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Oscillation of Second-Order Nonlinear Impulsive Dynamic Equations on Time Scales

H. A. Agwa, Ahmed M. M. Khodier and Heba M. Atteya*

Department of Mathematics, Faculty of Education, Ain Shams University, Roxy, Cairo, Egypt.

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Abstract: In this paper, we use Riccati transformation technique and the impulsive inequality to establish some new oscillation criteria for the second-order nonlinear impulsive dynamic equation on a time scale $\mathbb{T}$. Our results generalize and extend some pervious results [14,15,16,18,20]. Finally, we give some examples to show that impulses play a dominant part in the oscillations of dynamic equations on time scales and to illustrate our main results.

Keywords: Oscillation, time scales, impulsive dynamic equations.

1 Introduction

The theory of time scales was introduced by Hilger [12] in order to unify, extend and generalize ideas from discrete calculus, quantum calculus and continuous calculus to arbitrary time scale calculus. A time scale is an arbitrary closed subset of the reals. When time scale equals to the reals or to the integers, it represents the classical theories of differential and difference equations. Many other interesting time scales exist, e.g., $\mathbb{T} = q^{\mathbb{N}_0} := \{q^t : t \in \mathbb{N}_0 \text{ for } q > 1\}$ (which has important applications in quantum theory), $\mathbb{T} = h\mathbb{N}$ with $h > 0$, $\mathbb{T} = \mathbb{N}^2$ and $\mathbb{T} = \mathbb{T}^n$ (the space of the harmonic numbers). For an introduction to time scale calculus and dynamic equations, see Bohner and Peterson books [7,8].
Recently, many results have been obtained on the oscillation and nonoscillation of dynamic equations on time scales (see [9,10,13,19,21] and references cited therein).
Impulsive dynamic equations on time scales have been investigated by Agarwal et al. [1], Belarbi et al. [2], Benchohra et al. [3-6] and so forth. Benchohra et al. [6] considered the existence of extremal solutions for a class of second order impulsive dynamic equations on time scales.
The oscillation of impulsive differential equations and impulsive difference equations has been investigated by many authors and many results were obtained (see [11, 17] etc. and the references cited therein). But fewer papers are on the oscillation of impulsive dynamic equations on time scales.

Qiaoluan Li and Lina Zhou [18] studied the oscillation criteria for second-order impulsive dynamic equations of the form

$$\begin{cases} x^{\Delta\Delta}(t) + q(t)x(\sigma(t)) = e(t),\ t \in \mathbb{J} := [0,\infty) \cap \mathbb{T},\ t \neq t_k, \\ k = 1,2,..., \\ x(t_k^+) = a_k(x(t_k^-)),\ x^{\Delta}(t_k^+) = b_k(x^{\Delta}(t_k^-)), k = 1,2,..., \end{cases}$$

Liu and Xu [20] considered the forced super-linear impulsive ordinary differential equation

$$\begin{cases} (r(t)x'(t))' + p(t)|x(t)|^{\alpha-1}x(t) = e(t),\ t \in \mathbb{J} := [0,\infty) \cap \mathbb{T}, \\ t \neq t_k,\ k = 1,2,..., \\ x(t_k^+) = a_k(x(t_k^-)), x'(t_k^+) = b_k(x'(t_k^-)), k = 1,2,..., \end{cases}$$

Huang *et al.* [14,15] considered the second-order nonlinear impulsive dynamic equations

$$\begin{cases} x^{\Delta\Delta}(t) + f(t, x^{\sigma}(t)) = 0,\ t \in \mathbb{J} := [0,\infty) \cap \mathbb{T},\ t \neq t_k, \\ k = 1,2,..., \\ x(t_k^+) = g_k(x(t_k^-)),\ x^{\Delta}(t_k^+) = h_k(x^{\Delta}(t_k^-)), k = 1,2,..., \\ x(t_0^+) = x_0,\ x^{\Delta}(t_0^+) = x_0^{\Delta}, \end{cases}$$

* Corresponding author e-mail: hebamostafa480@gmail.com

Here, we are concerned with the oscillation of second-order nonlinear dynamic equation with impulses on a time scale $\mathbb{T}$ which is unbounded above

$$\begin{cases} (r(t)g(x^{\Delta}(t)))^{\Delta}+f(t,x^{\sigma}(t))=G(t,x^{\sigma}(t)), \\ t\in\mathbb{J}:=[0,\infty)\cap\mathbb{T},\ t\neq t_k,\ k=1,2,..., \\ x(t_k^+)=\xi_k(x(t_k^-)),\ x^{\Delta}(t_k^+)=h_k(x^{\Delta}(t_k^-)),k=1,2,..., \\ x(t_0^+)=x_0,\ x^{\Delta}(t_0^+)=x_0^{\Delta}, \end{cases} \tag{1}$$

where $\mathbb{T}$ is an unbounded above time scale with $0\in\mathbb{T}$, $t_k\in\mathbb{T}$, $0\leq t_0<t_1<t_2<...<t_k<...$, $\lim_{k\to\infty}t_k=\infty$.

Through out this paper we assumed the following conditions are satisfied:
(H_1) $r(t)>0$ and $f\in C_{rd}(\mathbb{T}\times\mathbb{R},\mathbb{R})$, $uf(t,u)>0$ $(u\neq 0)$ and $\frac{f(t,u)}{\varphi(u)}\geq p(t)$ $(u\neq 0)$, where $p(t)\in C_{rd}(\mathbb{T},[0,+\infty))$, $\varphi(u)\in C^1(\mathbb{R},\mathbb{R})$ and $u\varphi(u)>0$ $(u\neq 0)$, $\varphi'(u)\geq 0$.

(H_2) $G:\mathbb{T}\times\mathbb{R}\to\mathbb{R}$ is a function such that

$$uG(t,u)>0\, for\ u\neq 0 \text{ and } \frac{G(t,u)}{\varphi(u)}\leq e(t),u\neq 0,$$

(H_3) $\xi_k,h_k\in C(\mathbb{R},\mathbb{R})$ and there exist positive constants a_k,a_k^*,b_k *and* b_k^* such that

$$a_k^*\leq\frac{\xi_k(u)}{u}\leq a_k,\ b_k^*\leq\frac{h_k(u)}{u}\leq b_k,\ u\neq 0,\ k=1,2,....$$

(H_4) $g\in C_{rd}(\mathbb{R},\mathbb{R})$ is continuous and increasing function with $ug(u)>0$, $u\neq 0$, we have

(i) $g(uv)\leq g(u)g(v)$, $uv\neq 0$,
(ii) $\lambda_2g^{-1}(u)g^{-1}(v)\leq g^{-1}(uv)\leq\lambda_1g^{-1}(u)g^{-1}(v)$, $uv\neq 0,\lambda_1,\lambda_2>0$,

The purpose of this paper is to establish some new oscillation criteria for the second-order nonlinear impulsive dynamic equations (1.1) which is not studied before. Our results extend and improve some results established by [14,15,16,18,20] and can be applied to arbitrary time scales. Some examples are given to show that a dynamic equation is nonoscillatory, it may become oscillatory by adding some impulses to it. In this cases, impulses play a dominating part in oscillations of dynamic equations on time scales.
By a solution of (1.1), we mean that a nontrivial real valued function x satisfies (1.1) for $t\in\mathbb{T}$. A solution x of (1.1) is called oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is called nonoscillatory. Eq. (1.1) is said to be oscillatory if all of its solutions are oscillatory.

