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An Efficient Method for Solving Regular Variable Fractional Sturm-Liouville Problems

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Abstract: This article is devoted to both theoretical and numerical study of the eigenvalues of regular fractional Sturm-Liouville problem. The fractional derivative in this paper is in the conformable derivative sense. In this paper, we implement the reproducing kernel Hilbert space method to approximate the eigenvalues. Existence and uniformly convergent of the eigenfunctions of the considered problem are provided and proved. The main properties of the Sturm-Liouville problem are investigated. Numerical results demonstrate the accuracy of the present algorithm. Comparisons with other methods are presented.

Keywords: Eigenvalues, variable fractional second-order Sturm-Liouville problem, reproducing kernel Hilbert space method, conformable derivative.

1 Introduction

The Sturm-Liouville eigenvalue problem has played an important role in modeling many physical problems. The theory of the problem is well developed and many results have been obtained concerning the eigenvalues and corresponding eigenfunctions. It should be noted that since finding analytical solutions for this problem is an extremely difficult task, several numerical algorithms have been developed to seek approximate solutions. Several researchers discussed fractional Sturm-Liouville eigenvalue problem when the fractional derivative is constant. In 1970 and 1977, Djrbashian [1] and Nahusev [2] studied this type of problems. In [1], the existence of a solution to such boundary value problem was established. In [2], the aforementioned relation between eigenvalues and zeros of Mittag-Leffler function was shown. Al-Mdallal [3] used the Adomian decomposition while Abbasbandz [4] used the Homotopy Analysis method. Ertürk [5] used the fractional differential transform method to compute the eigenvalues of such problems. Luchko [6] used the Fourier series to solve this problem. Neamatz et al. [7] and Shi et al. [8] used the method of Haar wavelet operational matrix. In [9]-[11], [12], and [13], researchers extended the scope of some spectral properties of fractional Sturm-Liouville problem. Recently, Al-Refai [14] has established existence and non-existence results for a class of fractional Sturm-Liouville eigenvalue problems and estimated the eigenvalues. Variational Methods and Inverse Laplace transform method applied in [15] and [16], respectively. Recently P. Antunes and R. Ferreira [17] constructed numerical schemes using radial basis functions while B. Jin and et [18] used Galerkin finite element method and Syam and Siyyam [19] implemented the iterated variation method to solve such problem.

The numerical solution of eigenvalue problems have received considerable interest in recent years because they have large number of applications in different areas of physics and engineering. A few examples of such applications are pendulums, vibrating and rotating shafts, viscous flow between rotating cylinders, the thermal instability of fluid spheres and spherical shells, earth's seismic behavior and ring structures; for more details see [20], [21], [22], [23], [24], [25]. Note that such problems is often referred to as the circular ring structure with constraints which has rectangular cross-sections of constant width and parabolic variable thickness; see [26] and [27].

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In this paper, we develop a numerical technique for approximating the eigenvalues of the following regular fractional Sturm-Liouville problem of the form

$$D^{\alpha(x)}[p(x)y'(x)] + q(x)y = -\lambda w(x)y(x), x \in I = [0,1],\ 0 < \alpha \le 1 \tag{1.1}$$

subject to

$$a_0 y(0) + a_1 y'(0) = 0,$$

$$a_2 y(1) + a_3 y'(1) = 0, \tag{1.2}$$

where $p,\ p',\ q,$ and $w(x)$ are continuous functions on $[0,1]$ with $p(x), w(x) > 0$ for all $x \in [0,1]$. Here a_j (for $j = 0, \cdots, 3$) are real constants such that

$$a_0^2 + a_1^2 > 0,$$

$$a_2^2 + a_3^2 > 0.$$

If the domain is $[a,b]$, then we use the following change of variable to make it $[0,1]$

$$x = (b-a)t + a.$$

For this reason, we assume that the domain is $[0,1]$. The fractional derivative here is in the conformable derivative sense. Up to our knowledge, we are the first who discuss the regular variable fraction order Sturm-Liouville problem numerically.

Historically, problem (1.1) had been studied theoretically when $\alpha(x) \equiv 1$ by [28] who showed that it has an infinite sequence of eigenvalues $\{\lambda_0, \lambda_1, \lambda_2, ...\}$ with the following property

$$\eta < \lambda_0 \le \lambda_1 \le \lambda_2 \le ...$$

where

$$\lim_{n\to\infty} \lambda_n = \infty,$$

and η is a constant and each eigenvalue has multiplicity at most 3.

The present work is motivated by approximating the eigenvalues of problem (1) using the Reproducing kernel Hilbert space method (RKM). The RKM which accurately computes the series solution is of great interest to applied sciences. This technique gives the solution in a rapidly convergent series with components that can be easily computed. This method is used for the investigation of several scientific applications, see [29], [30], [31].

This paper is organized as follows. In section 2, we present some preliminaries which we will use in this paper. A description of the RKM for discretization of problem (1.1) is presented in section 3. In addition, the existence and the uniformly convergent of the eigenfunctions are given in this section. Several numerical examples and comparisons with Al-Mdallal [3] results are presented in Section 4. Conclusions and closing remarks are given in Section 5.

2 Preliminaries

In this section, we review the definition and some preliminary results of the conformable derivatives as well as the $\alpha-$fractional integral and their properties.

Definition 2.1. Given a function $f : [0,\infty) \to \Re$. Then the conformable derivative of f of order α is defined by

$$D^{\alpha} f(x) = \lim_{\varepsilon\to 0} \frac{f(x + \varepsilon\, x^{1-\alpha}) - f(x)}{\varepsilon}$$

for all $x > 0,\ 0 < \alpha < 1$. If f is $\alpha-$differentiable is some $(0,a),\ \alpha > 0$, and $\lim_{x\to 0^+} D^{\alpha} f(x)$ exists, then define

$$D^{\alpha} f(0) = \lim_{x\to 0^+} D^{\alpha} f(x).$$

Among the properties of the conformable derivatives, we mention the following properties. Let $0 < \alpha < 1$ and $f,\ g$ be $\alpha-$differentiable at a point $x > 0$. Then,

1.$D^{\alpha}[af + bg] = a\, D^{\alpha} f(x) + b\, D^{\alpha} g(x)$, for all $a, b \in \Re$.
2.$D^{\alpha} x^p = p\, x^{p-\alpha}$ for all $p \in \Re$.

3.$D^{\alpha}p=0$ for all $p\in\Re$.
4.$D^{\alpha}[f\ g]=g\ D^{\alpha}f(x)+f\ D^{\alpha}g(x)$.
5.$D^{\alpha}\left[\frac{f}{g}\right](x)=\frac{g(x)\ D^{\alpha}f(x)\ -\ f(x)\ D^{\alpha}g(x)}{g^{2}(x)}$ provided that $g(x)\neq 0$.
6.$D^{\alpha}f(x)=x^{1-\alpha}\ f'(x)$.
7.$D^{1}f(x)=f'(x)$.

Next, we define the $\alpha-$ fractional integral.
Definition 2.2. The $\alpha-$ fractional integral is defined by

$$I^{\alpha}f(x)=\int_{0}^{x}\frac{f(t)}{t^{(}1-\alpha)}dt,$$

where the integral is the Riemann improper integral and $\alpha\in(0,1)$. For more details, see [32,33,34].

