Progress in Fractional Differentiation and Applications
An International Journal

http://dx.doi.org/10.18576/pfda/030202

On the Fractional Legendre Equation and Fractional Legendre Functions

*Mohammed Al-Refai**, *Muhammed Syam and Qasem Al-Mdallal*

Department of Mathematical Sciences, United Arab Emirates University, UAE.

Received: 22 Jul. 2016, Revised: 12 Jan. 2017, Accepted: 22 Jan. 2017
Published online: 1 Apr. 2017

Abstract: In this paper we propose a fractional generalization of the well-known Legendre equation. We obtain a solution in the form of absolutely convergent power series with radius of convergence 1. We then truncate the power series to obtain the even and odd fractional Legendre functions in closed forms. These functions converge to the Legendre polynomials as the fractional derivative approaches 1, and new explicit formulas of the even and odd Legendre polynomials have been derived.

Keywords: Fractional differential equations, Legendre Equation, Caputo fractional derivative.

1 Introduction

In recent years, fractional differential equation (FDEs) caught the attention of many researchers because of their appearance in modeling several phenomenon in the physical sciences [1,2,3]. As many FDEs don't possess exact solutions on closed forms, analytical and numerical techniques have been implemented to study these equations [4,5,6, 7,8,9,10,11,12]. In the literature there are many functions that are inspired by the use of fractional calculus. They appear as solutions to certain fractional deferential equations which generalize well known differential equations with integer orders, or as generalized fractional calculus operators of basic functions. These functions include the Mittag-Liffler functions, the Fox-Wright function and the Fox H-function, etc; for more details the reader is referred to [13,14]. In recent years, there are interests to generalize several well-known differential equations and study their solutions. In [15] the solution of the fractional Bessel equation is derived in terms of the power series and asymptotic analysis of the solution for large arguments is obtained, where the fractional derivative is of the Riemman-Liouville type. In [16] a new mathematical model of cornel topography based on the solution of the modified Bessel fractional differential equation is obtained. The fractional Legendre equation is investigated in [17]. However, the conformable fractional derivative is used which satisfies same properties of the integer derivative. In this paper we consider the fractional Legendre eigenvalue problem

$$(1-x^{2\alpha})D_{0+}^{2\alpha}y - 2\alpha x^{\alpha}D_{0+}^{\alpha}y + \ell(\ell+1)y = 0, \ \frac{1}{2} < \alpha < 1, \ 0 < x < 1, \tag{1.1}$$

where $D_{0+}^{2\alpha} = D_{0+}^{\alpha}(D_{0+}^{\alpha})$ and D_{0+}^{α} is the left Caputo fractional derivative. The above equation is a fractional generalization of the well-known Legendre equation

$$((1-x^2)y')' + \lambda y = (1-x^2)y'' - 2xy' + \lambda y = 0, \ 0 < x < 1, \ \lambda = \ell(\ell+1).$$

The left Caputo fractional derivative is defined by

$$(D_{0+}^{\alpha}f)(t) = (I_{0+}^{n-\alpha}\tfrac{d^n}{dt^n}f)(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}f^{(n)}(s)\,ds, & n-1<\alpha<n\in\mathbb{N}, \\ f^{(n)}(t), & \alpha = n\in\mathbb{N}, \end{cases}$$

where Γ is the well-known Gamma function and I_{0+}^{α} is the left Riemann-Liouville fractional integral defined by

$$(I_{0+}^{\alpha}f)(t) = \begin{cases} \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)ds, & \alpha>0, \\ f(t), & \alpha=0. \end{cases} \tag{1.2}$$

* Corresponding author e-mail: m_alrefai@uaeu.ac.ae

For more details about the definition and properties of fractional derivatives, the reader is referred to [18]. As the fractional derivative α in Eq.(1.1) approaches 1, the proposed fractional eigenvalue problem will reduce to the Legendre equation.
This paper is organized as follows. In Section 2, we present the infinite series solution and then discuss the convergence of the series in Section 3. In Section 4, we present the fractional Legendre functions and obtain closed formulas for the even and odd fractional Legendre functions. In Section 5, we illustrate the convergence of the fractional Legendre functions to the well known Legendre polynomials as the fractional derivative α approaches 1. Finally, we close up with some concluding remarks in Section 6.