Throughout the remainder of the paper, we assume that, for each $k=1,2,...$, the points of impulses t_k are right-dense (rd for short). In order to define the solutions of (1.1), we introduce the spaces

$AC^i=\{x:\mathbb{J}_{\mathbb{T}}\to\mathbb{R}$ *is* $i-$ *times* $\Delta-$ *differetiable, whose ith delta derivative* $x^{\Delta^{(i)}}$ *is absolutely continuous*$\}$.

$PC=\{x:\mathbb{J}_{\mathbb{T}}\to\mathbb{R}$ *is right dense continuous except at* $t_k,k=1,2,...$ *for which* $x(t_k^-),x(t_k^+),x^{\Delta}(t_k^-)$ *and* $x^{\Delta}(t_k^+)$ *exist with* $x(t_k^-)=x(t_k),x^{\Delta}(t_k^-)=x^{\Delta}(t_k)\}$.

2 Main results

In this section, we use Riccati substitution on time scales and establish new oscillation criteria for Eq. (1). Before we state and prove our main oscillation results, we prove some lemmas which are important in proving our main results.

Lemma 1. *[15] Assume that* $m\in PC\cap AC^1$ $(\mathbb{J}_{\mathbb{T}}\setminus\{t_1,t_2,...\},\mathbb{R})$ *and*

$$\begin{cases} m^{\Delta}(t)\leq p(t)m(t)+q(t),t\in\mathbb{J}_{\mathbb{T}}:=[0,\infty)\cap\mathbb{T},\ t\neq t_k,k=1,2,..., \\ m(t_k^+)\leq d_km(t_k)+b_k,k=1,2,..., \end{cases} \tag{2}$$

then for $t\geq t_0$

$$\begin{aligned} m(t)\leq & m(t_0)\prod_{t_0<t_k<t}d_ke_p(t,t_0)+\sum_{t_0<t_k<t}\left(\prod_{t_k<t_j<t}d_je_p(t,t_k)\right)b_k \\ & +\int_{t_0}^{t}\prod_{s<t_k<t}d_ke_p(t,\sigma(s))q(s)\Delta s. \end{aligned} \tag{3}$$

Lemma 2. *Assume that* (H_1)*-*(H_4) *hold and there exists* $e(t)$ *such that* $(p(t)-e(t))\geq 0$ *and* $x(t)>0$, $t\geq T\geq t_0$ *is a nonoscillatory solution of (1). If*

$$\int_{t_0}^{\infty}g^{-1}\left(\frac{1}{r(s)}\right)\frac{g^{-1}\left(\prod_{t_0<t_k<s}g(b_k^*)\right)}{\prod_{t_0<t_k<s}a_k}\Delta s=\infty, \tag{4}$$

then $x^{\Delta}(t_k^+)\geq 0$ *and* $x^{\Delta}(t)\geq 0$ *for* $t\in(t_k,t_{k+1}]_{\mathbb{T}}$, *where* $t_k\geq T$.

Proof. First, we prove that $x^{\Delta}(t_k)\geq 0$ for $t_k\geq T$, otherwise, there exist some j such that $t_j\geq T$ and $x^{\Delta}(t_j)<0$, hence

$$x^{\Delta}(t_j^+)=h_j(x^{\Delta}(t_j))\leq b_j^*x^{\Delta}(t_j)<0.$$

Let $x^{\Delta}(t_j^+)=-\beta$ $(\beta>0)$. From Eq. (1), (H_1) and (H_2), we have for $t\in(t_{j+i-1},t_{j+i}]_{\mathbb{T}}$, $i=1,2,...,$

$$\begin{aligned} (r(t)g(x^{\Delta}(t)))^{\Delta}= & G(t,x^{\sigma}(t))-f(t,x^{\sigma}(t))\leq \\ & -\varphi(x^{\sigma}(t))(p(t)-e(t))\leq 0, \end{aligned}$$

i.e., $(r(t)g(x^{\Delta}(t)))$ is nonincreasing in $(t_{j+i-1},t_{j+i}]_{\mathbb{T}}$, $i=1,2,...$. Then, we get

$$g(x^{\Delta}(t_{j+1}))\leq\frac{r(t_j)}{r(t_{j+1})}g(x^{\Delta}(t_j^+)),$$

which implies that

$$x^{\Delta}(t_{j+1}) \leq \lambda_1 g^{-1}\left(\frac{r(t_j)}{r(t_{j+1})}\right)x^{\Delta}(t_j^+) = -\lambda_1\beta g^{-1}\left(\frac{r(t_j)}{r(t_{j+1})}\right) < 0, \tag{5}$$

$$\begin{aligned} x^{\Delta}(t_{j+2}) \leq& \lambda_1 g^{-1}\left(\frac{r(t_{j+1})}{r(t_{j+2})}\right)x^{\Delta}(t_{j+1}^+) \\ &= \lambda_1 g^{-1}\left(\frac{r(t_{j+1})}{r(t_{j+2})}\right)h_{j+1}(x^{\Delta}(t_{j+1})) \\ &\leq \lambda_1 b_{j+1}^* g^{-1}\left(\frac{r(t_{j+1})}{r(t_{j+2})}\right)x^{\Delta}(t_{j+1}) \\ &\leq -\lambda_1 b_{j+1}^*\beta g^{-1}\left(\frac{r(t_j)}{r(t_{j+2})}\right) < 0. \end{aligned} \tag{6}$$

By induction, we get

$$x^{\Delta}(t_{j+n}) \leq -\lambda_1\beta g^{-1}\left(\frac{r(t_j)}{r(t_{j+n})}\right)\prod_{i=1}^{n-1} b_{j+i}^* < 0. \tag{7}$$

Consider the following impulsive dynamic inequalities

$$\begin{cases} (r(t)g(x^{\Delta}(t)))^{\Delta} \leq 0, t \geq t_j,\ t \neq t_k, k = j+1, j+2, ..., \\ x^{\Delta}(t_k^+) \leq b_k^* x^{\Delta}(t_k), k = j+1, j+2, ..., \end{cases}$$

let $m(t) = r(t)g(x^{\Delta}(t))$, then

$$\begin{aligned} &m^{\Delta}(t) \leq 0, t \geq t_j,\ t \neq t_k, k = j+1, j+2, ..., \\ &m(t_k^+) \leq g(b_k^*)m(t_k),\ k = j+1, j+2, ... \end{aligned}$$

Applying Lemma 1, we get for $t > t_j$

$$m(t) \leq m(t_j^+)\prod_{t_j<t_k<t} g(b_k^*),$$

i.e.,

$$r(t)g(x^{\Delta}(t)) \leq r(t_j)g(x^{\Delta}(t_j^+))\prod_{t_j<t_k<t} g(b_k^*),$$

then, we get

$$\begin{aligned} x^{\Delta}(t) &\leq \lambda_1^2 g^{-1}\left(\frac{r(t_j)}{r(t)}\right)x^{\Delta}(t_j^+)g^{-1}\left(\prod_{t_j<t_k<t} g(b_k^*)\right) \\ &= -\lambda_1^2\beta g^{-1}\left(\frac{r(t_j)}{r(t)}\right)g^{-1}\left(\prod_{t_j<t_k<t} g(b_k^*)\right) \end{aligned}$$

Assuming $M = \lambda_1^2\beta$, then $M > 0$ and hence

$$\begin{cases} x^{\Delta}(t) \leq -Mg^{-1}\left(\frac{r(t_j)}{r(t)}\right)g^{-1}\left(\prod_{t_j<t_k<t} g(b_k^*)\right), \\ x(t_k^+) \leq a_k x(t_k),\ for\ k = j+1, j+2, ... \end{cases} \tag{8}$$