Among the properties of the $\alpha-$ fractional integral, we mention the following property. If $f(x)$ is any continuous function in the domain of I^{α} and $x\geq 0$,

$$D^{\alpha}I^{\alpha}f(x)=f(x).$$

The reproducing kernel is given by this definition.

Definition 2.3. Let A be a nonempty set. A function $K:A\times A\to C$ is a reproducing kernel of the Hilbert space H if and only if

1.$K(.,x)\in H$ for all $x\in A$,
2.$(\phi(.),K(.,x))=\phi(x)$ for all $x\in A$ and $\phi\in H$.

The second condition is called the reproducing property and a Hilbert space which possesses a reproducing kernel is called a reproducing kernel Hilbert space (RKHS).

3 Analysis of RKHSM for Solving the Eigenvalue Problem

In this section, we discuss how to solve the following regular variable fractional Sturm-Liouville problem of the form

$$D^{\alpha(x)}[p(x)y'(x)]+q(x)y=-\lambda w(x)y(x),x\in I=[0,1],\ 0<\alpha(x)\leq 1 \tag{3.1}$$

subject to

$$a_{0}y(0)+a_{1}y'(0)=0,$$

$$a_{2}y(1)+a_{3}y'(1)=0, \tag{3.2}$$

where $p,\ p',\ q$, and $w(x)$ are continuous functions on $[0,1]$ with $p(x),w(x)>0$ for all $x\in[0,1]$. Here a_{j} (for $j=0,\cdots,3$) are real constants such that

$$a_{0}^{2}+a_{1}^{2}>0,$$
$$a_{2}^{2}+a_{3}^{2}>0.$$

If $\alpha(1)=1$, we get regular Sturm-Liouville problem of the form

$$\frac{d}{dx}[p(x)y'(x)]+q(x)y=-\lambda w(x)y(x),x\in I=[0,1]. \tag{3.3}$$

The eigenvalues of the regular Sturm-Liouville problem (3.3) are well known. For this reason, we assume that $0<\alpha(x)<1$. Using the properties mentioned in the previous section, we can rewrite Equation (3.3) as

$$x^{1-\alpha(x)}\ p(x)y''(x)+x^{1-\alpha(x)}\ p'(x)y'(x)+q(x)y(x)=-\lambda w(x)y(x). \tag{3.4}$$

Assume that $y(0)=\mu_{1}$ and $y'(0)=\mu_{2}$. To homogenize these conditions, we assume that

$$f(x)=y(x)-\mu_{2}x-\mu_{1}.$$

Then, Equation (3.4) becomes

$$x^{1-\alpha(x)}\ p(x)f''(x)+x^{1-\alpha(x)}\ p'(x)f'(x)+\mu_2 x^{1-\alpha(x)}\ p'(x)+q(x)f(x)+q(x)(\mu_2 x+\mu_1)$$
$$=-\lambda w(x)f(x)-\lambda w(x)(\mu_2 x+\mu_1)$$

or

$$g_0(x)\ f''(x)+g_1(x)\ f'(x)+g_2(x)f(x)=h(x) \tag{3.5}$$

subject to

$$f(0)=f'(0)=0. \tag{3.6}$$

where

$$\begin{aligned}
g_0(x) &= x^{1-\alpha(x)}\ p(x),\\
g_1(x) &= x^{1-\alpha(x)}\ p'(x),\\
g_2(x) &= q(x)+\lambda w(x),\\
h(x) &= -(\mu_2 x+\mu_1)(\lambda w(x)+q(x))-\mu_2 x^{1-\alpha(x)}\ p'(x).
\end{aligned}$$

Since $a_0^2+a_1^2>0$ and $\mu_1\ a_0+\mu_2\ a_1=0$, we have the following two cases.

1. If $a_0=0,\ \mu_2=0$ and we have only one unknown which is μ_1.
2. If $a_0\neq 0,\ \mu_1=-\frac{\mu_2\ a_1}{a_0}$ and we have only one unknown which is μ_2.

Therefore, we can write the solution as a product of one of the constants μ_1 and μ_2 and a function which depends on x and λ only. To find the eigenvalues of Problem (3.1)-(3.2), we use the simple shooting method by forcing the solution to satisfy the condition

$$a_2y(1)+a_3y'(1)=0.$$

In order to solve problem (3.5)-(3.6), we construct the kernel Hilbert spaces $W_2^1[0,1]$ and $W_2^3[0,1]$ in which every function satisfy the boundary conditions (3.6). Let

$$\begin{aligned}
W_2^3[0,1] = \{f(s):\ &f\ ,f', \text{and } f'' \text{ are absolutely continuous real-valued functions,}\\
&f'''\in L^2[0,1], f(0)=f'(0)=0\}.
\end{aligned}$$

The inner product in $W_2^3[0,1]$ is defined as

$$(u(z),v(z))_{W_2^3[0,1]}=u(0)v(0)+u'(0)v'(0)+u(1)v(1)+u'(1)v'(1)+\int_0^1 u^{(3)}(z)v^{(3)}(z)dy,$$

and the norm $\|u\|_{W_2^3[0,1]}$ is given by

$$\|u\|_{W_2^3[0,1]}=\sqrt{(u(z),u(z))_{W_2^3[0,1]}}$$

where $u,v\in W_2^3[0,1]$.

Theorem 3.1. The space $W_2^3[0,1]$ is a reproducing kernel Hilbert space, i.e.; there exists $K(s,z)\in W_2^3[0,1]$ such that for any $u\in W_2^3[0,1]$ and each fixed $z,x\in[0,1]$, we have

$$(u(z),K(x,z))_{W_2^3[0,1]}=u(x).$$

In this case, $K(x,z)$ is given by

$$K(x,z)=\left\{\begin{array}{ll}\sum_{i=0}^5 c_i(x)z^i, & z\le x\\ \sum_{i=0}^5 d_i(x)z^i, & z>x\end{array}\right\}$$

where

$$
\begin{aligned}
&c_0 = 0,\ c_1 = 0,\ c_2 = \frac{1}{120}(5z^4 - 111z^2 - 10z^3 - z^5),\\
&c_3 =\ 0,\ c_4 = \frac{-z}{24},\ c_5 = \frac{1}{120}(1+z^2),\\
&d_0 = \frac{z^5}{120},\ d_1 = \frac{-z^4}{24},\ d_2 = \frac{1}{120}(5z^4 - 111z^2 - z^5),\\
&d_3 =\ -\frac{z^2}{12},\ d_4 = 0,\ d_5 =\ \frac{z^2}{120}.
\end{aligned}
$$

Proof: Using integration by parts, one can get

$$
\begin{aligned}
(u(z),K(x,z))_{W_2^3[0,1]} = {} & u(0)K(x,0) + u(1)K(x,1) + u'(0)K_z(x,0) + u'(1)K_z(x,1)\\
& + u''(1)\frac{\partial^3 K}{\partial z^3}(x,1) - u''(0)\frac{\partial^3 K}{\partial z^3}(x,0)\\
& - u'(1)\frac{\partial^4 K}{\partial z^4}(x,1) + u'(0)\frac{\partial^4 K}{\partial z^4}(x,0) + u(1)\frac{\partial^5 K}{\partial z^5}(x,1) - u(0)\frac{\partial^5 K}{\partial z^5}(x,0) - \int_0^1 u(z)\frac{\partial^6 K}{\partial z^6}(x,z)dz.
\end{aligned}
$$