2 The Infinite Series Solution

In this section we obtain the series solution of Eq. (1.1). We first rewrite Eq. (1.1) as

$$D_{0^+}^{2\alpha}y + p(x)D_{0^+}^{\alpha}y + q(x)y = 0,\ \ \frac{1}{2} < \alpha < 1, \tag{2.1}$$

where $p(x) = -\frac{2\alpha x^{\alpha}}{1-x^{2\alpha}}$ and $q(x) = \frac{\ell(\ell+1)}{1-x^{2\alpha}}$. Since p and q are analytic at the origin with radius of convergence 1, then $x=0$ is an ordinary point of Eq. (1.1). We expand the solution in a power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n\alpha}.$$

As the fractional derivative $D_{0^+}^{\alpha}(c) = 0$, for c being constant, we have

$$D_{0^+}^{\alpha}y = \sum_{n=1}^{\infty} a_n \frac{\Gamma(n\alpha+1)}{\Gamma(n\alpha+1-\alpha)} x^{n\alpha-\alpha},$$

$$D_{0^+}^{2\alpha}y = \sum_{n=2}^{\infty} a_n \frac{\Gamma(n\alpha+1)}{\Gamma((n-2)\alpha+1)} x^{(n-2)\alpha},$$

$$-x^{2\alpha}D_{0^+}^{2\alpha}y = -\sum_{n=2}^{\infty} a_n \frac{\Gamma(n\alpha+1)}{\Gamma((n-2)\alpha+1)} x^{n\alpha},$$

$$(1-x^{2\alpha})D_{0^+}^{2\alpha}y = \sum_{n=2}^{\infty} a_n \frac{\Gamma(n\alpha+1)}{\Gamma((n-2)\alpha+1)} x^{(n-2)\alpha} - \sum_{n=2}^{\infty} a_n \frac{\Gamma(n\alpha+1)}{\Gamma((n-2)\alpha+1)} x^{n\alpha},$$

and

$$x^{\alpha}D_{0^+}^{\alpha}y = \sum_{n=1}^{\infty} a_n \frac{\Gamma(n\alpha+1)}{\Gamma(n\alpha+1-\alpha)} x^{n\alpha}.$$

Substituting in Eq. (1.1) yields

$$\begin{aligned}
0 =& \sum_{n=2}^{\infty} a_n \frac{\Gamma(n\alpha+1)}{\Gamma((n-2)\alpha+1)} x^{(n-2)\alpha} - \sum_{n=2}^{\infty} a_n \frac{\Gamma(n\alpha+1)}{\Gamma((n-2)\alpha+1)} x^{n\alpha} \\
&-2\alpha \sum_{n=1}^{\infty} a_n \frac{\Gamma(n\alpha+1)}{\Gamma(n\alpha+1-\alpha)} x^{n\alpha} + \ell(\ell+1)\sum_{n=0}^{\infty} a_n x^{n\alpha} \\
=& \sum_{n=0}^{\infty} a_{n+2} \frac{\Gamma((n+2)\alpha+1)}{\Gamma(n\alpha+1)} x^{n\alpha} - \sum_{n=2}^{\infty} a_n \frac{\Gamma(n\alpha+1)}{\Gamma((n-2)\alpha+1)} x^{n\alpha} \\
&-2\alpha \sum_{n=1}^{\infty} a_n \frac{\Gamma(n\alpha+1)}{\Gamma(n\alpha+1-\alpha)} x^{n\alpha} + \ell(\ell+1)\sum_{n=0}^{\infty} a_n x^{n\alpha}.
\end{aligned}$$

Equating the last equation we have

$$\Gamma(2\alpha+1)a_2 + \ell(\ell+1)a_0 = 0,$$

and

$$\frac{\Gamma(3\alpha+1)}{\Gamma(\alpha+1)} a_3 - 2\alpha\Gamma(\alpha+1)a_1 + \ell(\ell+1)a_1 = 0,$$

which implies

$$a_2 = -\frac{\ell(\ell+1)}{\Gamma(2\alpha+1)}a_0, \tag{2.2}$$

$$a_3 = \Gamma(\alpha+1)\frac{2\alpha\Gamma(\alpha+1)-\ell(\ell+1)}{\Gamma(3\alpha+1)}a_1, \tag{2.3}$$

and

$$a_{n+2} = \frac{\Gamma(n\alpha+1)}{\Gamma((n+2)\alpha+1)}\left(\Gamma(n\alpha+1)[\frac{1}{\Gamma((n-2)\alpha+1)}+\frac{2\alpha}{\Gamma((n-1)\alpha+1)}]-\ell(\ell+1)\right)a_n,\ n\geq 2. \tag{2.4}$$