Applying Lemma 1 on (8), we get

$$\begin{aligned} x(t) &\leq x(t_j^+)\prod_{t_j<t_k<t} a_k - \int_{t_j}^{t}\prod_{s<t_k<t} a_k M g^{-1}\left(\frac{r(t_j)}{r(s)}\right)g^{-1} \\ &\quad \left(\prod_{t_j<t_k<s} g(b_k^*)\right)\Delta s \\ &= \prod_{t_j<t_k<t} a_k\Bigg[x(t_j^+) - \lambda_2 M g^{-1}(r(t_j)) \\ &\quad .\int_{t_j}^{t} g^{-1}\left(\frac{1}{r(s)}\right)\frac{g^{-1}\left(\prod_{t_j<t_k<s} g(b_k^*)\right)}{\prod_{t_j<t_k<s} a_k}\Delta s\Bigg]. \end{aligned} \tag{9}$$

From condition (4), we get a contradiction with $x(t) > 0$ as $t \to \infty$. Therefore, $x^{\Delta}(t_k) \geq 0$ for $t_k \geq T$. Since

$$x^{\Delta}(t_k^+) = h_k(x^{\Delta}(t_k)),$$

then from (H_3), we get for any $t_k \geq T$

$$x^{\Delta}(t_k^+) \geq b_k^* x^{\Delta}(t_k) \geq 0.$$

Since $r(t)g(x^{\Delta}(t))$ is decreasing in $(t_k, t_{k+1}]_{\mathbb{T}}$, $t_k \geq T$, we get

$$x^{\Delta}(t) \geq \lambda_2 g^{-1}\left(\frac{r(t_{k+1})}{r(t)}\right)x^{\Delta}(t_{k+1}) \geq 0, (t_k, t_{k+1}]_{\mathbb{T}}.$$

This completes the proof.

Remark

When $x(t)$ is eventually negative, under hypothesis (H_1)-(H_4) and (4), one can prove in a similar way that $x^{\Delta}(t_k^+) \leq 0$ and $x^{\Delta}(t) \leq 0$ for $t \in (t_k, t_{k+1}]_{\mathbb{T}}$, where $t_k \geq T \geq t_0$.

Lemma 3. *Assume that (H_1)-(H_4) hold and there exists $e(t)$ such that $(p(t) - e(t)) \geq 0$ and $x(t) > 0$, $t \geq T \geq t_0$ is a nonoscillatory solution of (1). If*

$$\begin{aligned} &\int_{t_0}^{t_1} g^{-1}\left(\frac{1}{r(t)}\right)\Delta t + \frac{b_1^*}{a_1}\int_{t_1}^{t_2} g^{-1}\left(\frac{1}{r(t)}\right)\Delta t + ... + \\ &\frac{b_1^* b_2^* ... b_n^*}{a_1 a_2 ... a_n}\int_{t_n}^{t_{n+1}} g^{-1}\left(\frac{1}{r(t)}\right)\Delta t + ... = \infty, \end{aligned} \tag{10}$$

then $x^{\Delta}(t_k^+) \geq 0$ and $x^{\Delta}(t) \geq 0$ for $t \in (t_k, t_{k+1}]_{\mathbb{T}}$, where $t_k \geq T$.

Proof. First, we prove that $x^{\Delta}(t_k) \geq 0$ for $t_k \geq T$, otherwise, there exist some j such that $t_j \geq T$ and $x^{\Delta}(t_j) < 0$. Proceeding as in the proof of Lemma 2, we get (5) and (7). Since $r(t)g(x^{\Delta}(t))$ is nonincreasing in $(t_j, t_{j+1}]_{\mathbb{T}}$, $t_j \geq T$, then

$$x^{\Delta}(t) \leq \lambda_1 g^{-1}\left(\frac{r(t_j)}{r(t)}\right)x^{\Delta}(t_j^+), t \in (t_j, t_{j+1}]_{\mathbb{T}}.$$

Integrating the above inequality from t_j to t_{j+1}, we get

$$\int_{t_j}^{t_{j+1}} x^{\Delta}(t)\Delta t \leq \lambda_1\lambda_2 g^{-1}(r(t_j))x^{\Delta}(t_j^+)\int_{t_j}^{t_{j+1}} g^{-1}\left(\frac{1}{r(t)}\right)\Delta t,$$

i.e.,

$$x(t_{j+1}) \leq x(t_j^+) - \lambda_1\lambda_2\beta g^{-1}(r(t_j)) \int_{t_j}^{t_{j+1}} g^{-1}\left(\frac{1}{r(t)}\right)\Delta t. \tag{11}$$

Using (H_3), we get

$$\begin{aligned} x(t_{j+2}) &\leq x(t_{j+1}^+) + \lambda_1\lambda_2 g^{-1}(r(t_{j+1}))x^\Delta(t_{j+1}^+)\int_{t_{j+1}}^{t_{j+2}} g^{-1}\left(\frac{1}{r(t)}\right)\Delta t \\ &= \xi_{j+1}(x(t_{j+1})) + \lambda_1\lambda_2 g^{-1}(r(t_{j+1}))h_{j+1}(x^\Delta(t_{j+1}))\int_{t_{j+1}}^{t_{j+2}} g^{-1}\left(\frac{1}{r(t)}\right)\Delta t \\ &\leq a_{j+1}x(t_{j+1}) + \lambda_1\lambda_2 g^{-1}(r(t_{j+1}))b_{j+1}^* x^\Delta(t_{j+1})\int_{t_{j+1}}^{t_{j+2}} g^{-1}\left(\frac{1}{r(t)}\right)\Delta t. \end{aligned}$$

From (5) and (11), we get

$$\begin{aligned} x(t_{j+2}) &\leq a_{j+1}\left[x(t_j^+) - \lambda_1\lambda_2\beta g^{-1}(r(t_j))\int_{t_j}^{t_{j+1}} g^{-1}\left(\frac{1}{r(t)}\right)\Delta t\right] - \\ &\quad \lambda_1^2\lambda_2 b_{j+1}^*\beta g^{-1}(r(t_{j+1}))g^{-1}\left(\frac{r(t_j)}{r(t_{j+1})}\right)\int_{t_{j+1}}^{t_{j+2}} g^{-1}\left(\frac{1}{r(t)}\right)\Delta t \\ &= a_{j+1}\left[x(t_j^+) - \lambda_1\lambda_2\beta g^{-1}(r(t_j))\left(\int_{t_j}^{t_{j+1}} g^{-1}\left(\frac{1}{r(t)}\right)\Delta t\right.\right. \\ &\quad \left.\left. + \frac{b_{j+1}^*}{a_{j+1}}\int_{t_{j+1}}^{t_{j+2}} g^{-1}\left(\frac{1}{r(t)}\right)\Delta t\right)\right]. \end{aligned}$$

By induction, we get

$$\begin{aligned} x(t_{j+n}) &\leq a_{j+1}...a_{j+n-1}\left[x(t_j^+) - \lambda_1\lambda_2\beta g^{-1}(r(t_j))\left(\int_{t_j}^{t_{j+1}} g^{-1}\left(\frac{1}{r(t)}\right)\Delta t + \right.\right. \\ &\quad \left.\left. \frac{b_{j+1}^*}{a_{j+1}}\int_{t_{j+1}}^{t_{j+2}} g^{-1}\left(\frac{1}{r(t)}\right)\Delta t + ... + \frac{b_{j+1}^*...b_{j+n-1}^*}{a_{j+1}^*...a_{j+n-1}^*}\int_{t_{j+n-1}}^{t_{j+n}} g^{-1}\left(\frac{1}{r(t)}\right)\Delta t\right)\right]. \end{aligned}$$

From condition (10) as $n \to \infty$, we get a contradiction with $x(t) > 0$, $t \geq T$. Therefore, for $t_k \geq T$, $x^\Delta(t_k) \geq 0$, and as in the proof of Lemma 2, we get

$$x^\Delta(t_k^+) \geq 0 and x^\Delta(t) \geq 0, t \in (t_k, t_{k+1}]_{\mathbb{T}}, \quad t_k \geq T.$$

This completes the proof.

