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Uniform Exponential Stability for Time Varying Linear Dynamic Systems over Time Scales

Syed Omar Shah[*] *and Akbar Zada*

Department of Mathematics, University of Peshawar, Peshawar 25000, Pakistan.

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Abstract: This paper proves the uniform exponential stability of the time varying linear dynamic system $x^{\Delta}(r) = G(r)x(r),\ r \in \overline{T}$ in terms of bounded-ness of solution of the following Cauchy problem:

$$\begin{cases} W^{\Delta}(r) = G(r)W(r) + \omega(r),\ 0 \le r \in \overline{T}, \\ W(0) = v_0, \end{cases}$$

where $\overline{T}$ denotes time scale, $G(r)$ is a matrix valued function, $\omega(r)$ is a bounded function on $\overline{T}$ and $v_0 \in \mathbb{C}^m$. In this note we prove the results that have the above result as an immediate corollaries.

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1 Introduction

The theory of dynamic equations on time scales was introduced by Hilger [9], in 1988, in order to unify the continuous and discrete calculus. Since then, this theory has been developing rapidly and has received a lot of attention in recent years. The basic theory of time scales and dynamic equations on time scales can be found in the recent monographes by Bohner and Peterson [2,3] and the references contained therein.

Recently, many researchers paid attention to the study of different types of stabilities of dynamic equations on time scales, with different approaches. For more details, see [1,4,5,7,8,10,11,12,13].

2 Preliminaries

The non-empty arbitrary closed subset of real numbers is called Time Scale denoted by $\overline{T}$. The forward jump operators, backward jump operators and graininess function denoted by $\theta : \overline{T} \to \overline{T}$, $\rho : \overline{T} \to \overline{T}$, $\nu : \overline{T} \to [0,\infty)$ are respectively defined as:

$$\theta(r) = \inf\{v \in \overline{T} : v > r\},\ \rho(r) = \sup\{v \in \overline{T} : v < r\},\ \nu(r) = \theta(r) - r.$$

A point $r \in \overline{T}$ is said to be left-scattered and left-dense if $r > \rho(r)$ and $\rho(r) = r$, respectively. If $r < \theta(r)$ and $\theta(r) = r$, then such a point $r \in \overline{T}$ will be called right-scattered and right-dense, respectively. The set known as derived form of time scale $\overline{T}$ denoted by $\overline{T}^z$ is defined as follows:

$$\overline{T}^z = \begin{cases} \overline{T} \backslash (\rho(\sup \overline{T}), \sup \overline{T}], & \text{if } \sup \overline{T} < \infty, \\ \overline{T}, & \text{if } \sup \overline{T} = \infty. \end{cases}$$

A function $\alpha : \overline{T} \to \overline{R}$ is said to be right-dense continuous if it is continuous at all right-dense points in $\overline{T}$ and its left-sided limits exist at all left-dense points in $\overline{T}$, where $\overline{R}$ denotes the set of real numbers. A function $\alpha : \overline{T} \to \overline{R}$ is said to be regressive if $1 + \nu(r)\alpha(r) \neq 0$ for all $r \in \overline{T}^z$ and if $1 + \nu(r)\alpha(r) > 0$, then the function α is said to be positively regressive. The set of all right-dense continuous and regressive functions, right-dense continuous and positively regressive functions respectively will be denoted by $\mathbb{REG}(\overline{T})$ and $\mathbb{REG}(\overline{T})^+$. A function $w^{\theta} : \overline{T} \to \overline{R}$ is defined as $w^{\theta}(r) = w(\theta(r)),\ \forall\, r \in \overline{T}$.

Definition 1. *If $G \in \mathbb{REG}(\overline{T})$, then generalized exponential function $e_G(r,u)$ on $\overline{T}$ is defined as*

$$e_G(r,u) = exp\left(\int_u^r \chi_{\nu(v)} G(v)\Delta v\right)\ \forall\, r,u \in \overline{T},$$

[*] Corresponding author e-mail: omarshah89@yahoo.com, omarshahstd@uop.edu.pk

with cylindrical transformation

$$\chi_{v(r)}G(r) = \begin{cases} \dfrac{Log(1+v(r)G(r))}{v(r)}, & \text{if } v(r) \neq 0, \\ G(r), & \text{if } v(r) = 0. \end{cases}$$

