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Recurrence Relation of Single and Product Moments of Order Statistics from Additive Weibull Distribution

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Abstract: In this paper we study the sampling distribution of order statistics of the additive Weibull distribution. We consider the single and product moment of order statistics and establish some recurrence relations for single and product moments of order statistics. These expressions are used to calculate the mean, variances, kurtosis, skewness and other statistical measures of additive Weibull distribution.

Keywords: Additive Weibull distribution, order statistics, single and product moment of order statistics.

1 Introduction

Order statistics have been used in wide range of problems, including robust statistical estimation and detection of outliers, characterization of probability distributions, goodness of fit-tests, quality control, analysis of censored sample. The subject of order statistics deals with the properties and applications of these ordered random variable and of functions involving them (see David and Nagaraja [7], Tahir et al [16]). Asymptotic theory of extremes and related developments of order statistics are well described in an applausive work of Galambos [10] and the references therein.

For improved form of these results, Samuel and Thomes [15], Arnold et al. [4] have reviewed many recurrence relations and identities for the moments of order statistics arising from several specific continuous distributions such as normal, Cauchy, logistic, gamma and exponential. Recurrence relations for the expected values of certain functions of two order statistics have been considered by Ali and Khan [3]. The use of recurrence relations for the moments of order statistics is quite well known in statistical literature (see for example Arnold et al., [4], Malik et al. [14]). More recently Dar and Abdullah [9], [12], Dar et.al [8] study the sampling distribution of order statistics of some well-known life time models and derived the exact analytical expressions of entropy, residual entropy and past residual entropy for order statistics of Lomax distribution and Islam-Mukherjee distribution.

Xie and Lai [17] proposed the additive Weibull model based on the simple idea of combining the failure rates of two Weibull distribution, one has a decreasing failure rate and the other one has an increasing failure rate. We say that a random variable X with range of values $(0,\infty)$ has the Additive Weibull distribution (now onwards AWD) if its pdf is given by

$$f(x)=\left(\alpha\theta x^{\theta-1}+\gamma\beta x^{\beta-1}\right)e^{-\alpha x^{\theta}-\gamma x^{\beta}},x>0 \tag{1.1}$$

where $\alpha,\gamma>0$ are scale parameters and $\theta>\beta>0$ are shape parameters.

The cumulative distribution function (cdf) and survival function (sf) associated with (1.1) is given by

$$F(x)=1-e^{-\alpha x^{\theta}-\gamma x^{\beta}} \tag{1.2}$$

$$\bar{F}(x)=e^{-\alpha x^{\theta}-\gamma x^{\beta}}, \tag{1.3}$$

respectively.

The following functional relationship exists between p.d.f and c.d.f of AWD :

$$f(x)=\left(\alpha\theta x^{\theta-1}+\gamma\beta x^{\beta-1}\right)\left(1-F(x)\right). \tag{1.4}$$

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2 Distribution of Order Statistics

Let $X_1, X_2, \ldots., X_n$ be a random sample of size n from the AWD and let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ denotes the corresponding order statistics. Then the pdf of $X_{r:n}, 1 \leq r \leq n$, is given by [see David and Nagaraja [7] and Arnold et al. [4]]

$$f_{r:n}(x) = C_{r:n}\{[F(x)]^{r-1}[1-F(x)]^{n-r}f(x)\}, 0 < x < \infty, \tag{2.1}$$

where $C_{r:n} = \frac{n!}{(r-1)!(n-r)!}$.

The probability density function of smallest $(r = 1)$ and largest $(r = n)$ order statistics can be easily obtained from (2.1) and is given by $f_{(1,n)}(x) = n[1-F(x)]^{n-1}f(x)$ and $f_{(n,n)}(x) = n[F(x)]^{n-1}f(x)$ respectively.