Remark
When $x(t)$ is eventually negative, under hypothesis (H_1)-(H_4) and (10), one can prove in a similar way that $x^\Delta(t_k^+) \leq 0$ and $x^\Delta(t) \leq 0$ for $t \in (t_k, t_{k+1}]_{\mathbb{T}}$, where $t_k \geq T \geq t_0$.

Theorem 1. *Assume that (H_1)-(H_4), (10) hold and there exists $e(t)$ such that $(p(t) - e(t)) \geq 0$ and $a_k^* \geq 1$, for $k \geq k_0$, k_0 is a positive integer. If*

$$\int_{t_0}^{t_1} (p(t) - e(t))\Delta t + \frac{1}{g(b_1)}\int_{t_1}^{t_2} (p(t) - e(t))\Delta t + \frac{1}{g(b_1)g(b_2)}\int_{t_2}^{t_3} (p(t) - e(t))\Delta t + ... + \frac{1}{g(b_1)g(b_2)...g(b_n)}\int_{t_n}^{t_{n+1}} (p(t) - e(t))\Delta t + ... = \infty, \tag{12}$$

then, Eq. (1) is oscillatory.

Proof. Assume that Eq. (1) has a nonoscillatory solution x. Without loss of generality, we assume that x is eventually positive solution of (1), i.e. $x(t) > 0$, $t \geq t_0$ and $k_0 = 1$. From Lemma 3, we have $x^\Delta(t) \geq 0$, $t \in (t_k, t_{k+1}]_{\mathbb{T}}$, $k = 1, 2, ...$ Define

$$w(t) = \frac{r(t)g(x^\Delta(t))}{\varphi(x(t))}, \tag{13}$$

then $w(t_k^+) \geq 0$, $k = 1, 2, ...$ and $w(t) > 0$, $t \geq t_0$. Using the delta derivative rules of the product and quotient of two functions and then chain rule (see [[7], Theorem 1.90]), we find that when $t \neq t_k$

$$\begin{aligned} w^\Delta(t) &= \frac{(r(t)g(x^\Delta(t)))^\Delta}{\varphi(x^\sigma(t))} \\ &\quad - \frac{r(t)g(x^\Delta(t))}{\varphi(x(t))\varphi(x^\sigma(t))}\int_0^1 \varphi'(x(t) + h\mu(t)x^\Delta(t))dh\, x^\Delta(t). \end{aligned}$$

From Eq. (1) we have

$$\begin{aligned} w^\Delta(t) &= \frac{G(t, x^\sigma(t)) - f(t, x^\sigma(t))}{\varphi(x^\sigma(t))} \\ &\quad - \frac{r(t)g(x^\Delta(t))x^\Delta(t)}{\varphi(x(t))\varphi(x^\sigma(t))}\int_0^1 \varphi'(x(t) + h\mu(t)x^\Delta(t))dh. \end{aligned}$$

Using (H_1) and (H_2), we get

$$\begin{aligned} w^\Delta(t) &\leq e(t) - p(t) - \frac{r(t)g(x^\Delta(t))x^\Delta(t)}{\varphi(x(t))\varphi(x^\sigma(t))}\int_0^1 \varphi'(x(t) + h\mu(t)x^\Delta(t))dh \\ &\leq -(p(t) - e(t)). \end{aligned} \tag{14}$$

Since $\varphi'(x(t)) \geq 0$ and $\varphi(x(t)) \geq 0$, then from (H_3) and $a_k^* \geq 1$ we get for $k = 1, 2, ...$

$$\begin{aligned} w(t_k^+) &= \frac{r(t_k^+)g(x^\Delta(t_k^+))}{\varphi(x(t_k^+))} = \frac{r(t_k)g(h_k(x^\Delta(t_k)))}{\varphi(\xi_k(x(t_k)))} \\ &\leq \frac{r(t_k)g(b_k x^\Delta(t_k))}{\varphi(a_k^*(x(t_k))} \leq \frac{r(t_k)g(b_k)g(x^\Delta(t_k))}{\varphi(x(t_k)} \\ &= g(b_k)w(t_k). \end{aligned} \tag{15}$$

Integrating (14), we get

$$w(t_1) \leq w(t_0^+) - \int_{t_0}^{t_1} (p(t) - e(t))\Delta t. \tag{16}$$

Using (15), we get

$$w(t_1^+) \leq g(b_1)w(t_1) \leq g(b_1)\left[w(t_0^+) - \int_{t_0}^{t_1} (p(t) - e(t))\Delta t\right].$$

Similarly, we get

$$\begin{aligned} w(t_2^+) &\leq g(b_2)w(t_2) \leq g(b_2)\left[w(t_1^+) - \int_{t_1}^{t_2} (p(t) - e(t))\Delta t\right] \\ &\leq g(b_1)g(b_2)\left[w(t_0^+) - \int_{t_0}^{t_1} (p(t) - e(t))\Delta t - \frac{1}{g(b_1)}\int_{t_1}^{t_2} (p(t) - e(t))\Delta t\right] \end{aligned}$$

By induction, for any positive integer n, we get

$$\begin{aligned} w(t_n^+) &\leq g(b_1)g(b_2)...g(b_n)\left[w(t_0^+) - \int_{t_0}^{t_1} (p(t) - e(t))\Delta t - \frac{1}{g(b_1)}\int_{t_1}^{t_2} (p(t) - e(t))\Delta t\right. \\ &\quad \left. - ... - \frac{1}{g(b_1)g(b_2)...g(b_{n-1})}\int_{t_{n-1}}^{t_n} (p(t) - e(t))\Delta t\right]. \end{aligned} \tag{17}$$

From condition (12) and $g(b_k) > 0$ $(b_k > 0)$, $k = 1, 2, ...,$ we get $w(t_n^+) \to -\infty$ as $n \to \infty$, which is a contradiction with $w(t_n^+) \geq 0$. This completes the proof.

Theorem 2. *Assume that (H_1)-(H_4), (4) hold and there exists $e(t)$ such that $(p(t) - e(t)) \geq 0$ and $a_k^* \geq 1$, for $k \geq k_0$, k_0 is a positive integer. If*

$$\int_{t_0}^{\infty} \prod_{t_0 < t_k < t} \frac{1}{g(b_k)}(p(t) - e(t))\Delta t = \infty, \tag{18}$$

then, Eq. (1) is oscillatory.

Proof. As in the proof of Theorem 1, we assume that x is eventually positive solution of (1), i.e. $x(t) > 0$, $t \geq t_0$ and $k_0 = 1$. From Lemma 2, we have $x^{\Delta}(t) \geq 0$, $t \in (t_k, t_{k+1}]_{\mathbb{T}}$, $k = 1, 2, \ldots$. Defining $w(t)$ as in (13), we find that (14) and (15) hold. Applying Lemma 1 on (14) and (15), we get

$$
\begin{aligned}
w(t) &\leq w(t_0) \prod_{t_0<t_k<t} g(b_k) - \int_{t_0}^{t} \prod_{s<t_k<t} g(b_k)(p(s)-e(s))\Delta s \\
&= \prod_{t_0<t_k<t} g(b_k) \left[w(t_0) - \int_{t_0}^{t} \prod_{t_0<t_k<s} \frac{1}{g(b_k)}(p(s)-e(s))\Delta s \right]. \qquad (19)
\end{aligned}
$$

From condition (18), we get a contradiction as $t \to \infty$. This completes the proof.