Lemma 1.[2] *Let $G, H \in \mathbb{REG}(\overline{T})$, then:*

1. $e_0(r,u) = 1$ *and* $e_G(r,r) = 1$.
2. $e_G(\theta(r), r) = (1+v(r)G(r))e_G(r,u)$.
3. $(e_G(r,u))^{\Delta} = G(r)e_G(r,u)$.
4. $e_G(r,u) = \frac{1}{e_G(u,r)}$.
5. $e_G(r,u)e_G(u,v) = e_G(r,v)$.
6. *If* $r,u,v \in \overline{T}$, *then*

$$\int_r^u G(\eta)e_G(v,\theta(\eta))\Delta\eta = e_G(v,r) - e_G(v,u).$$

Definition 2.*Let G be $m \times m$ matrix-valued function on a time scale $(\overline{T})$ such that $||G(r)|| \leq \gamma$, $\gamma > 0$. Then G is said to be rd-continuous on $\overline{T}$ if each entry of G is rd-continuous and G is regressive if the eigenvalues $\kappa_i(r)$ of $G(r)$ are regressive for all $1 \leq i \leq m$.*

*Remark.*Let G^* be the conjugate transpose of $m \times m$ matrix valued function G. If $G \in (\overline{R})^{m\times m}$, then $G^* = G^T$ and $G^* \in \mathbb{REG}(\overline{T})$. Moreover, the function defined by $\ominus G(r) = \frac{-G(r)}{1+v(r)G(r)}$ is also regressive.

Consider the matrix-valued IVP,

$$W^{\Delta}(r) = G(r)W(r), \; W(r_0) = \mathtt{I}_m, \tag{2.1}$$

where $\mathtt{I}_m$ is $m \times m$ identity matrix.

Definition 3.*The fundamental matrix is defined to be the general solution to the matrix dynamic equation (2.1) and is denoted by $\Phi_G(r, r_0)$.*

Keep in mind that Φ_G as a transition matrix can be replaced with e_G in the following lemma. The next lemma lists some properties of the matrix exponential function.

Lemma 2.[6] *Let $G \in \mathbb{REG}(\overline{T})$ be the matrix-valued function on $\overline{T}$, then the family $\mathbf{G} = \{\Phi_G(r,u) : \; r,u \in \overline{T}\}$ has the following properties:*

1. $\Phi_0(r,u) = 1$ *and* $\phi_G(r,r) = 1$.
2. $\Phi_G(\theta(r),u) = (1+v(r)G(r))\Phi_G(r,u)$.
3. $\Phi^{-1}{}_G(r,u) = \Phi^*{}_{\ominus G^*}(u,r)$.
4. $\Phi_G(r,u) = \Phi^{-1}{}_G(r,u) = \Phi^*{}_{\ominus G^*}(u,r)$.
5. $\Phi_G(r,u)\Phi_G(u,v) = \Phi_G(r,v)$.
6. $\Phi_G^{\Delta}(r,u) = G(r)\Phi_G(r,u)$.

Now the next theorem guarantees a unique solution to the regressive $m \times 1$ vector-valued dynamic IVP

$$W^{\Delta}(r) = G(r)W(r) + \omega(r), \; W(r_0) = v_0. \tag{2.2}$$

Theorem 1.[6] *Let $r_0 \in \overline{T}$ and $v_0 \in \overline{R}^m$. Then the regressive IVP (2.2) has a unique solution $W : \overline{T} \to \overline{R}^m$ given by*

$$W(r) = \Phi_G(r,r_0)v_0 + \int_{r_0}^r \Phi_G(r,\theta(\eta))\omega(\eta)\Delta\eta.$$

Consider the regressive time varying linear dynamic system

$$x^{\Delta}(r) = G(r)x(r); \; x(r_0) = v_0, \; r \in \overline{T}, \; v_0 \in \mathbb{C}^m. \quad (G(r))$$

Theorem 2.[6] *The time varying linear dynamic system $(G(r))$ is uniformly exponentially stable if and only if there exists an $\eta, \gamma > 0$ with $-\eta \in \mathbb{REG}(\overline{T})^+$ such that the transition matrix Φ_G satisfies*

$$||\Phi_G(r,r_0)|| \leq \gamma e_{-\eta}(r,r_0), \; \forall \, r \geq r_0, \; \text{with } r, r_0 \in \overline{T}.$$