Also, the cumulative distribution function of the largest and smallest order statistics is given by $F_{(n,n)}(x) = [F(x)]^n$ and $F_{(1,n)}(x) = 1-[1-F(x)]^n$ respectively

Using (1.1), (1.2) and taking $r = 1$ in (2.1), yields the pdf of the minimum order statistics for the AWD

$$f_{1:n}(x) = n\left(\alpha\theta x^{\theta-1} + \gamma\beta x^{\beta-1}\right)e^{-n\alpha x^{\theta} - n\gamma x^{\beta}}.$$

Similarly using (1.1), (1.2) and taking $r = n$ in (2.1), yields the pdf of the largest order statistics for the AWD

$$f_{n:n}(x) = n\left(\alpha\theta x^{\theta-1} + \gamma\beta x^{\beta-1}\right)\sum_{i=0}^{n-1}\binom{n-1}{i}(-1)^i e^{-\alpha x^{\theta} - \gamma x^{\beta}(1+i)}.$$

The joint pdf of $X_{r:n}$ and $X_{s:n}$ for $1 \leq r < s \leq n$ is given by [see Arnold et al. [4 et.al]

$$f_{r,s:n}(x) = C_{r,s:n}\{[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}f(x)f(y)\} \tag{2.2}$$

for $-\infty < x < y < \infty$ and $C_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}$.

Theorem 2.1: Let $F(x)$ and $f(x)$ be the cdf and pdf of the AWD. Then the density function of the r^{th} order statistics say $f_{r:n}(x)$ is given by

$$f_{r:n}(x) = C_{r:n}\left(\alpha\theta x^{\theta-1} + \gamma\beta x^{\beta-1}\right)\sum_{i=0}^{n-r+1}\sum_{j=0}^{r+i-1}\binom{n-r+1}{i}\binom{r+i-1}{j}(-1)^{i+j}e^{-j(\alpha x^{\theta} + \gamma x^{\beta})} \tag{2.3}$$

Proof: Substituting (1.4) in to (2.1), we get

$$f_{r:n}(x) = C_{r:n}\left(\alpha\theta x^{\theta-1} + \gamma\beta x^{\beta-1}\right)\sum_{i=0}^{n-r+1}\binom{n-r+1}{i}(-1)^i\,[F(x)]^{r+i-1}. \tag{2.4}$$

The proof follows by substituting (1.2) into (2.4).

Theorem 2.2: Let $X_{r:n}$ and $X_{s:n}$ for $1 \leq r < s \leq n$ be the r^{th} and s^{th} order statistics from the AWD. Then the joint pdf of $X_{r:n}$ and $X_{s:n}$ is given by

$$f_{r;s:n}(x) = C_{r;s:n}\left(\alpha\theta y^{\theta-1} + \gamma\beta y^{\beta-1}\right) \times \sum_{i=0}^{s-r-1}\sum_{j=0}^{n-s+1}\binom{s-r-1}{i}\binom{n-s+1}{j}(-1)^{i+j}[F(y)]^{s-r-1-i+j}[F(x)]^{r-1+i}.$$

Proof: Equation (2.2) can be written as

$$f_{r;s:n}(x) = C_{r:n}\sum_{i=0}^{s-r-1}\sum_{j=0}^{n-s}\binom{s-r-1}{i}\binom{n-s}{j}(-1)^{i+j}[F(y)]^{s-r-1-i+j} \times [F(x)]^{r+i-1}f(x)f(y). \tag{2.5}$$

The proof follows by substituting (1.4) into (2.5).

3 Single and Product Moments

In this section, we derive explicit expressions for both of the single and product moments of order statistics from the AWD.