Since the odd terms of the series depend on a_1 and the even terms depend on a_0, the general solution to the fractional Legendre equation is

$$y = c_1y_1+c_2y_2 = c_1\sum_{n=0}^{\infty}a_{2n+1}x^{(2n+1)\alpha}+c_2\sum_{n=0}^{\infty}a_{2n}x^{2n\alpha}. \tag{2.5}$$

It is worth to mention that for $\alpha = 1$, Equations (2.2-2.4) reduce to

$$a_2 = -\frac{\ell(\ell+1)}{2}a_0,\ a_3 = -\frac{(\ell+2)(\ell-1)}{3!}a_1$$

and

$$a_{n+2} = -\frac{(\ell+n+1)(\ell-n)}{(n+2)(n+1)}a_n,$$

which gives the power series expansion of the ordinary Legendre equation.

3 Radius of Convergence

In this section we show that the infinite series solution (2.5) obtained in the previous section converges absolutely with radius of convergence $\rho = 1$. Since the terms of the series goes up by steps of two in n, applying the D'Alembert Ratio test we have

$$R_n = \frac{|a_{n+2}\ x^{(n+2)\alpha}|}{|a_n\ x^{n\alpha}|} = \frac{|a_{n+2}|}{|a_n|}x^{2\alpha},\ \ x>0.$$

In the following we prove that

$$\lim_{n\to\infty}\frac{|a_{n+2}|}{|a_n|} = 1,$$

and thus the series converges absolutely for $0<x<1$.
We have

$$\frac{a_{n+2}}{a_n} = x_n(z_n-\frac{\ell(\ell+1)}{\Gamma(n\alpha+1)}),$$

where

$$x_n = \frac{\Gamma^2(n\alpha+1)}{\Gamma((n+2)\alpha+1)},$$

and

$$z_n = \frac{1}{\Gamma((n-2)\alpha+1)}+\frac{2\alpha}{\Gamma((n-1)\alpha+1)}.$$

For large n we have $\frac{a_{n+2}}{a_n}\sim x_nz_n$. We apply the Stirling's formula

$$\Gamma(n)\sim\sqrt{2\pi}e^{-n}n^{n-1/2},$$

for large n to prove that $x_n z_n \sim 1$. We have

$$
\begin{aligned}
x_n &\sim \frac{\left(\sqrt{2\pi}e^{-n\alpha-1}[n\alpha+1]^{n\alpha+1/2}\right)^2}{\sqrt{2\pi}e^{-(n+2)\alpha-1}\left((n+2)\alpha+1\right)^{(n+2)\alpha+1/2}} \\
&\sim \frac{\sqrt{2\pi}e^{-n\alpha+2\alpha-1}\left(n\alpha\right)^{2n\alpha+1}\left(1+\frac{1}{n\alpha}\right)^{2n\alpha+1}}{\left((n+2)\alpha\right)^{(n+2)\alpha+1/2}\left(1+\frac{1}{(n+2)\alpha}\right)^{(n+2)\alpha+1/2}} \\
&\sim \frac{\sqrt{2\pi}e^{-n\alpha+2\alpha-1}\left(n\alpha\right)^{2n\alpha+1}e^2}{\left((n+2)\alpha\right)^{(n+2)\alpha+1/2}e} \\
&\sim \frac{\sqrt{2\pi}e^{-(n-2)\alpha}\left(n\alpha\right)^{n\alpha+1/2}}{\left(1+\frac{2}{n}\right)^{n\alpha+1/2}\left((n+2)\alpha\right)^{2\alpha}} \\
&\sim \frac{\sqrt{2\pi}e^{-(n-2)\alpha}\left(n\alpha\right)^{n\alpha+1/2}}{e^{2\alpha}\left((n+2)\alpha\right)^{2\alpha}} \\
&\sim \frac{\sqrt{2\pi}e^{-n\alpha}\left(n\alpha\right)^{n\alpha+1/2}}{\left(n+2\right)^{2\alpha}\alpha^{2\alpha}},
\end{aligned}
$$