Since $u(z)$ and $K(x,z) \in W_2^3[0,1]$,

$$u(0) = u'(0) = 0$$

and

$$K(x,0) = K_z(x,0) = 0. \tag{3.7}$$

Thus,

$$
\begin{aligned}
(u(z),K(x,z))_{W_2^3[0,1]} = {} & u(1)K(x,1) + u'(1)K_z(x,1) + u''(1)\frac{\partial^3 K}{\partial z^3}(x,1) - u''(0)\frac{\partial^3 K}{\partial z^3}(x,0)\\
& - u'(1)\frac{\partial^4 K}{\partial z^4}(x,1) + u(1)\frac{\partial^5 K}{\partial z^5}(x,1) - \int_0^1 u(z)\frac{\partial^6 K}{\partial z^6}(x,z)dz.
\end{aligned}
$$

Since $K(x,z)$ is a reproducing kernel of $W_2^3[0,1]$,

$$(u(z),K(x,z))_{W_2^3[0,1]} = u(x)$$

which implies that

$$\frac{\partial^6 K}{\partial z^6}(x,z) = \delta(z - x\ ) \tag{3.8}$$

where δ is the Dirac-delta function and

$$K(x,1) + \frac{\partial^5 K}{\partial z^5}(x,1) = 0, \tag{3.9}$$

$$K_z(x,1) - \frac{\partial^4 K}{\partial z^4}(x,1) = 0, \tag{3.10}$$

$$\frac{\partial^3 K}{\partial z^3}(x,1) = 0, \tag{3.11}$$

$$\frac{\partial^3 K}{\partial z^3}(x,0) = 0. \tag{3.12}$$

Since the characteristic equation of $\frac{\partial^6 K}{\partial z^6}(x,z) = \delta(x-z)$ is $\lambda^6 = 0$ and its characteristic value is $\lambda = 0$ with 6 multiplicity roots, we write $K(x,z)$ as

$$K(x,z) = \left\{ \begin{array}{ll} \sum_{i=0}^{5} c_i(x) z^i, & z \leq x \\ \sum_{i=0}^{5} d_i(x) z^i, & z > x \end{array} \right\}.$$

Since $\frac{\partial^6 K}{\partial z^6}(x,z) = \delta(x-z)$, we have

$$\frac{\partial^m K}{\partial z^m}(x,x+0) = \frac{\partial^m K}{\partial z^m}(x,x-0),\ m = 0,1,...,4. \tag{3.13}$$

On the other hand,integrating $\frac{\partial^6 K}{\partial z^x}(x,z) = \delta(x-z)$ from $x-\varepsilon$ to $x+\varepsilon$ with expect to z and letting $\varepsilon \to 0$ to get

$$\frac{\partial^5 K}{\partial z^5}(x,x+0) - \frac{\partial^5 K}{\partial z^5}(x,x-0) = -1. \tag{3.14}$$

Using the conditions (3.7), and (3.9)-(3.14), we get the following system of equations

$$\begin{aligned} c_0(x) &= 0, c_1(x) = 0, c_3(x) = 0, \\ 6d_3(x) + 24d_4(x) + 60d_5(x) &= 0,\ \sum_{i=0}^{5} d_i(x) + 120 d_5(x) = 0, \end{aligned}$$

$$\sum_{i=i}^{5} i d_i(x) - 24 d_4(x) - 120 d_5(x) = 0,$$

$$\sum_{i=0}^{5} c_i(x) x^i = \sum_{i=0}^{5} d_i(x) x^i,$$

$$\begin{aligned} \sum_{i=1}^{5} i c_i(x) x^{i-1} &= \sum_{i=i}^{5} i d_i(x) x^{i-1}, \\ \sum_{i=2}^{5} i(i-1) c_i(x) x^{i-2} &= \sum_{i=1}^{5} i(i-1) d_i(x) x^{i-2}, \\ \sum_{i=3}^{5} i(i-1)(i-2) c_i(x) x^{i-3} &= \sum_{i=3}^{5} i(i-1)(i-2) d_i(x) x^{i-3}, \end{aligned}$$

$$\begin{aligned} \sum_{i=4}^{5} i(i-1)(i-2)(i-3) c_i(x) x^{i-4} &= \sum_{i=4}^{5} i(i-1)(i-2)(i-3) d_i(x) x^{i-4}, \\ 5! d_5(x) - 5! c_5(x) &= -1. \end{aligned}$$

We solved the last system using Mathematica to get

$$\begin{aligned} c_0 &= 0,\ c_1 = 0,\ c_2 = \frac{1}{120}(5z^4 - 111z^2 - 10z^3 - z^5), \\ c_3 &= 0,\ c_4 = \frac{-z}{24},\ c_5 = \frac{1}{120}(1+z^2), \\ d_0 &= \frac{z^5}{120},\ d_1 = \frac{-z^4}{24},\ d_2 = \frac{1}{120}(5z^4 - 111z^2 - z^5), \\ d_3 &= -\frac{z^2}{12},\ d_4 = 0,\ d_5 = \frac{z^2}{120}. \end{aligned}$$

which completes the proof of the theorem.

Next, we study the space $W_2^1[0,1]$. Let

$$W_2^1[0,1] = \{u(x):\ u \ \text{are absolutely continuous real-valued functions}, u' \in L^2[0,1]\}.$$

The inner product in $W_2^1[0,1]$ is defined as

$$(u(z),v(z))_{W_2^1[0,1]} = u(0)v(0)\ + \int_0^1 u'(z)v'(z)dz,$$

and the norm $\|u\|_{W_2^1[0,1]}$ is given by

$$\|u\|_{W_2^1[0,1]} = \sqrt{(u(z),u(z))_{W_2^1[0,1]}}$$

where $u,v \in W_2^1[0,1]$.

