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The Odd Burr-III Family of Distributions

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Abstract: In this article, we propose a new family of distributions called *odd Burr-III family of distributions* generated from the logit of Burr-III random variable. We display density and hazard rate plots of four special distributions of this new family and found it very flexible with respect to density and hazard rate shapes. The family density can also be expressed as a linear combination of exponentiated-G densities of the baseline distribution. We obtain some mathematical properties of this new family such as quantile function, moments and incomplete moments, moment generating function, mean deviations, Shannon entropy, stress-strength reliability and the density of order statistics. The model parameters are obtained by employing the method of maximum likelihood. The mathematical properties of a special model of this family, the *odd Burr-III-Lomax (OBIIILx) distribution* are obtained and its usefulness is illustrated for uncensored and censored data sets.

Keywords: Burr-III distribution, generalized family, Lomax distribution, moments, Shannon entropy.

1 Introduction

Statistical distributions are very useful in describing real-world phenomenons. Although many distributions have been developed but there is always a room for new distributions which are either more flexible in term of fitting a specific real-world scenario. This attempt has motivated researchers to seek and develop new flexible distributions. As a result, many new distributions have been developed and studied in literature. From the past several years, there is a growing trend of generating new families of distributions from existing distribution by adding one or more additional parameter(s) to the baseline distribution to study the behavior of the shapes of density and hazard rate, and for checking the goodness-of-fit of proposed distributions.

If $g(x)$, $G(x)$ and $1-G(x)$ are the probability density function, cumulative distribution function and reliability function of the baseline distribution. Then, Eugene et al. (2002) first introduced *beta-G* family from the the logit of beta distribution, and studied beta-normal distribution. Cordeiro and de-Catro (2011) proposed a very flexible generalized family by adding two-additional parameters from the logit of Kumaraswamy distribution. Alexander et al. (2012) extended beta-G family and introduced *McDonald-G* family of distributions. Torabi and Montazeri (2012) used generator $G(x)/[1-G(x)]$ and proposed odd gamma generalized family from the logit of gamma distribution. Bourguignon et al. (2014) also used generator $G(x)/[1-G(x)]$ and introduced *Weibull-G* family of distribution from Weibull distribution logit. Zografos and Balakrishnan (2009) proposed *gamma-G* family using generator $-\log[1-G(x)]$. Ristić and Balakrishnan (2012) introduced another *gamma-G* family from generator $-\log[G(x)]$. Amini et al. (2012) introduced two *log-gamma-G* families from generators $-\log[1-G(x)]$ and $-\log[G(x)]$ with motivation to upper and lower records. Cordeiro et al. (2013) proposed *exponentiated-generalized-G* family of distributions. Alzaatreh et al. (2013) pioneered a very general approach, the *transformed-transformer (T-X)* family. Alzaghal et al. (2013) further extended T-X family and proposed *exponentiated T-X* family of distributions. Aljarrah et al. (2014) introduced T-X family based on quantile function approach.

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Very recently, some new families of distributions have been proposed in literature. The peculiar ones are: odd Burr-G family (Alizadeh et al., 2017), generalized odd log-logistic family (Cordeiro et al., 2017), new generalized odd log-logistic family (Haghbin et al., 2016), and odd log-logistic Lindley Poisson family (Özel et al., 2016).

Burr (1942) gave a system of twelve cumulative distribution functions for the purpose of fitting data. From which Burr XII and Burr X models have received considerable attention and different extended Burr XII and Burr X models have been proposed in literature. The Burr III model has received comparatively less attention in statistical literature. Therefore, in this paper we propose and study a new generalized family from Burr III logit.

The cumulative distribution function (cdf) and probability density function (pdf) of Burr III distribution are, respectively, given by

$$\Pi(x;c,k) = \left(1+x^{-c}\right)^{-k} \tag{1}$$

and

$$\pi(x;c,k) = ckx^{-c-1}\left(1+x^{-c}\right)^{-k-1}, \quad x>0, \quad c,k>0, \tag{2}$$

where c and k are both shape parameters.

Now, we introduce a new family of distributions from Burr-III density (2) by replacing x with the odds $G(x)/[1-G(x)]$. The cdf of odd Burr III generalized (OBIII-G) family of distributions is defined by

$$F(x;c,k,\xi) = \int_0^{\frac{G(x)}{1-G(x)}} ckt^{-c-1}\left(1+t^{-c}\right)^{-k-1} dt = \left\{1+\left(\frac{1-G(x,\xi)}{G(x,\xi)}\right)^c\right\}^{-k}. \tag{3}$$

The pdf corresponding to Eq. (3) is

$$f(x;c,k,\xi) = ckg(x,\xi)\,\frac{[1-G(x,\xi)]^{c-1}}{G(x,\xi)^{c+1}}\left[1+\left(\frac{1-G(x,\xi)}{G(x,\xi)}\right)^c\right]^{-k-1}, \tag{4}$$

where $G(x;\xi)$ and $g(x;\xi)$ is the cdf and pdf of any baseline distribution, and ξ is the vector of parameters in a baseline distribution. Henceforth, a random variable X having density (4) with parameters c, k and ξ is denoted by $X \sim \text{OBIIIG}(c,k,\xi)$.