Theorem 3.[6] *Suppose that there exists a constant γ such that for all $r \in \overline{T}$, $||G(r)|| \leq \gamma$. Then the time varying linear dynamic system $(G(r))$ is uniformly exponentially stable if and only if there exists a constant $\beta > 0$ such that*

$$\int_{r_0}^r ||\Phi_G(r,\theta(\eta))||\Delta\eta \leq \beta, \; \forall \, r \geq \theta(\eta), \; \text{with } r,\eta \in \overline{T}.$$

3 Main Results

Our main result concerning the uniform exponential stability of the system $(G(r))$ is stated as follows:

Theorem 4.*The system $(G(r))$ is uniformly exponentially stable if and only if for each $v_0 \in \mathbb{C}^m$ and each bounded function $\omega(r)$, the unique solution of the following Cauchy problem*

$$\begin{cases} W^{\Delta}(r) = G(r)W(r) + \omega(r), \; r \geq 0 \\ W(0) = v_0, \end{cases} \quad (G(r), \omega, v_0)$$

is bounded.

*Proof.***Necessity:** Let the system $(G(r))$ is uniformly exponentially stable, then by Theorem 2, we have

$$||\Phi_G(r,r_0)|| \leq \gamma e_{-\eta}(r,r_0), \; \forall \, r \geq r_0, \; \text{with } r, \; r_0 \in \overline{T}.$$

Consider the solution of the Cauchy problem $(G(r), \omega, v_0)$,

$$\begin{aligned} W(r) &= \Phi_G(r,0)v_0 + \int_0^r \Phi_G(r,\theta(\eta))\omega(\eta)\Delta\eta \\ ||W(r)|| &\leq ||\Phi_G(r,0)||v_0 + \int_0^r ||\Phi_G(r,\theta(\eta))||||\omega(\eta)||\Delta\eta \\ &\leq e_{-\eta}(r,0) + C\int_0^r ||\Phi_G(r,\theta(\eta))||\Delta\eta \\ &\leq e_{-\eta}(r,0) + C\beta. \end{aligned}$$

Hence, the unique solution of $(G(r),\omega,v_0)$ is bounded.
Sufficiency: Suppose on contrary the system $(G(r))$ is not uniformly exponentially stable and set $\omega(r) = \frac{\Phi_G(r,0)}{1+\nu(r)\ominus G^*(r)}$. Obviously $\omega(r)$ is bounded function. Now consider the solution of $(G(r),\omega,v_0)$,

$$\begin{aligned} W(r) &= \Phi_G(r,0)v_0 + \int_0^r \Phi_G(r,\theta(\eta))\omega(\eta)\Delta\eta \\ &= \Phi_G(r,0)v_0 + \int_0^r \Phi^*{}_{\ominus G^*}(\theta(\eta)),r)\frac{\Phi_G(\eta,0)}{1+\nu(\eta)\ominus G^*(\eta)}\Delta\eta \\ &= \Phi_G(r,0)v_0 + \int_0^r \Phi^*{}_{\ominus G^*}(\eta,r)(1+\nu(\eta)\ominus G^*(\eta))\frac{\Phi_G(\eta,0)}{1+\nu(\eta)\ominus G^*(\eta)}\Delta\eta \\ &= \Phi_G(r,0)v_0 + \int_0^r \Phi_G(r,\eta)\Phi_G(\eta,0)\Delta\eta \\ &= \Phi_G(r,0)v_0 + \Phi_G(r,0)r. \end{aligned}$$

If we take $v_0 = 0$, then we have a contradiction because the map

$$r \mapsto \Phi_G(r,0)r$$

is unbounded. But since $z_0 \neq 0$, then as the system $G(r)$ is not uniformly exponentially stable, so by using Theorem 2 we can find $-\eta,\gamma > 0$ with $\eta \in \mathbb{REG}(\overline{T})^+$ such that

$$||\Phi_G(r,0)|| \geq \gamma e_\eta(r,0),$$

i.e. in this case again the solution will be unbounded and thus we arrived at a contradiction. So the system $(G(r))$ is uniformly exponentially stable.