Theorem 3.2. The space $W_2^1[0,1]$ is a reproducing kernel Hilbert space, i.e.; there exists $R(s,z) \in W_2^1[0,1]$ such that for any $u \in W_2^1[0,1]$ and each fixed $z,x \in [0,1]$, we have

$$(u(z),R(x,z))_{W_2^1[0,1]} = u(x).$$

In this case, $R(x,z)$ is given by

$$R(x,z) = \left\{ \begin{array}{l} 1+z,\ \ z \le x \\ 1+x\ ,\ \ z > x \end{array} \right\}.$$

Proof: Using integration by parts, one can get

$$\begin{aligned}(u(z),R(x,z))_{W_2^1[0,1]} &= u(0)R(x,0)\ + \int_0^1 u'(z)\frac{\partial R}{\partial z}(x,z)dz \\ &= u(0)R(x,0) + u(1)\frac{\partial R}{\partial z}(x,1) - u(0)\frac{\partial R}{\partial z}(x,0) - \int_0^1 u(z)\frac{\partial^2 R}{\partial z^2}(x,z)dz.\end{aligned}$$

Since $R(x,z)$ is a reproducing kernel of $W_2^1[0,1]$,

$$(u(z),R(x,z))_{W_2^1[0,1]} = u(x)$$

which implies that

$$-\frac{\partial^2 R}{\partial z^2}(x,z) = \delta(z-x\ ) \tag{3.15}$$

and

$$R(x,0) - \frac{\partial R}{\partial z}(x,0)\ = 0, \tag{3.16}$$

$$\frac{\partial R}{\partial z}(x,1)\ = 0. \tag{3.17}$$

Since the characteristic equation of $-\frac{\partial^2 R}{\partial z^2}(x,z) = \delta(z-x\ )$ is $\lambda^2 = 0$ and its characteristic value is $\lambda\ = 0$ with 2 multiplicity roots, we write $R(x,z)$ as

$$R(x,z) = \left\{ \begin{array}{l} c_0(x)+c_1(x)z,\ \ z \le x \\ d_0(x)+d_1(x)z,\ \ z > x \end{array} \right\}.$$

Since $\frac{\partial^2 R}{\partial z^2}(x,z) = -\delta(z-x\ )$, we have

$$R(x,x+0) - R(x,x+0) = 0 \tag{3.18}$$

$$\frac{\partial R}{\partial z}(x,x+0) - \frac{\partial R}{\partial z}(x,x+0)\ \ = -1 \tag{3.19}$$

Using the conditions (3.17)-(3.19), we get the following system of equations

$$\begin{aligned} c_0(x) - c_1(x) &= 0, \\ d_1(x) &= 0, \\ c_0(x) + c_1(x)\ x &= d_0(x) + d_1(x)\ x, \\ d_1(x) -\ c_1(x) &= -1, \end{aligned} \tag{3.20}$$

which implies that

$$c_0(x) = 1,\ c_1(x) = 1, d_0(x) = 1 + x,\ d_1(x) = 0.$$

This completes the proof of the theorem.

Now, we present how to solve Problem (3.5)-(3.6) using the reproducing kernel method. Let

$$\sigma_i(x) = R(x_i, x)$$

for $i = 1, 2, \ldots$. It is clear that $L : W_2^3[0,1] \to W_2^1[0,1]$ is bounded linear operator. Let

$$\psi_i(x) = L^* \sigma_i(x)$$

where $L(\sigma_i(x)) = g_0(x)\ \sigma_i''(x) + g_1(x)\ \sigma_i'(x) + g_2(x)\sigma_i(x)$ and L^* is the adjoint operator of L. Using Gram-Schmidt orthonormalization to generate orthonormal set of functions $\{\bar{\psi}_i(x)\}_{i=1}^{\infty}$ where

$$\bar{\psi}_i(x) = \sum_{j=1}^{i} \alpha_{ij} \psi_j(x) \tag{3.21}$$

and α_{ij} are coefficients of Gram-Schmidt orthonormalization. In the next theorem, we show the existence of the solution of Problem (3.5)-(3.6).

Theorem 3.3. If $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$, then

$$f(x) = \sum_{i=1}^{\infty} \sum_{j=1}^{i} \alpha_{ij} h(x_j) \bar{\psi}_i(x). \tag{3.22}$$

Proof: First, we want to prove that $\{\psi_i(x)\}_{i=1}^{\infty}$ is the complete system of $W_2^3[0,1]$ and $\psi_i(x) = L(K(x, x_i))$. It is clear that $\psi_i(x) \in W_2^3[0,1]$ for $i = 1, 2, \ldots$. Simple calculations implies that

$$\begin{aligned} \psi_i(x) &= L^* \sigma_i(x) = (L^* \sigma_i(x), K(x,z))_{W_2^3[0,1]} \\ &= (\sigma_i(x), L(K(x,z)))_{W_2^3[0,1]} = L(K(x, x_i)). \end{aligned}$$

For each fixed $f(x) \in W_2^3[0,1]$, let

$$(f(x), \psi_i(x))_{W_2^3[0,1]} = 0,\ i = 1, 2, \ldots.$$

Then

$$\begin{aligned} (f(x), \psi_i(x))_{W_2^3[0,1]} &= (f(x), L^* \sigma_i(x))_{W_2^3[0,1]} \\ &= (Lf(x), \sigma_i(x))_{W_2^3[0,1]} \\ &= Lf(x_i) = 0. \end{aligned}$$

Since $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$, $Lf(x) = 0$. Since L^{-1} exists, $u(x) = 0$. Thus, $\{\psi_i(x)\}_{i=1}^{\infty}$ is complete system of $W_2^3[0,1]$. Second, we prove equation (3.22). Simple calculations implies that

$$
\begin{aligned}
f(x) &= \sum_{i=1}^{\infty} (f(x), \bar{\psi}_i(x))_{W_2^3[0,1]} \bar{\psi}_i(x) \\
&= \sum_{i=1}^{\infty} \sum_{j=1}^{i} \alpha_{ij} (f(x), L^*(K(x,x_j)))_{W_2^3[0,1]} \bar{\psi}_i(x) \\
&= \sum_{i=1}^{\infty} \sum_{j=1}^{i} \alpha_{ij} (Lf(x), K(x,x_j))_{W_2^3[0,1]} \bar{\psi}_i(x) \\
&= \sum_{i=1}^{\infty} \sum_{j=1}^{i} \alpha_{ij} (h(x), K(x,x_j))_{W_2^3[0,1]} \bar{\psi}_i(x) \\
&= \sum_{i=1}^{\infty} \sum_{j=1}^{i} \alpha_{ij} h(x_j) \bar{\psi}_i(x)
\end{aligned}
$$

and the proof is complete.

Let the approximate solution of Problem (3.5)-(3.6) be given by

$$
f_N(x) = \sum_{i=1}^{N} \sum_{j=1}^{i} \alpha_{ij} h(x_j) \bar{\psi}_i(x). \tag{3.23}
$$

In the next theorem, we show the uniformly convergent of the $\left\{ \frac{d^m f_N(x)}{dx^m} \right\}_{N=1}^{\infty}$ to $\frac{df(x)}{dx}$ for $m = 0,1,2$.

Theorem 3.4. If $f(x)$ and $f_N(x)$ are given as in (3.22) and (3.23), then $\left\{ \frac{d^m f_N(x)}{dx^m} \right\}_{N=1}^{\infty}$ converges uniformly to $\frac{df(x)}{dx}$ for $m = 0,1,2$.