Theorem 3. *Assume that (H_1)-(H_4), (10) hold and there exists $e(t)$ such that $(p(t) - e(t)) \geq 0$ and $\varphi(ab) \geq \varphi(a)\varphi(b)$, for any $ab > 0$. If*

$$
\begin{aligned}
&\int_{t_0}^{t_1}(p(t)-e(t))\Delta t + \frac{\varphi(a_1^*)}{g(b_1)}\int_{t_1}^{t_2}(p(t)-e(t))\Delta t + \frac{\varphi(a_1^*)\varphi(a_2^*)}{g(b_1)g(b_2)}\int_{t_2}^{t_3}(p(t)-e(t))\Delta t \\
&+ \ldots + \frac{\varphi(a_1^*)\ldots\varphi(a_n^*)}{g(b_1)g(b_2)\ldots g(b_n)}\int_{t_n}^{t_{n+1}}(p(t)-e(t))\Delta t + \ldots = \infty, \qquad (20)
\end{aligned}
$$

then, Eq. (1) is oscillatory.

Proof. Assume that Eq. (1) has a nonoscillatory solution x. Without loss of generality, we assume that x is eventually positive solution of (1), i.e. $x(t) > 0$, $t \geq t_0$ and $k_0 = 1$. From Lemma 3, we have $x^{\Delta}(t) \geq 0$, $t \in (t_k, t_{k+1}]_{\mathbb{T}}$, $k = 1, 2, \ldots$. Defining $w(t)$ as in (13), we get $w(t) \geq 0$, $t \geq t_0$, $w(t_k^+) \geq 0$, $k = 1, 2, \ldots$. From Theorem 1, we find that (14) holds for $t \neq t_k$ and

$$
\begin{aligned}
w(t_k^+) &= \frac{r(t_k^+)g(x^{\Delta}(t_k^+))}{\varphi(x(t_k^+))} = \frac{r(t_k)g(h_k(x^{\Delta}(t_k)))}{\varphi(\xi_k(x(t_k)))} \leq \frac{r(t_k)g(b_k x^{\Delta}(t_k))}{\varphi(a_k^* x(t_k)} \\
&\leq \frac{r(t_k)g(b_k)g(x^{\Delta}(t_k))}{\varphi(a_k^*)\varphi(x(t_k))} = \frac{g(b_k)}{\varphi(a_k^*)}w(t_k). \qquad (21)
\end{aligned}
$$

As in the proof of Theorem 1, by induction, we get for any positive integer n,

$$
\begin{aligned}
w(t_n^+) &\leq \frac{g(b_1)\ldots g(b_n)}{\varphi(a_1^*)\ldots\varphi(a_n^*)}\left[w(t_0^+) - \int_{t_0}^{t_1}(p(t)-e(t))\Delta t - \frac{\varphi(a_1^*)}{g(b_1)}\int_{t_1}^{t_2}(p(t)-e(t))\Delta t \right. \\
&\left. - \ldots - \frac{\varphi(a_1^*)\ldots\varphi(a_{n-1}^*)}{g(b_1)\ldots g(b_{n-1})}\int_{t_{n-1}}^{t_n}(p(t)-e(t))\Delta t \right]. \qquad (22)
\end{aligned}
$$

From condition (20) as $n \to \infty$, we get a contradiction. This completes the proof.

Theorem 4. *Assume that (H_1)-(H_4), (4) hold and there exists $e(t)$ such that $(p(t) - e(t)) \geq 0$ and $\varphi(ab) \geq \varphi(a)\varphi(b)$, for any $ab > 0$. If*

$$
\int_{t_0}^{\infty} \prod_{t_0<t_k<t} \frac{\varphi(a_k^*)}{g(b_k)}(p(t)-e(t))\Delta t = \infty, \qquad (23)
$$

then, Eq. (1) is oscillatory.

Proof. Similar to the proof of Theorem 3. So it is omitted.

In the following, we use the hypothesis:

(H_5) $\int_{\pm\varepsilon}^{\pm\infty} g^{-1}\left(\frac{1}{\varphi(u)}\right)\Delta u < \infty$, for any $\varepsilon > 0$,

where $\int_{\pm\varepsilon}^{\pm\infty} g^{-1}\left(\frac{1}{\varphi(u)}\right)\Delta u < \infty$ means $\int_{\varepsilon}^{\infty} g^{-1}\left(\frac{1}{\varphi(u)}\right)\Delta u < \infty$ and $\int_{-\varepsilon}^{-\infty} g^{-1}\left(\frac{1}{\varphi(u)}\right)\Delta u < \infty$.

Theorem 5. *Assume that (H_1)-(H_5), (4) hold and there exists $e(t)$ such that $(p(t) - e(t)) \geq 0$ and $a_k^* \geq 1$, for $k \geq k_0$, k_0 is a positive integer. If*

$$
\sum_{k=0}^{\infty}\int_{t_k}^{t_{k+1}} g^{-1}\left(\frac{1}{r(s)}\right) g^{-1}\left(\int_s^{\infty}\prod_{s<t_k<\theta}\frac{1}{g(b_k)}(p(\theta)-e(\theta))\Delta\theta\right)\Delta s = \infty, \qquad (24)
$$

then, Eq. (1) is oscillatory.

Proof. Assume that $x(t) > 0$, $t \geq t_0$ be a nonoscillatory solution of (1) and $k_0 = 1$. From Lemma 2, we have $x^{\Delta}(t_k^+) \geq 0$, $k = 1, 2, \ldots$ and $x^{\Delta}(t) \geq 0$, $t \geq t_0$. By (H_3) and $a_k^* \geq 1$, $k = 1, 2, \ldots$, we get

$$
x(t_0^+) \leq x(t_1) \leq x(t_1^+) \leq x(t_2) \leq x(t_2^+) \leq \ldots.
$$

It follows that $x(t)$ is nondecreasing in $[t_0, \infty)_{\mathbb{T}}$. From (1), we have

$$
\begin{cases}
(r(t)g(x^{\Delta}(t)))^{\Delta} \leq -(p(t)-e(t))\varphi(x^{\sigma}(t)), t \neq t_k, k = 1, 2, \ldots, \\
x^{\Delta}(t_k^+) \leq b_k x^{\Delta}(t_k), k = 1, 2, \ldots,
\end{cases}
$$

Let $m(t) = r(t)g(x^{\Delta}(t))$. Then

$$
\begin{cases}
m(t)^{\Delta} \leq -(p(t)-e(t))\varphi(x^{\sigma}(t)), t \neq t_k, k = 1, 2, \ldots, \\
m(t_k^+) \leq g(b_k)m(t_k), k = 1, 2, \ldots.
\end{cases}
$$

Applying Lemma 1, we get

$$
m(t) \leq m(s)\prod_{s<t_k<t} g(b_k) - \int_s^t \prod_{\theta<t_k<t} g(b_k)(p(\theta)-e(\theta))\varphi(x^{\sigma}(\theta))\Delta\theta, t_0 \leq s \leq t,
$$

i.e.,

$$
r(t)g(x^{\Delta}(t)) \leq r(s)g(x^{\Delta}(s))\prod_{s<t_k<t} g(b_k) - \int_s^t \prod_{\theta<t_k<t} g(b_k)(p(\theta)-e(\theta))\varphi(x^{\sigma}(\theta))\Delta\theta, t_0 \leq s \leq t,
$$