Theorem 3.1: Let $X_1, X_2, \dots, X_n$ be a random sample of size n from the AWD and let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ denote the corresponding order statistics. Then the k^{th} moments of the rth order statistics for $k = 1, 2, \dots$ denoted by $\mu_{r:n}^{(k)}$ is given by

$$\mu_{r:n}^{(k)} = \alpha C_{r:n} \frac{\sum_{i=0}^{n-r+1} \sum_{j=0}^{r+i-1} \sum_{m=0}^{\infty} \binom{n-r+1}{i} \binom{r+i-1}{j} (\gamma j)^m}{m!} (-1)^{i+j+m} \frac{\Gamma\left(\frac{k+\beta m}{\theta} + 1\right)}{(\alpha j)^{\frac{k+\beta m}{\theta}+1}}$$

$$+\gamma C_{r:n} \frac{\sum_{i=0}^{n-r+1} \sum_{j=0}^{r+i-1} \sum_{m=0}^{\infty} \binom{n-r+1}{i} \binom{r+i-1}{j} (\alpha j)^l}{l!} (-1)^{i+j+l} \frac{\Gamma\left(\frac{k+l\theta}{\beta} + 1\right)}{(\gamma j)^{\frac{k+l\theta}{\beta}+1}},$$

where Γ is a gamma function.

Proof: We have

$$\mu_{r:n}^{(k)} = \int_0^{\infty} x^k f_{r:n}(x) dx$$

Using (2.1) and then (1.4), we get

$$\mu_{r:n}^{(k)} = \alpha\theta C_{r:n} I_1 + \gamma\beta C_{r:n} I_2, \tag{3.1}$$

where

$$I_1 = \int_0^{\infty} x^{k+\theta-1} [F(x)]^{r-1} [1 - F(x)]^{n-r+1} dx \tag{3.2}$$

and

$$I_2 = \int_0^{\infty} x^{k+\beta-1} [F(x)]^{r-1} [1 - F(x)]^{n-r+1} dx \tag{3.3}$$

Now simplifying (3.2), (3.3) and then substituting I_1 and I_2 into (3.1), we get the desired result

Remark 3.1: The k^{th}moment for smallest order statistics i.e $r = 1$ is given by

$$\mu_{1:n}^{(k)} = \alpha n \frac{\sum_{m=0}^{\infty} \gamma^m n^m}{m!} (-1)^m \frac{\Gamma\left(\frac{k+\beta m}{\theta} + 1\right)}{(\alpha n)^{\frac{k+\beta m}{\theta}+1}} + \gamma n \frac{\sum_{l=0}^{\infty} \alpha^l n^l}{l!} (-1)^l \frac{\Gamma\left(\frac{k+l\theta}{\beta} + 1\right)}{(\gamma n)^{\frac{k+l\theta}{\beta}+1}}.$$

Taking $k = 1$, one can obtain the mean of smallest order statistics

$$\mu_{1:n} = \alpha n \frac{\sum_{m=0}^{\infty} \gamma^m n^m}{m!} (-1)^m \frac{\Gamma\left(\frac{1+\beta m}{\theta} + 1\right)}{(\alpha n)^{\frac{1+\beta m}{\theta}+1}} + \gamma n \frac{\sum_{l=0}^{\infty} \alpha^l n^l}{l!} (-1)^l \frac{\Gamma\left(\frac{1+l\theta}{\beta} + 1\right)}{(\gamma n)^{\frac{1+l\theta}{\beta}+1}}$$

Also, second order moment of the smallest order statistic can be obtained as

$$\mu_{1:n}^{(2)} = \alpha n \frac{\sum_{m=0}^{\infty} \gamma^m n^m}{m!} (-1)^m \frac{\Gamma\left(\frac{2+\beta m}{\theta} + 1\right)}{(\alpha n)^{\frac{2+\beta m}{\theta}+1}} + \gamma n \frac{\sum_{m=0}^{\infty} \alpha^l n^l}{l!} (-1)^l \frac{\Gamma\left(\frac{2+l\theta}{\beta} + 1\right)}{(\gamma n)^{\frac{2+l\theta}{\beta}+1}}.$$

Therefore the variance of the smallest order statistic can be obtained by using the relation

$$V(X_{1:n}) = \mu_{1:n}^{(2)} - (\mu_{1:n})^2.$$

Remark 3.2: Similarly one can obtain the mean, second order moment and hence variance of the largest order statistics $(r = n)$.