and

$$
\begin{aligned}
z_n &\sim \frac{1}{\sqrt{2\pi}}\left(\frac{e^{(n-2)\alpha+1}}{((n-2)\alpha+1)^{(n-2)\alpha+1/2}}+\frac{2\alpha e^{(n-1)\alpha+1}}{((n-1)\alpha+1)^{(n-1)\alpha+1/2}}\right) \\
&\sim \frac{e^{(n-2)\alpha+1}}{\sqrt{2\pi}}\left(\frac{1}{\left((n-2)\alpha\right)^{(n-2)\alpha+1/2}\left(1+\frac{1}{(n-2)\alpha}\right)^{(n-2)\alpha+1/2}}\right. \\
&\quad \left.+\frac{2\alpha e^{\alpha}}{\left((n-1)\alpha\right)^{(n-1)\alpha+1/2}\left(1+\frac{1}{(n-1)\alpha}\right)^{(n-1)\alpha+1/2}}\right) \\
&\sim \frac{e^{(n-2)\alpha+1}}{\sqrt{2\pi}}\left(\frac{1}{\left((n-2)\alpha\right)^{(n-2)\alpha+1/2}e}+\frac{2\alpha e^{\alpha}}{\left((n-1)\alpha\right)^{(n-1)\alpha+1/2}e}\right) \\
&\sim \frac{e^{(n-2)\alpha}}{\sqrt{2\pi}\left((n-2)\alpha\right)^{(n-2)\alpha+1/2}}\left(1+\frac{2\alpha e^{\alpha}}{\left((n-1)\alpha\right)^{\alpha}\left(1+\frac{1}{n-2}\right)^{(n-1)\alpha+1/2}}\right) \\
&\sim \frac{e^{(n-2)\alpha}}{\sqrt{2\pi}\left((n-2)\alpha\right)^{(n-2)\alpha+1/2}}\left(1+\frac{2\alpha e^{\alpha}}{\left((n-1)\alpha\right)^{\alpha}e^{\alpha}}\right) \\
&\sim \frac{e^{(n-2)\alpha}}{\sqrt{2\pi}\left((n-2)\alpha\right)^{(n-2)\alpha+1/2}}\left(1+\frac{2\alpha}{\left((n-1)\alpha\right)^{\alpha}}\right) \\
&\sim \frac{e^{(n-2)\alpha}}{\sqrt{2\pi}\left((n-2)\alpha\right)^{(n-2)\alpha+1/2}}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
x_n z_n &\sim \frac{\sqrt{2\pi}e^{-n\alpha}\left(n\alpha\right)^{n\alpha+1/2}}{\left(n+2\right)^{2\alpha}\alpha^{2\alpha}}\,\frac{e^{(n-2)\alpha}}{\sqrt{2\pi}\left((n-2)\alpha\right)^{(n-2)\alpha+1/2}} \\
&\sim \frac{e^{-2\alpha}}{\left(n+2\right)^{2\alpha}\alpha^{2\alpha}\left(1-\frac{2}{n}\right)^{n\alpha+1/2}\left(n-2\right)^{-2\alpha}\alpha^{-2\alpha}} \\
&\sim \left(\frac{n-2}{n+2}\right)^{2\alpha}\sim\left(1-\frac{4}{n+2}\right)^{2\alpha}\sim 1,
\end{aligned}
\tag{3.1}
$$

which proves the result. In the above analysis we used the well known fact that for large n

$$(1+\frac{a}{n})^n \sim e^a.$$

4 The Fractional Legendre Functions

For certain values of ℓ, one of the two infinite series in (2.5) will truncate to obtain a finite sum, the fractional Legendre functions. In Eq. (2.4), let

$$\beta_\alpha^n = \frac{\Gamma(n\alpha+1)}{\Gamma((n+2)\alpha+1)}, \tag{4.1}$$

$$s_\alpha^n = \Gamma(n\alpha+1)\left(\frac{1}{\Gamma((n-2)\alpha+1)} + \frac{2\alpha}{\Gamma((n-1)\alpha+1)}\right), \tag{4.2}$$

then

$$a_{n+2} = \beta_\alpha^n\left(s_\alpha^n - \ell(\ell+1)\right)a_n, \ \ n \geq 2. \tag{4.3}$$

If we choose ℓ such that

$$\ell(\ell+1) = s_\alpha^m,$$

for certain m, then

$$a_{m+2k} = 0, \ \ \text{for } k \geq 1,$$

and the infinite series will truncate to obtain the finite sum

$$P_\alpha^m = \sum_{n=0}^{m} a_n x^{n\alpha},$$

the fractional Legendre function of degree $m\alpha$. In the following we derive a general formula for obtaining P_α^m for m being even and odd.