The hazard rate function (hrf) of OBIII-G family is given by

$$h(x;c,k,\xi) = \frac{ckg(x)\,\frac{[1-G(x)]^{c-1}}{G(x)^{c+1}}\left[1+\left(\frac{1-G(x)}{G(x)}\right)^c\right]^{-k-1}}{1-\left[1+\left(\frac{1-G(x)}{G(x)}\right)^c\right]^{-k}}. \tag{5}$$

The main motivation of this study are: (i) to obtain more flexible model with less number of parameters and to get goodness-of-fit on the real life survival data, (ii) to make the kurtosis more flexible, (iii) to generate distributions with symmetric, left-skewed, right-skewed, J, reversed-J shaped and bimodal, (iv) to make a skewness for symmetrical distributions, (v) to build heavy-tailed distributions that are not longer-tailed for modeling real data, (vi) to describe special models with all types of the hrf, and (vii) to provide consistently better fits than other generated models under the same baseline distribution.

This paper is organized as follows. In Section 2, four special models of OBIII-G family are described, and the plots of their densities and hazard rate functions are displayed. In Section 3, some important mathematical properties of the new family such as the quantile function, asymptotics, shapes of the density and hazard rate functions, a useful expansion of OBIII-G family, ordinary and incomplete moments, mean deviations, generating function, Rényi and Shannon entropies, stress-strength reliability parameter and order statistics are obtained. In Section 4, the family parameters are estimated by the method of maximum likelihood. The properties of a special model, that is the *Odd Burr III Lomax* (OBIIILx) distribution are given in Section 5. In Section 6, a simulation is conducted to assess the performance of maximum likelihood estimators. Three real-life data are analyzed to illustrate the performance of the OBIIILx model in Section 7. Section 8 offers some concluding remarks.

2 Special models of OBIII-G family

In this section, we discuss four special models of OBIII-G family and display their plots of density and hazard rate functions.

2.1 Odd Burr III-uniform distribution

Let uniform is the baseline distribution with parameter $\theta > 0$ having cdf and pdf $G(x,\theta) = x/\theta$ and $g(x,\theta) = 1/\theta$, respectively. Then the cdf and pdf of odd Burr-III-uniform (OBIIIU) distribution are, respectively, given by

$$F(x;c,k,\theta) = \left\{1 + \left[\frac{\theta - x}{x}\right]^{c}\right\}^{-k} \tag{6}$$

and

$$f(x) = \frac{ck}{\theta}\,\frac{\left\{1 - \frac{x}{\theta}\right\}^{c+1}}{\left\{\frac{x}{\theta}\right\}^{c-1}}\left\{1 + \left[\frac{\theta - x}{x}\right]^{c}\right\}^{-k-1}. \tag{7}$$

A random variable having density (7) is denoted by $X \sim \text{OBIIIU}(c,k,\theta)$. In Figure 1, the plots of density and hazard rate of OBIIIU distribution are displayed. The density can produce shapes such as left-skewed, symmetrical, J, reversed-J and U, and the hazard rate exhibits increasing and bathtub shapes.

(a) (b)

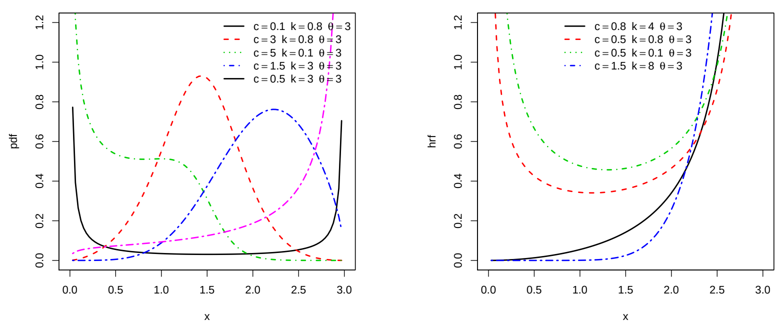


Fig. 1: Plots of (a) densities and (b) hazard rates of OBIIIU distribution.