Now we derive recurrence relation for single moments.

Theorem 3.2: Let $X_1, X_2, \dots, X_n$ be a random sample of size n from the AWD and let $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$ denote the corresponding order statistics. Then for $1 \le r \le n$, we have the following moment relation:

$$\mu_{r:n}^{(k)} = (n-r+1)\left\{\frac{\alpha\theta}{k+\theta}\left(\mu_{r:n}^{k+\theta} - \mu_{r-1:n}^{k+\theta}\right) + \frac{\gamma\beta}{k+\beta}\left(\mu_{r:n}^{k+\beta} - \mu_{r-1:n}^{k+\beta}\right)\right\}.$$

Proof: We have

$$\mu_{r:n}^{(k)} = \int_0^\infty x^k f_{r:n}(x)dx$$

$$= C_{r:n}\int_0^\infty x^k [F(x)]^{r-1}[1-F(x)]^{n-r} f(x)dx$$

Using (1.4), we get

$$\mu_{r:n}^{(k)} = \alpha\theta C_{r:n}\left\{\int_0^\infty x^{k+\theta-1}[F(x)]^{r-1}[1-F(x)]^{n-r+1}dx + \gamma\beta C_{r:n}\int_0^\infty x^{k+\beta-1}[F(x)]^{r-1}[1-F(x)]^{n-r+1}dx\right\}.$$

$$\mu_{r:n}^{(k)} = \alpha\theta C_{r:n} I_1 + \gamma\beta C_{r:n} I_2 \quad (3.4)$$

Where,

$$I_1 = \int_0^\infty x^{k+\theta-1}[F(x)]^{r-1}[1-F(x)]^{n-r+1}dx$$

and

$$I_2 = \int_0^\infty x^{k+\beta-1}[F(x)]^{r-1}[1-F(x)]^{n-r+1}dx$$

By using integration by parts, one obtains

$$I_1 = \frac{n-r+1}{k+\theta}\int_0^\infty x^{k+\theta}[F(x)]^{r-1}[1-F(x)]^{n-r} f(x)dx - \frac{r-1}{k+\theta}\int_0^\infty x^{k+\theta}[F(x)]^{r-2}[1-F(x)]^{n-r+1} f(x)dx$$

and

$$I_2 = \frac{n-r+1}{k+\beta}\int_0^\infty x^{k+\beta}[F(x)]^{r-1}[1-F(x)]^{n-r} f(x)dx - \frac{r-1}{k+\beta}\int_0^\infty x^{k+\beta}[F(x)]^{r-2}[1-F(x)]^{n-r+1} f(x)dx$$

Substituting I_1 and I_2 in (3.4), we get the desired result.

Theorem 3.3: For $1 \le r \le s \le n, n \in N$, we have

$$\mu_{r;s:n}^{(k_1,k_2)} = \frac{\alpha\theta(n-s+1)}{k_2+\theta}\left\{\mu_{r;s:n}^{(k_1,k_2+\theta)} - \mu_{r;s-1:n}^{(k_1,k_2+\theta)}\right\} + \frac{\gamma\beta(n-s+1)}{k_2+\beta}\left\{\mu_{r;s:n}^{(k_1,k_2+\beta)} - \mu_{r;s-1:n}^{(k_1,k_2+\beta)}\right\}.$$

Proof: We know that

$$\mu_{r;s:n}^{(k_1,k_2)} = C_{r:s;n}\int_0^\infty\int_x^\infty x^{k_1}y^{k_2}[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s} f(x)f(y)dydx$$

or

$$\mu_{r;s:n}^{(k_1,k_2)} = C_{r:s;n}\int_0^\infty x^{k_1}[F(x)]^{r-1} f(x) I_x dx \quad (3.5)$$

where,

$$I_X = \int_x^\infty y^{k_2}[F(y) - F(x)]^{s-r-1}[1 - F(y)]^{n-s} f(y)dy.$$

Using (1.4), we get

$$I_X = \alpha\theta \int_x^\infty y^{k_2+\theta-1}[F(y) - F(x)]^{s-r-1}[1 - F(y)]^{n-s+1}dy + \gamma\beta \int_x^\infty y^{k_2+\beta-1}[F(y) - F(x)]^{s-r-1}[1 - F(y)]^{n-s+1}dy.$$

Now integrating by parts and then substituting I_X in (3.5), we get the desired result.