Proposition 1.*The following holds true*

$$\prod_{j=1}^{n} \beta_\alpha^{2j} = \frac{\Gamma(2\alpha+1)}{\Gamma(2(n+1)\alpha+1)}, \tag{4.4}$$

$$\prod_{j=2}^{n} \beta_\alpha^{2j-1} = \frac{\Gamma(3\alpha+1)}{\Gamma((2n+1)\alpha+1)}, \tag{4.5}$$

where β_α^n and s_α^n are defined in Eq.'s (4.1-4.2).

*Proof.*We have

$$\begin{aligned}\prod_{j=1}^{n} \beta_\alpha^{2j} &= \beta_\alpha^2\beta_\alpha^4\beta_\alpha^6\cdots\beta_\alpha^{2(n-1)}\beta_\alpha^{2n} \\ &= \frac{\Gamma(2\alpha+1)}{\Gamma(4\alpha+1)}\frac{\Gamma(4\alpha+1)}{\Gamma(6\alpha+1)}\frac{\Gamma(6\alpha+1)}{\Gamma(8\alpha+1)}\cdots\frac{\Gamma(2(n-2)\alpha+1)}{\Gamma(2n\alpha+1)}\frac{\Gamma(2n\alpha+1)}{\Gamma((2n+2)\alpha+1)} \\ &= \frac{\Gamma(2\alpha+1)}{\Gamma(2(n+1)\alpha+1)}.\end{aligned}$$

The proof of the result in Eq. (4.5) is analogous.

Theorem 1.*The fractional Legendre function of order $2m$ is given by*

$$P_\alpha^{2m} = 1 - \frac{s_\alpha^{2m}}{\Gamma(2\alpha+1)}x^{2\alpha} - s_\alpha^{2m}\sum_{n=2}^{m}\frac{x^{2n\alpha}}{\Gamma(2n\alpha+1)}\prod_{j=1}^{n-1}(s_\alpha^{2j} - s_\alpha^{2m}). \tag{4.6}$$

Proof. We have

$$P_\alpha^{2m} = a_0 + a_2 x^{2\alpha} + \sum_{n=2}^{m} a_{2n} x^{2n\alpha}.$$

For $n \geq 2$, it holds that

$$\begin{aligned}
a_{2n} &= \beta_\alpha^{2n-2}(s_\alpha^{2n-2} - s_\alpha^{2m}) a_{2n-2} \\
&= \underbrace{\beta_\alpha^{2n-2}\beta_\alpha^{2n-4}\cdots\beta_\alpha^{2}}_{n-1}\underbrace{(s_\alpha^{2n-2} - s_\alpha^{2m})(s_\alpha^{2n-4} - s_\alpha^{2m})\cdots(s_\alpha^{2} - s_\alpha^{2m})}_{n-1} a_2 \\
&= a_2 \prod_{j=1}^{n-1} \beta_\alpha^{2j}(s_\alpha^{2j} - s_\alpha^{2m}) \\
&= -\frac{s_\alpha^{2m}}{\Gamma(2\alpha+1)} \frac{\Gamma(2\alpha+1)}{\Gamma(2n\alpha+1)} a_1 \prod_{j=1}^{n-1} (s_\alpha^{2j} - s_\alpha^{2m}) \\
&= -\frac{s_\alpha^{2m}}{\Gamma(2n\alpha+1)} a_1 \prod_{j=1}^{n-1} (s_\alpha^{2j} - s_\alpha^{2m})
\end{aligned}$$

which proves the result.

Theorem 2. *The fractional Legendre function of order $2m+1$ is given by*

$$P_\alpha^{2m+1} = x^\alpha + \frac{w_\alpha}{\Gamma(3\alpha+1)} x^{3\alpha} + w_\alpha \sum_{n=2}^{m} \frac{x^{(2n+1)\alpha}}{\Gamma((2n+1)\alpha+1)} \prod_{j=2}^{n} (s_\alpha^{2j-1} - s_\alpha^{2m+1}), \tag{4.7}$$

where s_α^n is defined in Eq. (4.2) and

$$w_\alpha = \Gamma(\alpha+1)\Big(2\alpha\Gamma(\alpha+1) - s_\alpha^{2m+1}\Big).$$

Proof. We have

$$\begin{aligned}
P_\alpha^{2m+1} &= a_1 x^\alpha + a_3 x^{3\alpha} + \sum_{n=2}^{m} a_{2n+1} x^{(2n+1)\alpha}, \\
&= a_1 x^\alpha + a_1 \frac{w_\alpha}{\Gamma(3\alpha+1)} x^{3\alpha} + \sum_{n=2}^{m} a_{2n+1} x^{(2n+1)\alpha}
\end{aligned}$$