Then for $t_0 \leq s \leq t$, we have

$$
x^{\Delta}(s) \geq \lambda_2 g^{-1}\left(\frac{1}{r(s)}\right) g^{-1}\left(\int_s^t \prod_{s<t_k<\theta}\frac{1}{g(b_k)}(p(\theta)-e(\theta))\varphi(x^{\sigma}(\theta))\Delta\theta\right).
$$

Since $\varphi(x) > 0$ $(x \neq 0)$ and $\varphi(x)$ is nondecreasing, we get

$$
\begin{aligned}
g^{-1}\left(\frac{1}{\varphi(x(s))}\right)x^{\Delta}(s) &\geq \frac{\lambda_2}{\lambda_1} g^{-1}\left(\frac{1}{r(s)}\right) g^{-1}\left(\int_s^t \prod_{s<t_k<\theta}\frac{1}{g(b_k)}(p(\theta)-e(\theta))\frac{\varphi(x^{\sigma}(\theta))}{\varphi(x(s))}\Delta\theta\right) \\
&\geq \frac{\lambda_2}{\lambda_1} g^{-1}\left(\frac{1}{r(s)}\right) g^{-1}\left(\int_s^t \prod_{s<t_k<\theta}\frac{1}{g(b_k)}(p(\theta)-e(\theta))\Delta\theta\right) \qquad (25)
\end{aligned}
$$

for $s \in (t_k, t_{k+1}]_{\mathbb{T}}$, $k = 1, 2, \ldots$. Then

$$
\int_{t_k}^{t_{k+1}} g^{-1}\left(\frac{1}{\varphi(x(s))}\right)x^{\Delta}(s)\Delta s = \int_{x(t_k^+)}^{x(t_{k+1})} g^{-1}\left(\frac{1}{\varphi(\theta)}\right)\Delta\theta. \qquad (26)
$$

Using (26) in (25), we get

$$
\begin{aligned}
&\frac{\lambda_2}{\lambda_1}\sum_{k=0}^{\infty}\int_{t_k}^{t_{k+1}} g^{-1}\left(\frac{1}{r(s)}\right) g^{-1}\left(\lim_{t\to\infty}\int_s^t \prod_{s<t_k<\theta}\frac{1}{g(b_k)}(p(\theta)-e(\theta))\Delta\theta\right)\Delta s \\
&\leq \sum_{k=0}^{\infty}\int_{x(t_k^+)}^{x(t_{k+1})} g^{-1}\left(\frac{1}{\varphi(\theta)}\right)\Delta\theta.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\frac{\lambda_2}{\lambda_1}\sum_{k=0}^{\infty}\int_{t_k}^{t_{k+1}} g^{-1}\left(\frac{1}{r(s)}\right) g^{-1}\left(\lim_{t\to\infty}\int_s^t \prod_{s<t_k<\theta}\frac{1}{g(b_k)}(p(\theta)-e(\theta))\Delta\theta\right)\Delta s \\
&\leq \int_{x(t_0^+)}^{\infty} g^{-1}\left(\frac{1}{\varphi(\theta)}\right)\Delta\theta.
\end{aligned}
$$

From condition (24), we get a contradiction. This completes the proof.

Theorem 6. *Assume that (H_1)-(H_5), (4) hold and there exists $e(t)$ such that $(p(t)-e(t)) \geq 0$ and $\varphi(ab) \geq \varphi(a)\varphi(b)$, for any $ab > 0$. If*

$$\sum_{k=0}^{\infty}\int_{t_k}^{t_{k+1}} g^{-1}\left(\frac{1}{r(s)}\right) g^{-1}\left(\int_s^{\infty}\prod_{s<t_k<\theta}\frac{\varphi(a_k^*)}{g(b_k)}(p(\theta)-e(\theta))\Delta\theta\right)\Delta s = \infty, \quad (27)$$

then, Eq. (1) is oscillatory.

Proof. We assume that $x(t) > 0$, $t \geq t_0$ be a nonoscillatory solution of (1) and $k_0 = 1$. From Lemma 2, we have $x^{\Delta}(t) \geq 0$, $t \geq t_0$. Defining $w(t)$ as in (13), we find that (14) holds for $t \neq t_k$ and (21) holds. Then

$$\begin{cases} w^{\Delta}(t) \leq -(p(t)-e(t)), t \neq t_k, k = 1,2,..., \\ w(t_k^+) \leq \frac{g(b_k)}{\varphi(a_k^*)} w(t_k), k = 1,2,..., \end{cases}$$

Applying Lemma 1, we get

$$w(t) \leq w(s)\prod_{s<t_k<t}\frac{g(b_k)}{\varphi(a_k^*)} - \int_s^t \prod_{\theta<t_k<t}\frac{g(b_k)}{\varphi(a_k^*)}(p(\theta)-e(\theta))\Delta\theta, t_0 \leq s \leq t.$$

It yields that

$$w(s) \geq \int_s^t \prod_{s<t_k<\theta}\frac{\varphi(a_k^*)}{g(b_k)}(p(\theta)-e(\theta))\Delta\theta,$$

i.e.,

$$g^{-1}\left(\frac{1}{\varphi(x(s))}\right)x^{\Delta}(s) \geq \frac{\lambda_2}{\lambda_1} g^{-1}\left(\frac{1}{r(s)}\right) g^{-1}\left(\int_s^t \prod_{s<t_k<\theta}\frac{\varphi(a_k^*)}{g(b_k)}(p(\theta)-e(\theta))\Delta\theta\right) \quad (28)$$

for $s \in (t_k, t_{k+1}]_{\mathbb{T}}$, $k = 1,2,....$ Hence

$$\int_{t_k}^{t_{k+1}} g^{-1}\left(\frac{1}{\varphi(x(s))}\right)x^{\Delta}(s)\Delta s = \int_{x(t_k^+)}^{x(t_{k+1})} g^{-1}\left(\frac{1}{\varphi(\theta)}\right)\Delta\theta. \quad (29)$$

Using (29) in (28), we get

$$\frac{\lambda_2}{\lambda_1}\sum_{k=0}^{\infty}\int_{t_k}^{t_{k+1}} g^{-1}\left(\frac{1}{r(s)}\right) g^{-1}\left(\lim_{t\to\infty}\int_s^t \prod_{s<t_k<\theta}\frac{\varphi(a_k^*)}{g(b_k)}(p(\theta)-e(\theta))\Delta\theta\right)\Delta s$$
$$\leq \sum_{k=0}^{\infty}\int_{x(t_k^+)}^{x(t_{k+1})} g^{-1}\left(\frac{1}{\varphi(\theta)}\right)\Delta\theta,$$

thus, we have

$$\frac{\lambda_2}{\lambda_1}\sum_{k=0}^{\infty}\int_{t_k}^{t_{k+1}} g^{-1}\left(\frac{1}{r(s)}\right) g^{-1}\left(\lim_{t\to\infty}\int_s^t \prod_{s<t_k<\theta}\frac{\varphi(a_k^*)}{g(b_k)}(p(\theta)-e(\theta))\Delta\theta\right)\Delta s$$
$$\leq \int_{x(t_0^+)}^{\infty} g^{-1}\left(\frac{1}{\varphi(\theta)}\right)\Delta\theta.$$

From condition (27), we get a contradiction. This completes the proof.

