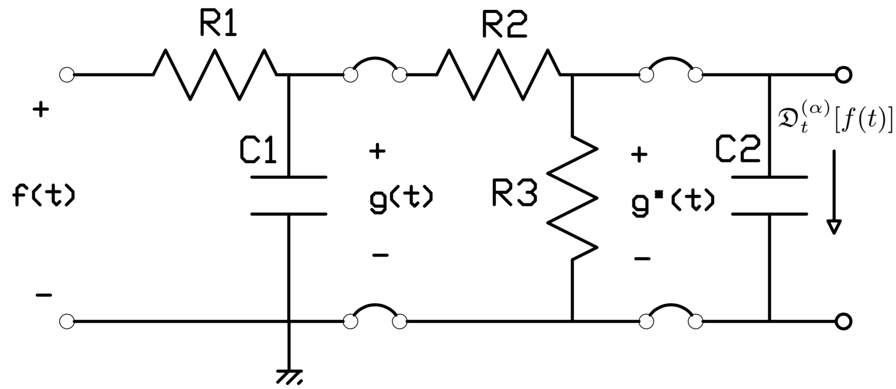

Fig. 1: A physical system for the calculation of the NFD_t

$$M(\gamma) = \left(\frac{1-\gamma}{\gamma}\right)\left(\frac{R_3}{R_2+R_3}\right).$$

The term, $\frac{R_3}{R_2+R_3}$, represents the attenuation on the output signal $g(t)$:

$$g^*(t) = \left(\frac{\gamma}{1-\gamma}\right) M(\gamma)\, g(t).$$

The current through the second capacitor is the derivative of the output signal from RC circuit, attenuated and multiplied by the capacity, C_2:

$$\left(\frac{\gamma}{1-\gamma}\right) C_2 M(\gamma)\, \dot{g}(t).$$

It coincides with the new fractional derivative, $\mathfrak{D}_t^{(\gamma)}[f(t)]$, putting $T = R_1$, i.e.: $C_1 = \frac{1-\gamma}{\gamma}(Farad)$ e $C_2 = \frac{1}{\gamma}(Farad)$. Finally, from Figure 1, we observe the new fractional derivative of the input tension, $f(t)$, is a electric current which depends on the parameters of the system. There are a number of new fractional time derivative applications some of which are reported in [3,4,5,6,7,8].

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