For $n \geq 2$, it holds that

$$\begin{aligned}
a_{2n+1} &= \beta_\alpha^{2n-1}(s_\alpha^{2n-1} - s_\alpha^{2m+1}) a_{2n-1} \\
&= \underbrace{\beta_\alpha^{2n-1}\beta_\alpha^{2n-3}\cdots\beta_\alpha^{3}}_{n-1}\underbrace{(s_\alpha^{2n-1} - s_\alpha^{2m+1})(s_\alpha^{2n-3} - s_\alpha^{2m+1})\cdots(s_\alpha^{3} - s_\alpha^{2m+1})}_{n-1} a_3 \\
&= a_3 \prod_{j=2}^{n} \beta_\alpha^{2j-1}(s_\alpha^{2j-1} - s_\alpha^{2m+1}) \\
&= \frac{\Gamma(3\alpha+1)}{\Gamma((2n+1)\alpha+1)} \frac{w_\alpha}{\Gamma(3\alpha+1)} a_1 \prod_{j=2}^{n} (s_\alpha^{2j-1} - s_\alpha^{2m+1}) \\
&= \frac{w_\alpha}{\Gamma((2n+1)\alpha+1)} a_1 \prod_{j=2}^{n} (s_\alpha^{2j-1} - s_\alpha^{2m+1}),
\end{aligned}$$

which proves the result.

Applying the above formulas for $m = 1$, we have

$$P_\alpha^2 = 1 - \frac{\Gamma(\alpha)+2}{\Gamma(\alpha)} x^{2\alpha}, \tag{4.8}$$

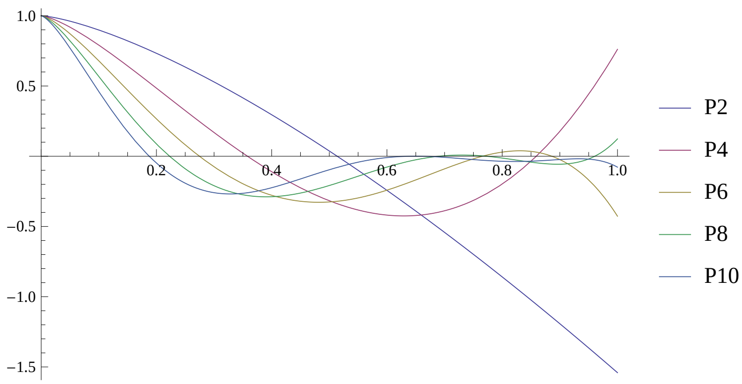

Fig. 1 A plot of $P^2, P^4, P^6, P^8, P^{10}$ for $\alpha = 0.7$

and

$$P_\alpha^3 = x^\alpha - \Gamma(\alpha+1)\frac{2\alpha\Gamma(\alpha+1) - \Gamma(3\alpha+1)[\frac{1}{\Gamma(\alpha)+1} + \frac{1}{\Gamma(2\alpha)}]}{\Gamma(3\alpha+1)}x^{3\alpha}. \tag{4.9}$$

As the order of the fractional derivative α approaches 1, we have

$$P_1^2 = 1 - 3x^2,$$

and

$$P_1^3 = x - \frac{5}{3}x^3, \tag{4.10}$$

which are constant multiples of the Legendre polynomials of order 2 and 3, respectively. Figures 1-4 depict the even and odd fractional Legendre functions for several values of α.

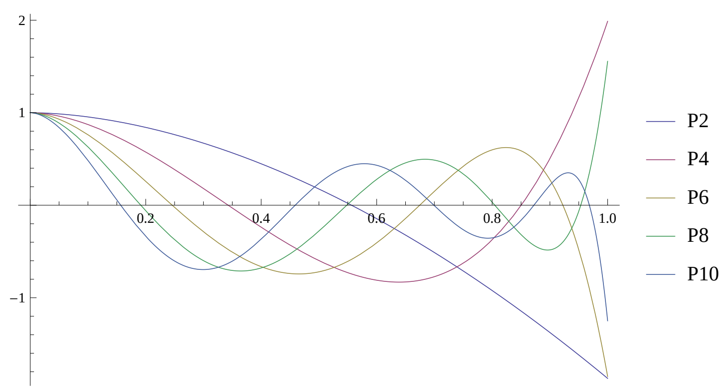

Fig. 2 A plot of $P^2, P^4, P^6, P^8, P^{10}$ for $\alpha = 0.9$

5 Comparison with the Legendre Polynomials

In the following we illustrate that the even and odd Legendre functions obtained in Eq.'s (4.6-4.7) converge to the odd and even Legendre polynomials as the fractional derivative α approaches 1. For $\alpha = 1$, we have $s_\alpha^n = n(n+1)$ and