Corollary 1.*The system $(G(r))$ is uniformly exponentially stable if and only if for each v, $v_0 \in \mathbb{C}^m$ and each bounded function $\omega(r)$, the unique solution of the following Cauchy problem*

$$\begin{cases} W^\Delta(r) = G(r)W(r) + \omega(r),\ r \geq 0 \\ W(0) = v - v_0, \end{cases} \qquad (G(r),\omega,v,v_0)$$

is bounded.

Corollary 2.*The system $(G(r))$ is uniformly exponentially stable if and only if for each bounded function $\omega(r)$, the unique solution of the following Cauchy problem*

$$\begin{cases} W^\Delta(r) = G(r)W(r) + \omega(r),\ r \geq 0 \\ W(0) = 0, \end{cases} \qquad (G(r),\omega)$$

is bounded.

4 Conclusion

The idea of uniform exponential stability is generalized for time varying linear dynamic systems on time scale. The uniform exponential stability of the system $(G(r))$ is proved in terms of boundedness of solution of the following Cauchy problems $(G(r),\omega,v_0)$, $(G(r),\omega,v,v_0)$ and $(G(r),\omega)$. Moreover, the uniform exponential stability of the system $(G(r))$ is proved with the help of Theorem 2 and Theorem 3.

References

[1] S. András and A. R. Mészáros, *Ulam-Hyers stability of dynamic equations on time scales via Picard operators*, Appl. Math. Comput., 2013, **219**, 4853–4864.
[2] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales, An Introduction with Applications*, Birkhäuser, Boston, Mass, USA, 2001.
[3] M. Bohner and A. Peterson, *Advances in Dynamics Equations on time scales*, Birkhäuser, Boston, Mass, USA, 2003.
[4] J. J. Dachunha , *Stability for time varying linear dynamic systems on time scales*, J. Comput. Appl. Math., 2005, **176**, 381–410.
[5] N. H. Du and L. H. Tien, *On the exponential stability of dynamic equations on time scales*, J. Math. Anal. Appl., 2007, **331**, 1159–1174.
[6] V. Dündar, *Dynamical Systems on Time Scales*, Master of Science thesis, İzmir Institute of Technology, June 2007.
[7] T. Gard and J. Hoffacker, *Asymptotic behavior of natural growth on time scales*, Dynam. Systems Appl., 2003, **12**, 131–147.
[8] A. Hamza and K. M. Oraby, *Stability of abstract dynamic equations on time scales*, Adv. Difference Equ., 2012.
[9] S. Hilger, *Analysis on measure chains-A unified approach to continuous and discrete calculus*, Result math., 1990, **18**, 18–56.
[10] V. Lupulescu and A. Zada, *Linear impulsive dynamic systems on time scales*, Electron. J. Qual. Theory Differ. Equ., 2010, 1–30.
[11] A.C. Peterson and N.Y. Raffoul, *Exponential stability of dynamic equations on time scales*, Adv. Difference Equ., 2005, 133–144.
[12] C. Pötzsche, S. Siegmund and F. Wirth, *A Spectral Characterization of Exponential Stability for Liner Time-Invariant Systems on time scales*, Discrete Contin. Dyn. Sys., 2003, **9**, 1223–1241.
[13] A. Zada, T. Li, S. Ismail and O. Shah, *Exponential dichotomy of linear autonomous systems over time scales*, Diff. Equa. Appl., 2016, **8**, 123–134.

Syed Omar Shah received the M.S. degree in Mathematics from University of Peshawar, Peshawar, Pakistan. Currently, he is PhD scholar at University of Peshawar, Peshawar, Pakistan. He is also serving as a lecturer of Mathematics at Qurtuba University of Science & IT Peshawar, Peshawar, Pakistan. His research interests include qualitative theory of differential equations, difference equations and dynamic equations on time scales. He published several research articles in reputed international journals of mathematics.

Akbar Zada is an assistant professor in University of Peshawar, Peshawar, Pakistan. He obtained his PhD from Abdus Salam School of Mathematical Sciences, GCU, Lahore Pakistan (2010). He is an active researcher in the field of qualitative theory of linear differential and difference systems, especially the asymptotic behavior of semigroup of operators and evolution families, arises in the solutions of non-autonomous systems. He published several research articles in reputed international journals of mathematics.