Proof: First, we prove the theorem for $m = 0$. For any $x \in [0,1]$,

$$
\begin{aligned}
\|f(x) - f_N(x)\|^2_{W_2^4[0,1]} &= (f(x) - f_N(x), f(x) - f_N(x))_{W_2^3[0,1]} \\
&= \sum_{i=N+1}^{\infty} ((f(x), \bar{\psi}_i(x))_{W_2^3[0,1]} \bar{\psi}_i(x), (f(x), \bar{\psi}_i(x))_{W_2^3[0,1]} \bar{\psi}_i(x))_{W_2^3[0,1]} \\
&= \sum_{i=N+1}^{\infty} (f(x), \bar{\psi}_i(x))^2_{W_2^3[0,1]}.
\end{aligned}
$$

Thus,

$$
\underset{x\in[0,1]}{xup} \|f(x) - f_N(x)\|^2_{W_2^3[0,1]} = \underset{x\in[0,1]}{xup} \sum_{i=N+1}^{\infty} (f(x), \bar{\psi}_i(x))^2_{W_2^3[0,1]}.
$$

From Theorem (3.3), one can see that $\sum_{i=1}^{\infty} (f(x), \bar{\psi}_i(x))_{W_2^3[0,1]} \bar{\psi}_i(x)$ converges uniformly to $f(x)$. Thus,

$$
\underset{N\to\infty}{Lim} \underset{x\in[0,1]}{xup} \|f(x) - f_N(x)\|_{W_2^3[0,1]} = 0
$$

which implies that $\{f_N(x)\}_{N=1}^{\infty}$ converges uniformly to $f_N(x)$.

Second, we prove the uniformly convergence for $m = 1,2$. Since $\frac{d^m K(x,z)}{dx^m}$ is bounded function on $[0,1] \times [0,1]$,

$$
\left\| \frac{d^m K(x,z)}{dx^m} \right\|_{W_2^3[0,1]} \le \chi_m, \ m = 1,2.
$$

Thus, for any $x \in [0,1]$,

$$\left|f^{(m)}(x)-f_N^{(m)}(x)\right| = \left|(f(x)-f_N(x),\frac{d^mK(x,z)}{dx^m})_{W_2^3[0,1]}\right|$$
$$\leq \|f(x)-f_N(x)\|_{W_2^3[0,1]}\left\|\frac{d^mK(x,z)}{dx^m}\right\|_{W_2^3[0,1]}$$
$$\leq \chi_m \|f(x)-f_N(x)\|_{W_2^3[0,1]}$$
$$\leq \chi_m \underset{x\in[0,1]}{xup}\|f(x)-f_N(x)\|_{W_2^3[0,1]}.$$

Hence,

$$\underset{x\in[0,1]}{xup}\left\|f^{(m)}(x)-f_N^{(m)}(x)\right\|_{W_2^3[0,1]} \leq \chi_m \underset{x\in[0,1]}{xup}\|f(x)-f_N(x)\|_{W_2^3[0,1]}$$

which implies that

$$\underset{N\to\infty}{Lim}\ \underset{x\in[0,1]}{xup}\left\|f^{(m)}(x)-f_N^{(m)}(x)\right\|_{W_2^3[0,1]} = 0.$$

Therefore, $\left\{\frac{d^m f_N(x)}{dx^m}\right\}_{N=1}^{\infty}$ converges uniformly to $\frac{d^{(m)}f(x)}{dx^m}$ for $m=1,2$.

4 Numerical Results

In this section, we apply the RKM outlined in the previous sections to solve numerically the following three examples. Note that the maximum number of terms in the series solution is taken as $N=12$ for all examples considered in this paper.

Example 4.1. Consider the following regular fractional eigenvalue problem

$$D^{\frac{2-\nu\sin x}{4}}y'(x) = -\lambda y(x),\ 0<x<1,\ 0\leq\nu\leq 1,$$

subject to

$$y'(0)=0,\ y(1)=0.$$

Al-Mdallal [3] solved this problem using the Adomian decomposition method in the Caputo fractional derivative sense when $\nu=0$ and he found the first three eigenvalues only for $N=25$. These eigenvalues are

$$\lambda_1 = 2.11027708,\ \lambda_2 = 13.76538223,\ \lambda_3 = 24.24328676.$$

Using the conformable derivative sense, the corresponding problem to Equations (6) is

$$x^{1-\alpha(x)}\,y''(x) = -\lambda y(x)$$

subject to

$$y(0)=\mu_1,\ y'(0)=0.$$

Using the change of variable $f(x)=y(x)-\mu_1$, we get

$$x^{1-\alpha(x)}\ f''(x)+\lambda f(x) = -\lambda\mu_1 \tag{4.1}$$

subject to

$$f(0)=0,\ f'(0)=0. \tag{4.2}$$

We report the first five eigenvalues in Table 1 for $\nu=0$. The eigenfunctions corresponding to these eigenvalues are shown in Figure 1.

It worth mention that when we take $n=40$, we find the following eigenvalues

Table 1: Eigenvalues of Example (4.1) for $v = 0$

i	λ_i
1	4.790491770421957
2	37.18737319624638
3	100.4162368272831
4	194.4864297458961
5	319.3895777330032

Table 2: $\delta_{i,j}$ for Example (4.1) when $v = 0$

j	$\delta_{1,j}$	$\delta_{2,j}$	$\delta_{3,j}$	$\delta_{4,j}$	$\delta_{5,j}$
1		$5.7*10^{-16}$	$2.0*10^{-14}$	$2.8*10^{-14}$	$9.3*10^{-14}$
2	$5.7*10^{-16}$		$7.6*10^{-14}$	$8.5*10^{-14}$	$9.5*10^{-14}$
3	$2.0*10^{-14}$	$7.6*10^{-14}$		$2.9*10^{-14}$	$5.1*10^{-14}$
4	$2.8*10^{-14}$	$8.5*10^{-14}$	$2.9*10^{-14}$		$6.2*10^{-14}$
5	$9.3*10^{-14}$	$9.5*10^{-14}$	$5.1*10^{-14}$	$6.2*10^{-14}$	

Table 3: Eigenvalues of Example (4.1) for $v = 1$

i	λ_i
1	5.189982714917013
2	39.58147104205038
3	106.91354150495103
4	182.2275092159661

$$4.790491770421957, 37.18737319624638, 100.4162368272831, 194.4864297458961,$$
$$319.3895777330032, 475.1536472780752, 661.7509213990331, 879.1907367345706,$$
$$1127.4718520205686, 1406.6251455569052, 1715.8056579318209, 2086.2321659794966,$$
$$4090.860440344304, 5129.869022158444, 6836.407711802377, 7267.32559030844.$$

We notice that these eigenvalues satisfy the property

$$4.790491770421957 = \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \ldots.$$

Let

$$\delta_{i,j} = \left| \int_0^1 y_i(x)\, y_j(x)\, w(x) dx \right|.$$

In Table 2, we report the values of $\delta_{i,j}$ for $i, j = 1, 2, \ldots, 5$ with $i \neq j$.

However, we compute $\delta_{i,j}$ for Al-Mdallal results [3] when $n = 25$ and the results are

$$\delta_{1,2} = 0.0011366,\ \delta_{1,3} = 0.00904938,\ \delta_{2,3} = 0.0270058.$$

In addition, We take $n = 60$ but we do not get any new eigenvalue using his technique.

For $v = 1$, we can write the problem under study as follow

$$x^{\frac{2+\sin x}{4}}\, y''(x) = -\lambda y(x)$$

subject to

$$y(0) = \mu_1,\ y'(0) = 0.$$

For $v = 1$, the first four eigenvalues are reported in Table 3. The eigenfunctions corresponding to these eigenvalues are shown in Figure 2.