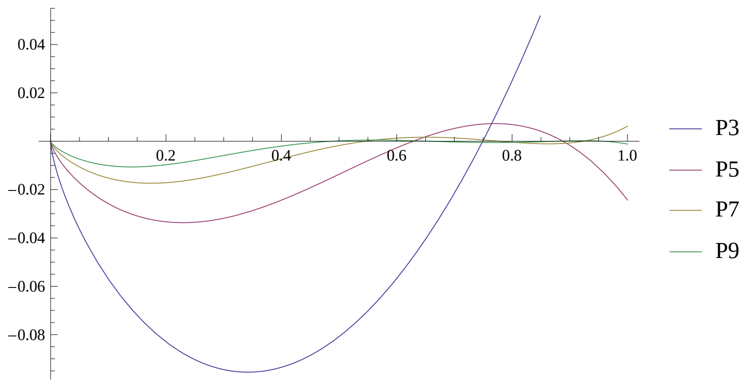

Fig. 3 A plot of P^3, P^5, P^7, P^9 for $\alpha = 0.7$

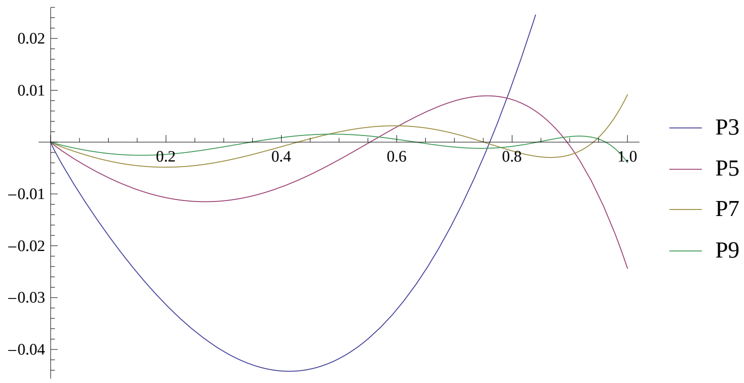

Fig. 4 A plot of P^3, P^5, P^7, P^9 for $\alpha = 0.9$

$w_\alpha = 2-(2m+1)(2m+1) = -4m(m+1)$. Substituting in Eq.'s (4.6-4.7) yields

$$\begin{aligned} P^{2m} &= 1-m(2m+1)x^2 - 2m(2m+1)\sum_{n=2}^{m}\frac{x^{2n}}{(2n)!}\prod_{j=1}^{n-1}(2j(2j+1)-2m(2m+1)), \\ &= 1-m(2m+1)\left(x^2-\sum_{n=2}^{m}\frac{2^{n-1}x^{2n}}{(2n)!}\prod_{j=1}^{n-1}(j(2j+1)-m(2m+1))\right), \end{aligned} \tag{5.1}$$

and

$$\begin{aligned} P^{2m+1} &= x-2m(2m+3)\left(\frac{x^3}{6}+\sum_{n=2}^{m}\frac{x^{2n+1}}{(2n+1)!}\prod_{j=2}^{n}(2j(2j-1)-(2m+1)(2m+2))\right), \\ &= x-m(2m+3)\left(\frac{x^3}{3}+\sum_{n=2}^{m}\frac{2^n x^{2n+1}}{(2n+1)!}\prod_{j=2}^{n}(j(2j-1)-(2m+1)(m+1))\right), \end{aligned} \tag{5.2}$$

and we obtain new formulas for the Legendre polynomials. Applying the above formulas we have

$$p^4 = 1 - 10x^2 + \frac{35}{3}x^4$$
$$P^5 = x - \frac{14}{3}x^3 + \frac{21}{5}x^5$$
$$P^6 = 1 - 21x^2 + 63x^4 - \frac{462}{10}x^6$$
$$p^7 = x - 9x^3 + \frac{99}{5}x^5 - \frac{429}{35}x^7$$
$$p^8 = 1 - 36x^2 + 198x^4 - \frac{1716}{5}x^6 + \frac{1287}{7}x^8$$
$$p^9 = x - \frac{44}{3}x^3 + \frac{286}{5}x^5 - \frac{572}{7}x^7 + \frac{2431}{63}x^9$$

which are, up to constants, the Legendre polynomials of degrees 4 to 9.