Corollary 1. *Assume that (H_1)-(H_4), (4) hold and there exists $e(t)$ such that $(p(t)-e(t)) \geq 0$ and $a_k^* \geq 1$, $b_k \leq 1$ for $k \geq k_0$, k_0 is a positive integer. If $\int^{\infty}(p(t)-e(t))\Delta t = \infty$, then, Eq. (1) is oscillatory.*

Proof. Without loss of generality, let $k_0 = 1$, by $b_k \leq 1$ and $g(b_k) \leq 1$, we get $\frac{1}{g(b_k)} \geq 1$, therefore

$$\int_{t_0}^t \prod_{t_0<t_k<s}\frac{1}{g(b_k)}(p(s)-e(s))\Delta s \geq \int_{t_0}^t (p(s)-e(s))\Delta s.$$

As $t \to \infty$, using $\int^{\infty}(p(t)-e(t))\Delta t = \infty$ and Theorem 2, we get that Eq. (1) is oscillatory.

Corollary 2. *Assume that (H_1)-(H_5), (4) hold and there exists $e(t)$ such that $(p(t)-e(t)) \geq 0$ and $a_k^* \geq 1$, $b_k \leq 1$ for $k \geq k_0$, k_0 is a positive integer. If $\int_{t_0}^{\infty} g^{-1}\left(\frac{1}{r(s)}\right) g^{-1}\left(\int_s^{\infty}(p(t)-e(t))\Delta t\right)\Delta s = \infty$, then, Eq. (1) is oscillatory.*

Proof. Using Theorem 5, the proof is similar to the proof of Corollary 1.

Corollary 3. *Assume that (H_1)-(H_4), (10) hold and there exists $e(t)$ such that $(p(t)-e(t)) \geq 0$. Also, assume that there exist a positive integer k_0 and a constant $\gamma > 0$ such that*

$$a_k^* \geq 1, \frac{1}{g(b_k)} \geq \left(\frac{t_{k+1}}{t_k}\right)^{\gamma} for\ k \geq k_0. \quad (30)$$

If

$$\int^{\infty} t^{\gamma}(p(t)-e(t))\Delta t = \infty, \quad (31)$$

then, Eq. (1) is oscillatory.

Proof. Without loss of generality, let $k_0 = 1$, then

$$\int_{t_0}^{t_1}(p(t)-e(t))\Delta t + \frac{1}{g(b_1)}\int_{t_1}^{t_2}(p(t)-e(t))\Delta t + ... + \frac{1}{g(b_1)g(b_2)...g(b_n)}\int_{t_n}^{t_{n+1}}(p(t)-e(t))\Delta t$$
$$\geq \frac{1}{t_1^{\gamma}}\left[\int_{t_1}^{t_2} t_2^{\gamma}(p(t)-e(t))\Delta t + \int_{t_2}^{t_3} t_3^{\gamma}(p(t)-e(t))\Delta t + ... + \int_{t_n}^{t_{n+1}} t_{n+1}^{\gamma}(p(t)-e(t))\Delta t\right]$$
$$\geq \frac{1}{t_1^{\gamma}}\left[\int_{t_1}^{t_2} s^{\gamma}(p(s)-e(s))\Delta s + \int_{t_2}^{t_3} s^{\gamma}(p(s)-e(s))\Delta s + ... + \int_{t_n}^{t_{n+1}} s^{\gamma}(p(s)-e(s))\Delta s\right]$$
$$= \frac{1}{t_1^{\gamma}}\int_{t_n}^{t_{n+1}} s^{\gamma}(p(s)-e(s))\Delta s. \quad (32)$$

As $n \to \infty$, using (31) and (32) yield that (12) holds. According to Theorem 1, we obtain that Eq. (1) is oscillatory.

Corollary 4. *Assume that (H_1)-(H_4), (10) hold and there exists $e(t)$ such that $(p(t)-e(t)) \geq 0$ and $\varphi(ab) \geq \varphi(a)\varphi(b)$ for $ab > 0$. Suppose there exist a positive integer k_0 and a constant $\gamma > 0$ such that $\frac{\varphi(a_k^*)}{g(b_k)} \geq \left(\frac{t_{k+1}}{t_k}\right)^{\gamma}$, for $k \geq k_0$. If $\int^{\infty} t^{\gamma}(p(t)-e(t))\Delta t = \infty$, then, Eq. (1) is oscillatory.*

Proof. Similar to the proof of Corollary 3, and so it is omitted.

Remark

(1) When $g(x) = x^{\alpha}$ and $G(t, x^{\sigma}(t)) = 0$, we get the main result of Huang [16].
(2) When $g(x) = x$, $r(t) = 1$ and $G(t, x^{\sigma}(t)) = 0$, we get the main result of Huang [14,15].

Example 1. Consider the equation ($\mathbb{T} = \mathbb{R}$)

$$\begin{cases} \left(\frac{1}{t}x'(t)\right)' + \left(\frac{1}{t\ln t} + t^2\right)x^{\nu}(t) = t^2 x^{\nu}, \ t \geq \frac{3}{2}, \ \ t \neq k \\ x(k^+) = \left(1 + \frac{1}{k}\right)x(k), x'(k^+) = x'(k), \ k = 1,2,..., \\ x(\frac{3}{2}) = x_0, \ x'(\frac{3}{2}) = x_0', \end{cases} \quad (33)$$

where ν is the quotient of odd positive integers.
Here, $r(t)=\frac{1}{t}$, $p(t)=\left(\frac{1}{t\ln t}+t^2\right)$, $a_k=a_k^*=1+\frac{1}{k}$, $b_k=b_k^*=1$, $t_k=k$ and $e(t)=t^2$. To apply Corollary 1, take $\phi(x)=x^\nu$ and $g(x)=x$. Note that

$$\begin{aligned}\int_{t_0}^{\infty} g^{-1}\left(\frac{1}{r(s)}\right)\frac{g^{-1}\left(\prod_{t_0<t_k<s} g(b_k^*)\right)}{\prod_{t_0<t_k<s} a_k}ds &= \int_{t_0}^{\infty} s\prod_{t_0<t_k<s}\frac{g(b_k^*)}{a_k}ds\\ &= \int_{t_0}^{\infty} s\prod_{t_0<t_k<s}\frac{b_k^*}{a_k}ds=\int_{t_0}^{\infty} s\prod_{t_0<t_k<s}\frac{k}{k+1}ds\\ &= \int_{t_0}^{t_1} s\prod_{t_0<t_k<s}\frac{k}{k+1}ds+\int_{t_1}^{t_2} s\prod_{t_0<t_k<s}\frac{k}{k+1}ds\\ &\quad+\int_{t_2}^{t_3} s\prod_{t_0<t_k<s}\frac{k}{k+1}ds+\ldots=\infty,\end{aligned}$$

Thus condition (4) is satisfied. Let $k_0=1$. Then

$$a_k^*\geq 1 \text{ and } b_k=1 \text{ for } k\geq 1,$$

and

$$\int_{\frac{3}{2}}^{\infty}(p(s)-e(s))ds=\int_{\frac{3}{2}}^{\infty}\frac{1}{s\ln s}ds=\infty.$$

Hence, every solution of Eq. (33) is oscillatory.