Example 4.2. Consider the following regular variable fractional eigenvalue problem

$$D^{\frac{2+v\ x}{4}} y'(x) + y(x) = -\lambda y(x),\ 0 < x < 1, 0 \leq v \leq 1,$$

Table 4: Eigenvalues of Example (4.2) for $\nu = 0$

i	λ_{1i}
1	13.051713056570495
2	57.778106764331230
3	133.34327399186424
4	239.75031067251480
5	376.99966723778533

Table 5: $\delta_{i,j}$ for Example (4.1) when $\nu = 0$

j	$\delta_{1,j}$	$\delta_{2,j}$	$\delta_{3,j}$	$\delta_{4,j}$	$\delta_{5,j}$
1		$1.6*10^{-16}$	$1.4*10^{-15}$	$8.4*10^{-15}$	$1.0*10^{-14}$
2	$1.6*10^{-16}$		$9.4*10^{-15}$	$1.1*10^{-14}$	$1.3*10^{-14}$
3	$1.4*10^{-15}$	$9.4*10^{-15}$		$2.2*10^{-14}$	$4.8*10^{-14}$
4	$8.4*10^{-15}$	$1.1*10^{-14}$	$2.2*10^{-14}$		$3.2*10^{-14}$
5	$1.0*10^{-14}$	$1.3*10^{-14}$	$4.8*10^{-14}$	$3.2*10^{-14}$	

subject to

$$y(0) = 0,\ y'(1) = 0.$$

Using the conformable derivative sense, the corresponding problem to Equations (6) is

$$x^{1-\alpha(x)}\, y''(x) + y(x) = -\lambda y(x)$$

subject to

$$y(0) = 0,\ y'(0) = \mu_2.$$

Using the change of variable $f(x) = y(x) - \mu_2 x$, we get

$$x^{1-\alpha(x)}\, f''(x) + (1+\lambda)\, f(x) = -\,(1+\lambda x)\,\mu_2 \tag{4.3}$$

subject to

$$f(0) = 0,\ \ f'(0) = 0. \tag{4.4}$$

We report the first five eigenvalues in Table 4 for $\nu = 0$. The eigenfunctions corresponding to these eigenvalues are shown in Figure 3.

It worth mention that when we take $n = 40$, we find the following eigenvalues

$$\begin{aligned}&13.051713056570495, 57.77810676433123, 133.34327399186424, 239.7503106725148,\\&376.99966723778533, 545.0914600437688, 744.025733056811, 973.8023947543952,\\&1234.4247413197409, 1525.7973831367005, 1850.4310652365637, 2164.852262741093,\\&2438.535229165145, 2863.299602198962, 4826.808779889705, 5495.563649719205, 7953.451176291645\end{aligned}$$

We notice that these eigenvalues satisfy the property

$$13.051713056570495 = \lambda_0 \le \lambda_1 \le \lambda_2 \le \ldots.$$

Let

$$\delta_{i,j} = \left| \int_0^1 y_i(x)\, y_j(x)\, w(x) dx \right|.$$

In Table 5, we report the values of $\delta_{i,j}$ for $i, j = 1, 2, ..., 5$ with $i \neq j$.

For $\nu = 1$, the first four eigenvalues are reported in Table 6. The eigenfunctions corresponding to these eigenvalues are shown in Figure 4.

Example 4.3. Consider the following regular variable fractional eigenvalue problem

$$D^{\frac{2+\nu\, x^2}{4}} y'(x) + y(x) = -\lambda y(x),\ 0 < x < 1, 0 \le \nu \le 1,$$

Table 6: Eigenvalues of Example (4.2) for $v = 1$

i	λ_i
1	12.07671301571908
2	54.14588306274719
3	125.617837878429016
4	175.197982492775378

Table 7: Eigenvalues of Example (4.3) for $v = 0$

i	λ_{1i}
1	14.60628075017210128211
2	65.05773017966611660092
3	150.5957338546744860472
4	271.2247472417288673558
5	426.9454126371902860653

subject to

$$y(0) = 0,\ y'(1) = 0.$$

Using the conformable derivative sense, the corresponding problem to Equations (6) is

$$x^{1-\alpha(x)}\, y''(x) + y(x) = -\lambda y(x)$$

subject to

$$y(0) = 0,\ y'(0) = \mu_2.$$

Using the change of variable $f(x) = y(x) - \mu_2 x$, we get

$$x^{1-\alpha(x)} f''(x) + (1+\lambda)\, f(x) = -(1+\lambda x)\,\mu_2 \tag{4.5}$$

subject to

$$f(0) = 0,\ f'(0) = 0. \tag{4.6}$$

We report the first five eigenvalues in Table 7 for $v = 0$. The eigenfunctions corresponding to these eigenvalues are shown in Figure 5.

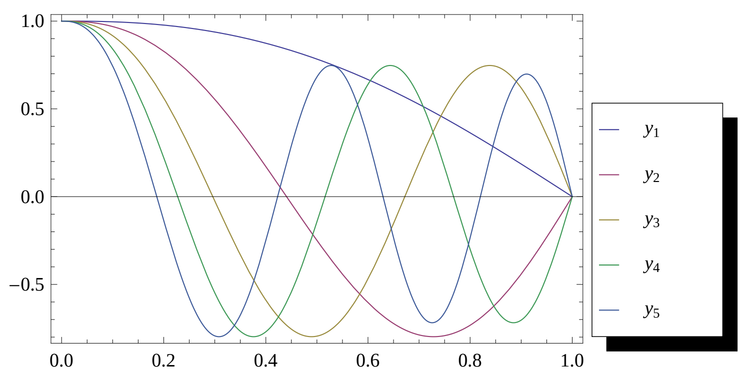

Fig. 1: The first five eigenfunctions of Example 4.1 for $v = 0$.

It worth mention that when we take $n = 40$, we find the following eigenvalues

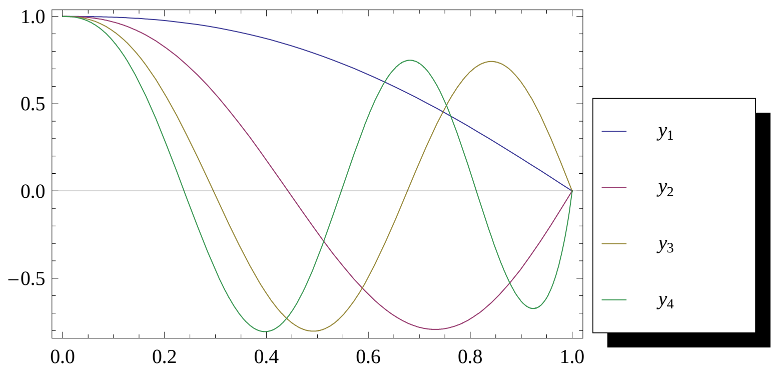

Fig. 2: The first four eigenfunctions of Example 4.1 for $\nu = 1$.