4 Conclusions

In this paper we study the sampling distribution from the order statistics of Additive Weibull distribution (AWD). Also we consider the single and product moment of order statistics from AWD and establish recurrence relation for single and product moments of order statistics. These relations are useful in computing mean, variance and other statistical measures of the order statistics of the AWD.

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References

[1] Abdul-Moniem. I B, Abdel-Hameed. H F, On exponentiated Lomax distribution, International J. Math Arch, (3) 2144-2150, 2012

[2] Al-Awadhi. S A, Ghitany. M E, Statistical properties of Poisson-Lomax distribution and its application to repeated accidents data, Journal of Applied Statistical Science, (10) 365-372, 2001

[3] Ali. A A, Khan. A H, Recurrence relations for the expected values of the certain function of two order statistics, Metron, LVI, 107-119, 1998.

[4] Arnold. B C, Blakrishan, N and Nagraja. N. H, A first course in order Statistics, John Wiley and Sons, New York, 1992.

[5] Behboodian. J, Tahmasebi. S, Some properties of entropy for the exponentiated pareto distribution (EPD) based on order statistics, Journal of Mathematical Extension, (3) 43-53, 2008.

[6] Beirlant. J et.al, Non parametric entropy estimation: An overview, Int. J. Math. Statist. Sci., (6) 17-39, 1997.

[7] David. H A, Order Statistics, Second Edition, John Willey and Sons New York, 1981.

[8] Dar. J G, Al-Zahrani. B, Sobhi, M. and Baig. M A K, On Some Order Statistics Properties of the Mukherjee- Islam Distribution, Journal of Mathematical and Computational Science, (6) No.4, 555-567, 2016.

[9] Dar. J G, Hossain. A , Order Statistics Properties of the Two Parameter Lomax Distribution, Pakistan journal of statistics and operation research Vol.XI (2) 181-194, 2015.

[10] Galambos. J D, The asymptotic theory of extreme order statistics, Krieger Publishing Company, Florida, 1987.

[11] Ghitany. M E, Al-Awadhi. F A, Alkhalfan. L A, Marshall-Olkin extended Lomax distribution and its application to censored data, Commun. in Statistics- Theory and Method, (36) 1855-1866, 2007.

[12] Hossain. A , Dar. J G, Some Results On Moment Of Order Statistics For The Quadratic Hazard Rate Distribution, J. Stat. Appl. Pro. 5 (2) 185-201, 2016.

[13] Khan. A H, Yaqub. M , Parvez. S, Recurrence relations between moments of order statistics, Naval. Res. Logist. Quart. (30) 419-441, 1983.

[14] Mailk. H J, Balakrishnan. N, Ahmad. S E, Recurrence relations and identities for moments of order statistics 1: Arbitrary continuous distributions." Commun. Statist.-Theo.Meth., (17)2623-2655, 1998.

[15] Samuel. P, Thomas. P Y, An improved form of a recurrence relation on the product moment of order statistics, Commun. Statist.-Theo.Meth.,(29)1559-1564, 2000.

[16] Tahir. M H et.,al, The Weibull-Lomax distribution: properties and applications, Hacettepe Journal of Mathematics and Statistics, (44) , 455 – 474, 2015.

[17] Xie. M , Lai. C.D, Reliability analysis using an additive Weibull model with bath tube shaped failure rate function, Reliability Engineering and System Safety. (52), No. 1, 87-94, 1996.