*Example 2.*Consider the equation $(\mathbb{T}=\mathbb{N})$

$$\begin{cases}\Delta\left(\frac{1}{(\sigma(t))^\beta}\Delta(x(t))^\beta\right)+2t(\sigma(t))(x^\sigma(t))^{2n-1}=t(x^\sigma(t))^{2n-1}, t\geq t_0,\ \ t\neq k+1\\ x(t_k^+)=\left(\frac{k+2}{k+1}\right)x(k),\Delta(x(t_k^+))=\Delta(x(k)), k=1,2,\ldots,\\ x(t_0)=x_0,\Delta(x(t_0))=\Delta(x_0),\end{cases}\tag{34}$$

where $\beta\geq 1$, $n\geq 2$, $f^\Delta(t)=\Delta f(t)=f(t+1)-f(t)$ and $\sigma(t)=t+1$.
Here, $r(t)=\frac{1}{(\sigma(t))^\beta}$, $p(t)=2t(\sigma(t))$, $a_k=a_k^*=\frac{k+2}{k+1}$, $b_k=b_k^*=1$ and $e(t)=t$.
To apply Corollary 4, take $\phi(x)=x^{2n-1}$ and $g(x)=x^\beta$. It is easy to see that the assumption (10) holds. Let $k_0=1$ and $\gamma=2$, then

$$\frac{\phi(a_k^*)}{g(b_k)}=\left(\frac{k+2}{k+1}\right)^{2n-1}=\left(\frac{t_{k+1}}{t_k}\right)^{2n-1}\geq\left(\frac{t_{k+1}}{t_k}\right)^2$$

and

$$\int_{t_0}^{\infty}s^\gamma(p(s)-e(s))\Delta s=\sum_{t=n_0}^{\infty}(2s^4+s^3)=\infty.$$

Hence, every solution of Eq. (34) is oscillatory.

*Example 3.*Consider the equation $(\mathbb{T}=2^{\overline{\mathbb{Z}}})$

$$\begin{cases}\Delta_2(\Delta_2 x(t))+\frac{\lambda(\sigma(t))^{\alpha-1}}{t^\alpha}(x^\sigma(t))^5=\frac{x(\sigma(t))^5}{(x(\sigma(t)))^{10}+1)^{\frac{1}{2}}\sigma(t)}, t\geq t_0:=2,\ \ t\neq t_k\\ x(t_k^+)=\frac{2(k+1)}{k}x(k),\ \Delta_2 x(t_k^+)=\Delta_2 x(k), k=1,2,\ldots,\\ x(2)=x_0,\ \Delta_2 x(2)=\Delta_2 x_0,\end{cases}\tag{35}$$

where $\alpha\geq 1$, $f^\Delta(t)=\Delta_2 f(t)=[f(2t)-f(t)]/(2t-t)$ and $\sigma(t)=2t$.
Here, $r(t)=1$, $p(t)=\frac{\lambda(\sigma(t))^{\alpha-1}}{t^\alpha}$, $a_k=a_k^*=\frac{2(k+1)}{k}$, $b_k=b_k^*=1$, $t_k=2^k$ and $e(t)=\frac{1}{\sigma(t)}$. To apply Theorem 2, take $\phi(x)=x^5$ and $g(x)=x$. Note that

$$\begin{aligned}\int_{t_0}^{\infty} g^{-1}\left(\frac{1}{r(s)}\right)\frac{g^{-1}\left(\prod_{t_0<t_k<s} g(b_k^*)\right)}{\prod_{t_0<t_k<s} a_k}\Delta s &= \int_{t_0}^{\infty}\prod_{t_0<t_k<s}\frac{g(b_k^*)}{a_k}\Delta_2 s\\ &= \int_{t_0}^{\infty}\prod_{t_0<t_k<s}\frac{b_k^*}{a_k}\Delta_2 s=\int_{t_0}^{\infty}\prod_{t_0<t_k<s}\frac{k}{2(k+1)}\Delta_2 s\\ &= \int_{t_0}^{t_1}\prod_{t_0<t_k<s}\frac{k}{2(k+1)}\Delta_2 s+\int_{t_1}^{t_2}\prod_{t_0<t_k<s}\frac{k}{2(k+1)}\Delta_2 s\\ &\quad+\int_{t_2}^{t_3}\prod_{t_0<t_k<s}\frac{k}{2(k+1)}\Delta_2 s+\ldots=\infty,\end{aligned}$$

Thus condition (4) is satisfied. Let $k_0=1$. Then

$$a_k^*\geq 1 \text{ for } k\geq 1,$$

and

$$\int_{t_0}^{\infty}\prod_{t_0<t_k<s}\frac{1}{g(b_k)}(p(s)-e(s))\Delta_2 s=\int_{t_0}^{\infty}\left(\frac{\lambda(2s)^{\alpha-1}}{s^\alpha}-\frac{1}{2s}\right)\Delta_2 s=\left(\lambda 2^{\alpha-1}-\frac{1}{2}\right)\int_{t_0}^{\infty}\frac{1}{s}\Delta_2 s=\infty.$$

Hence, every solution of Eq. (35) is oscillatory if $\lambda>\frac{1}{2^\alpha}$.

*Example 4.*Consider the second order impulsive dynamic equation

$$\begin{cases}((x^\Delta(t))^\alpha)^\Delta+\left(\sigma(t)+\frac{1}{t^{\gamma-1}}\right)x^\sigma(t)(1+(x^\sigma(t))^2)=\sigma(t)x^\sigma(t)(1+(x^\sigma(t))^2), t\geq 0,\ \ t\neq k\\ x(k^+)=x(k),\ x^\Delta(k^+)=\left(\frac{k}{k+1}\right)^{\left(\frac{1}{\alpha}\right)}x^\Delta(k), k=1,2,\ldots,\\ x(t_0)=x_0,\ x^\Delta(t_0)=x_0^\Delta,\end{cases}\tag{36}$$

where γ is the quotient of odd positive integers.
Here, $r(t)=1$, $p(t)=\left(\sigma(t)+\frac{1}{t^{\gamma-1}}\right)$, $a_k=a_k^*=1$, $b_k=b_k^*=\left(\frac{k}{k+1}\right)^{\left(\frac{1}{\alpha}\right)}$ and $e(t)=\sigma(t)$.
To apply Corollary 3, take $\phi(x)=(1+x^2(t))x(t)$ and $g(x)=x^\alpha$. It is easy to see that the assumption (10) holds. Let $k_0=1$, then

$$\frac{1}{g(b_k)}=\frac{1}{b_k^\alpha}=\frac{1}{\left(\left(\frac{k}{k+1}\right)^{\frac{1}{\alpha}}\right)^\alpha}=\frac{k+1}{k}=\frac{t_{k+1}}{t_k}$$

and

$$\int_{t_0}^{\infty}s^\gamma(p(s)-e(s))\Delta s=\int_{t_0}^{\infty}s\Delta s=\infty.$$

Hence, every solution of Eq. (36) is oscillatory.

3 Conclusion

In this paper, we use Riccati transformation technique and the impulsive inequality to establish some new oscillation criteria for the second-order nonlinear impulsive dynamic equation on a time scale $\mathbb{T}$. Our results extend and improve some results established by [14,15,16,18,20] and can be applied to arbitrary time scales. The results of [20] can not be applied to Eq. (34). But, according to Corollary 4, this equation is oscillatory. Also, when

$g(x) = x^{\alpha}$ and $G(t, x^{\sigma}(t)) = 0$, we get the main result of Huang [16] and when $g(x) = x$, $r(t) = 1$ and $G(t, x^{\sigma}(t)) = 0$, we get the main result of Huang [14, 15]. So the results of [14, 15, 16] can be considered as special cases of our results.

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H. A. Agwa is professor of pure mathematics and vice dean for Education and Student Affairs at faculty of education Ain shams university. He has published several papers in many international journals. His main research interests are: functional differential equations, dynamic equations, time scales, inequalities.

Ahmed M. M. Khodier was lecturer of pure mathematics in department of mathematics and director of the open education center, Faculty of Education, Ain Shams university. He has published more than 25 research papers in International journals. His main research interests are: Numerical analysis, functional differential equations, dynamic equations, time scales. He died in May 2017

Heba M. Atteya is lecturer assistant in mathematics department, Faculty of Education, Ain shams university. She has gained her master degree in 2015. She has published 03 research papers in International journals. Her main research interests are: functional differential equations, dynamic equations, time scales.