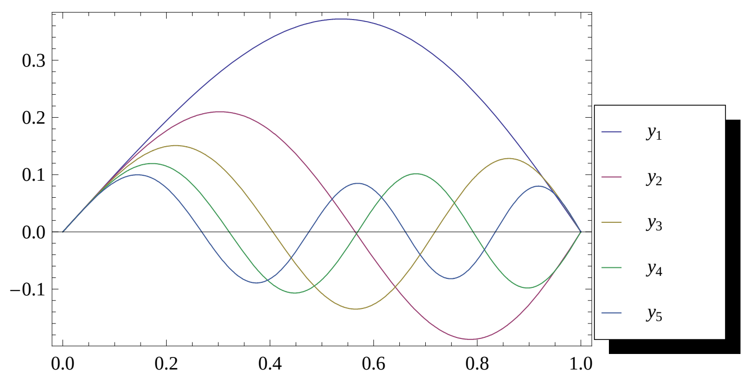

Fig. 3: The first five eigenfunctions of Example 4.2 for $\nu = 0$.

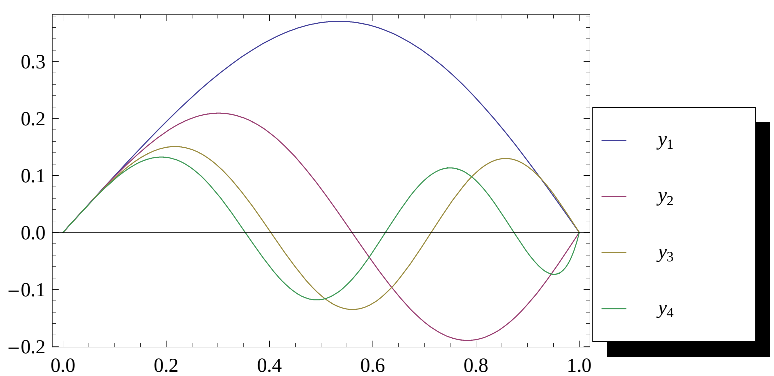

Fig. 4: The first four eigenfunctions of Example 4.2 for $\nu = 1$.

14.60628075017210128211442561107, 65.05773017966611660092310974683, 150.5957338546744860472,
271.2247472417288673558, 426.9454126371902860653, 617.7578953196759670319,
843.6622528272925835772, 1104.6585093617758632779, 1400.7466765219458657824,
1731.9267604254752142777, 2098.1987645450013065074, 2499.5626909698101854646,
2936.018541017728289761, 3407.5663155531630992203, 3914.2060151621494835994,
4455.9376403535404422933, 5032.7600365544535838088, 5653.4207870076875568754,
5884.4185706095808307898

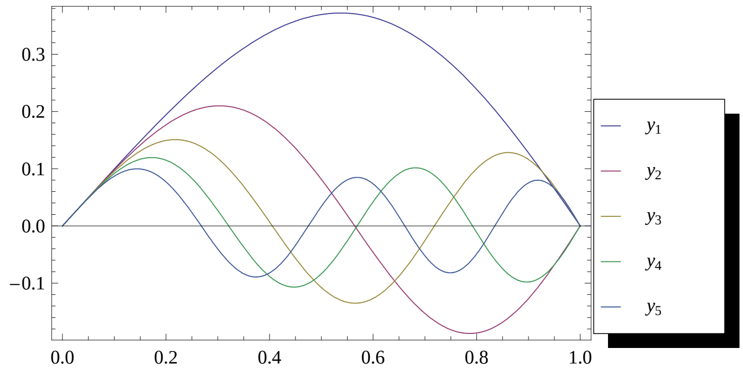

Fig. 5: The first five eigenfunctions of Example 4.3 for $\nu = 0$.

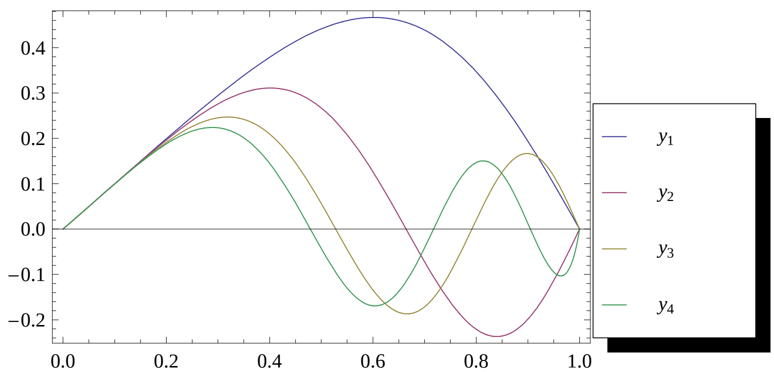

Fig. 6: The first four eigenfunctions of Example 4.3 for $\nu = 1$.

Table 8: $\delta_{i,j}$ for Example (4.1) when $\nu = 0$

j	$\delta_{1,j}$	$\delta_{2,j}$	$\delta_{3,j}$	$\delta_{4,j}$	$\delta_{5,j}$
1		$1.9*10^{-16}$	$2.1*10^{-15}$	$2.6*10^{-15}$	$3.8*10^{-15}$
2	$1.9*10^{-16}$		$2.2*10^{-15}$	$2.4*10^{-15}$	$3.1*10^{-15}$
3	$2.1*10^{-15}$	$2.2*10^{-15}$		$2.9*10^{-15}$	$4.9*10^{-15}$
4	$2.6*10^{-15}$	$2.4*10^{-15}$	$2.9*10^{-15}$		$3.8*10^{-15}$
5	$3.8*10^{-15}$	$3.1*10^{-15}$	$4.1*10^{-15}$	$3.8*10^{-14}$	

We notice that these eigenvalues satisfy the property

$$14.606280750172101282114425611107 = \lambda_0 \le \lambda_1 \le \lambda_2 \le \ldots.$$

Let

$$\delta_{i,j} = \left| \int_0^1 y_i(x)\, y_j(x)\, w(x) dx \right|.$$

In Table 8, we report the values of $\delta_{i,j}$ for $i, j = 1, 2, \ldots, 5$ with $i \neq j$.

For $\nu = 1$, the first four eigenvalues are reported in Table 9. The eigenfunctions corresponding to these eigenvalues are shown in Figure 6.

Table 9: Eigenvalues of Example (4.3) for $v = 1$

i	λ_i
1	24.28225719853922514678
2	112.6205306176597434665
3	265.1194498587259278269
4	381.2966761564513870887

5 Conclusion

In this paper, we have developed a numerical technique to approximate the eigenvalues of regular variable fractional Sturm-Liouville problem. The method of solution is based on RKM. The numerical results for the examples demonstrate the efficiency and accuracy of the present method. From the three examples which we mentioned in the previous section, we notice that our technique is very efficient for computing the eigenvalues of the regular variable fractional problems. It is competes the method in [3] and gives better and faster results. We end this section by the following remarks.

1. From Examples (4.1), (4.2), and (4.3), we can find as much eigenvalues as the model requires with the following property
$$\lambda_1 < \lambda_2 < \lambda_3 < ... < \lambda_n < ....$$
while in [3] only three eigenvalue can be found.
2. From Examples (4.1), (4.2), and (4.3), the orthogonality property
$$int_0^1 y_i(x)\ y_j(x)\ w(x) = 0,\ i \neq j$$
holds while in [3], we get $\delta_{1,2} = 0.0011366,\ \delta_{1,3} = 0.00904938,\ \delta_{2,3} = 0.0270058.$
3. From Figures 1-6, we see that corresponding to each eigenvalue λ_i is a unique (up to a normalization constant) eigenfunctions $y_i(x)$ which has exactly i - 1 zeros in $(0,1)$.
4. We notice that the conformable derivative sense is more suitable to study the regular variable fractional Sturm-Liouville problems than the Caputo fractional derivative sense.
5. The results in this paper confirm that RKM is a powerful and efficient method for solving regular variable fractional Sturm-Liouville problems in different fields of sciences and engineering.
6. RKM is excellent tool due to rapid convergent.
7. The existence and uniformly convergent are proven in Theorems (3.3) and (3.4).