6 Conclusion

In this paper we have proposed a fractional generalization of the well-known Legendre equation. We obtained its solution in the form of a power series that converges absolutely for $0 < x < 1$. As the fractional derivative α approaches 1, the obtained power series solution coincides with the one for the Legendre equation. For certain non integer values of ℓ, the power series expansion truncates to obtain the fractional Legendre functions. We obtained closed forms of the odd and even fractional Legendre functions. These functions generalize the Legendre polynomials. As the fractional derivative α approaches 1, a new interesting formula of the Legendre polynomials is obtained.

References

[1] J. H. He, Some applications of nonlinear fractional differential equations and their approximations, *Bull. Sci. Technol.* **15**(2), 86-90 (1999).

[2] F. Mainardi, Fractional calculus: Some basic problems in continum and statistical mechanics, in Fractals Fractional calculus in Continuum Mechanics, A. Carpinteri and F. Mainardi, Eds., New York: Springer-Verlag, 291-348,1997.

[3] G. O. Young, Definition of physical consistent damping laws with fractional derivatives, *Z. Angrew. Math. Mech.* **75**, 623-635 (1995).

[4] Q. Al-Mdallal, M. Syam and M. N. Anwar, A collocation-shooting method for solving fractional boundary value problems, *Commun. Nonlin. Sci. Numer. Sim.*, **15**(12), 3814–3822 (2010).

[5] Q. Al-Mdallal and M. A. Hajji, A convergent algorithm for solving higher-order nonlinear fractional boundary value problems, *Fract. Calc. Appl. Anal.* **18**(6), 1423-1440(2015).

[6] M. Al-Refai and M. A. Hajji, Monotone iterative sequences for nonlinear boundary value problems of fractional order, *Nonlin. Anal. Ser. A Theor. Meth. Appl.* **74**(11), 3531-3539 (2011).

[7] M. Al-Refai, Basic results on nonlinear eigenvalue problems of fractional order, *Electr. J. Differ. Equ.* **2012**(191), 1-12 (2012).

[8] M. Al-Refai and Yu. Luchko, Maximum principle for the fractional diffusion equations with the Riemann-Liouville fractional derivative and its applications, *Fract. Calc. Appl. Anal.* **17**(2), 483-498(2014).

[9] M. Al-Refai, M. Hajji and M. Syam, An efficient series solution for fractional differential equations, *Abstr. Appl. Anal.* **2014**, http://dx.doi.org/10.1155/2014/891837.

[10] A. H. Bhrawy, E. H. Doha, D. Baleanu and S. S. Ezz-Eldien, A spectral tau algorithm based on Jacobi operational matrix for numerical solution of time fractional diffusion-wave equations, *J. Comput. Phys.* **293**, 142-156 (2015).

[11] M. Syam and M. Al-Refai, Solving fractional diffusion equation via the collocation method based on fractional Legendre functions, *J. Comput. Meth. Phys.* **2014**, http://dx.doi.org/10.1155/2014/381074, (2014).

[12] G. C. Wu and D. Baleanu, Variational iteration method for the Burgers flow with fractional derivativesnew Lagrange multipliers, *Appl. Math. Modell.* **37**(9), 6183-6190 (2013).

[13] V. Kiryakova, The special functions of fractional calculus as a generalized fractional calculus operators of some basic functions, *Comput. Math. Appl.* **59**, 1128-1141 (2010).

[14] H. Srivastava, R. K. Raina and X. J. Yang, *Special Functions in Fractional Calculus and Related Fractional Differintegral Equations*, World Scientific Publishing Company, 2015.

[15] W. Okrasinski and L. Plociniczak, A note of fractional Bessel equation and its asymptotics, *Fract. Calc. Appl. Anal.* **16**(3), 559-572 (2013).
[16] W. Okrasinski and L. Plociniczak, On fractional Bessel equation and the description of corneal topography, arXiv:1201.2526v2 [math.CA], (2012).
[17] M. A. Hammad and R. Khalil, Legendre fractional differential equation and Legendre fractional polynomials, *Int. J. Appl. Math. Res.* **3**(3), 214-219 (2014).
[18] I. Podlubny, *Fractional Differential Equations*, Academic Press, New York, 1999.