References

[1] M. M. Djrbashian, A boundary value problem for a Sturm-Liouville type differential operator of fractional order, *Izv. Akad. Nauk Armjan. SSR Ser. Mat.* **5**(2), 71–96 (1970).
[2] A. M. Nahusev, The Sturm-Liouville problem for a second order ordinary differential equation with fractional derivatives in the lower terms, *Dokl. Akad. Nauk SSSR* **234**, 308–311 (1977).
[3] Q. M. Al-Mdallal, An efficient method for solving fractional Sturm-Liouville problems, *Chaos, Solit. Fract.* **40**, 183–189, (2009).
[4] S. Abbasbandy and A. Shirzadi, Homotopy analysis method for multiple solutions of the fractional Sturm-Liouville problems, *Numer. Algorit.* **54**(4), 521-532 (2010).
[5] V. S. Ertürk, Computing eigenelements of Sturm-Liouville Problems of fractional order via fractional differential transform method, *Math. Comput. Appl.* **16**(3), 712-720 (2011).
[6] Y. Luchko, Initial-boundary-value problems for the one-dimensional time-fractional diffusion equation, *Fract. Calc. Appl. Anal.* **15**(1), 141–160, (2012).
[7] A. Neamaty, R. Darzi, S. Zaree and B. Mohammad Zadeh, Haar wavelet operational matrix of fractional order integration and its application for eigenvalues of fractional Sturm– Liouville problem, *World Appl. Sci. J.* **16**(12), 1668–167 (2012).
[8] Z. Shi and Y. Y. Cao, Application of Haar wavelet method to eigenvalue problems of higher order, *Appl. Math. Model. 36*(9), 4020–4026 (2012).
[9] E. Bas and F. Metin, Spectral properties of fractional Sturm–Liouville problem for diffusion operator, arXiv:1212.4761, (2012).
[10] E. Bas and F. Metin, A note basis properties for fractional hydrogen atom equation, arXiv:1303.2839v2, (2013).
[11] E. Bas, Fundamental spectral theory of fractional singular Sturm-Liouville operator, *J. Funct. Space. Appl.*, Article ID 915830, 7 pages, Volume (2013).
[12] M. Zayernouri and G. E. Karniadakis, Fractional Sturm–Liouville eigen-problems: theory and numerical approximation, *J. Comput. Phys.* **252**, 495–517 (2013).

[13] T. Aboelenen and H. M. El-Hawary, Spectral theory and numerical approximation for singular fractional Sturm-Liouville eigenproblems on unbounded domain, arXiv:1410.1583v4, (2014).

[14] M. Al-Refai, Non-existence results and analytical bounds of eeigenvalues for a class of fractional Sturm-Liouville eigenvalue problems, *Fract. Differ. Calc.*, to appear, (2017).

[15] M. Klimek, T. Odzijewicz and A. B. Malinowska, Variational methods for the fractional Sturm–Lio uville problem, *J. Math. Anal. Appl.* **416**(1), 402–426 (2014).

[16] F. D. Saei, S. Abbasi and Z. Mirzayi, Inverse Laplace transform method for multiple solutions of the fractional Sturm-Liouville problems, *Comput. Meth. Differ. Equ.* **2**(1), 56-61 (2014).

[17] P. Antunes and R. Ferreira, An augmented – RBE method for solving fractional Sturm-Liouville eigenvalues problems, *SIAM J. Sci. Comput.* **37**(1), A515-A535 (2015).

[18] B. Jin, R. Lazarov, X. Lu and Z. Zhou, A simple finite element method for boundary value problems with a Riemann-Liouville derivative, *J. Comput. Appl. Math.* **293**, 94-111 (2015).

[19] M. Syam and H. Siyyam, An efficient technique for finding the eigenvalues of sixth-order Sturm-Liouville problems, *Chaos Solit. Fract.***39**, 659–665 (2009).

[20] A. Boutayeb and E. H. Twizell, Finite-difference methods for twelfth-order boundary-value problems, *J. Comput. Appl. Math.* **35**, 133–138 (1991).

[21] K. Djidjeli, E. H. Twizell and A. Boutayeb, Numerical methods for special nonlinear boundary-value problems of order 2m, *J. Comput. Appl. Math.* **47**, 35–45 (1993).

[22] W. Z. Huang and D. M. Sloan, The pseudospectral method for solving differential eigenvalue problems, *J. Comput. Phys.* **111**, 399–409 (1994).

[23] C. P. Gupta, Existence and uniqueness theorems for a bending of an elastic beam equation at resonance, *J. Math. Anal. Appl.* **135**, 208–225 (1988).

[24] D. O'Regan, Solvability of some fourth (and higher) order singular boundary value problems, *J. Math. Anal. Appl. 161*, 78–116 (1991).

[25] Y. Wang, Y. B. Zhao and G. W. Wei, A note on the numerical solution of higher-order differential equations, *J. Comput. Appl. Math.* **159**, 387–398 (2003).

[26] R. H. Gutierrez and P. A. A. Laura, Vibrations of non-uniform rings studied by means of the differential quadrature method, *J. Sound Vib.* **185**(3), 507–513 (1995).

[27] T. Y. Wu and G. R. Liu, The generalized differential quadrature rule for fourth-order differential equations, *Int. J. Numer. Meth. Eng.* **50**, 1907–1929 (2001).

[28] L. Greenberg, An oscillation method for fourth order self-adjoint two-point boundary value problems with nonlinear eigenvalues, *SIAM J. Math. Anal.* **22**, 1021–1042 (1991).

[29] F. Geng and M. Cui, Solving a nonlinear system of second order boundary value problems, *J. Math. Anal. Appl.* **327**, 1167–1181 (2007).

[30] J. Du and M. Cui, Solving the forced Duffing equations with integral boundary conditions in the reproducing kernel space, *Int. J. Comp. Math.* **87**, 2088–2100 (2010).

[31] H. Yao and M. Cui, A new algorithm for a class of singular boundary value problems, Appl. Math. Comput. **186**, 1183–1191 (2007).

[32] R. Khalil, M. Al Horani, A. Yousef and M. Sababheh, A new definition of fractional derivative, *J. Comput. Math. Appl.* **264**, 65–70 (2014).

[33] O. Acan, O. Firat, Y. Keskin and G. Oturanc, Conformable variational iteration method, *NTMSCI* **5**(1), 172–178 (2017).

[34] O. Iyiola and E. Nwaeze, Some new results on the new conformable fractional calculus with application using D'Alambert approach, *Progr. Fract. Differ. Appl.* **2**(2), 1–7 (2016).