2.2 Odd Burr III-exponential distribution

Let exponential is the baseline distribution with parameter $\alpha > 0$ having cdf and pdf $G(x) = 1 - e^{-\alpha x}$ and $g(x) = \alpha e^{-\alpha x}$, respectively. Then the cdf and pdf of odd Burr III-exponential (OBIIIE) distribution are, respectively, given by

$$F(x) = \left[1 + \left\{\frac{e^{-\alpha x}}{1 - e^{-\alpha x}}\right\}^{c}\right]^{-k} \tag{8}$$

and

$$f(x) = ck\alpha e^{-\alpha x}\frac{\{e^{-\alpha x}\}^{c+1}}{\{1 - e^{-\alpha x}\}^{c-1}}\left[1 + \left\{\frac{e^{-\alpha x}}{1 - e^{-\alpha x}}\right\}^{c}\right]^{-k}. \tag{9}$$

A random variable having density (7) is denoted by $X \sim \text{OBIIIE}(c,k,\alpha)$. In Figure 2, the plots of density and hazard rate of OBIIIE distribution are given. The density can produced shapes such as right-skewed, symmetrical and reverse- J and the shapes of the hazard rate are increasing, decreasing, constant and bathtub.

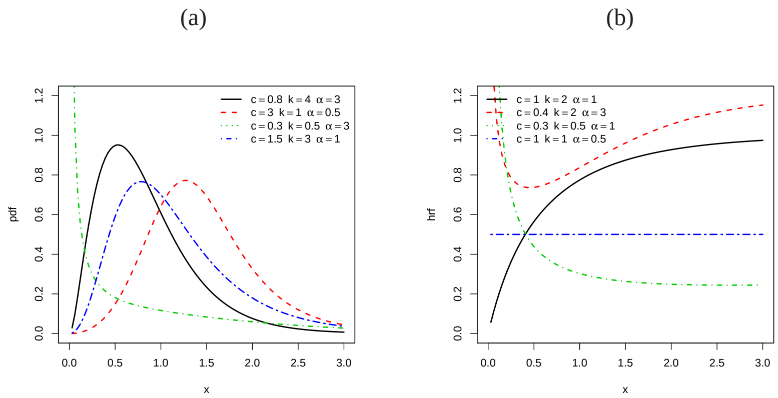


Fig. 2: Plots of (a) densities and (b) hazard rates of OBIIIE distribution.

2.3 Odd Burr III-Lomax distribution

Let Lomax is the baseline distribution with parameters $\alpha > 0$ and $\beta > 0$ having cdf and pdf $G(x) = [1+(x/\beta)]^{-\alpha}$ and $g(x) = (\alpha/\beta)\,[1+(x/\beta)]^{-\alpha-1}$, respectively. Then the cdf and pdf of odd Burr III-Lomax (OBIIILx) distribution are, respectively, given by

$$F(x) = \left[1+\left(\frac{\left(1+\frac{x}{\beta}\right)^{-\alpha}}{1-\left(1+\frac{x}{\beta}\right)^{-\alpha}}\right)^{c}\right]^{-k} \tag{10}$$

and

$$f(x) = ck\frac{\alpha}{\beta}\left(1+\frac{x}{\beta}\right)^{-\alpha-1}\frac{\left[\left(1+\frac{x}{\beta}\right)^{-\alpha}\right]^{c-1}}{\left[1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right]^{c+1}}\left[1+\left(\frac{\left(1+\frac{x}{\beta}\right)^{-\alpha}}{1-\left(1+\frac{x}{\beta}\right)^{-\alpha}}\right)^{c}\right]^{-k-1}. \tag{11}$$

A random variable having density (11) is denoted by $X \sim \text{OBIIILx}(c,k,\alpha,\beta)$. In Figure 3, the plots of density and hazard rate of OBIIILx distribution are presented. The density can produce right-skewed and reverse-J shapes, and hazard rate exhibits decreasing and upside-down bathtub shapes.

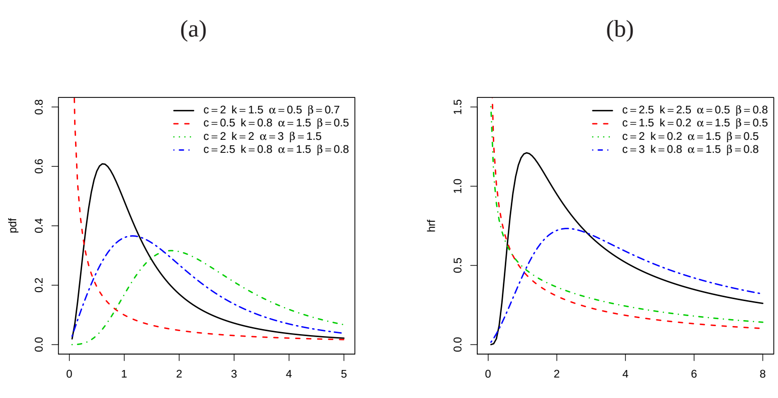


Fig. 3: Plots of (a) densities and (b) hazard rates of OBIIILx distribution.

2.4 Odd Burr III-logistic distribution

Let logistic is the baseline distribution with parameter $\lambda > 0$ having cdf and pdf $G(x) = (1+e^{-\alpha x})^{-1}$ and $g(x) = \alpha e^{-\alpha x}(1+e^{-\alpha x})^{-2}$, respectively. Then the cdf and pdf of odd Burr-III-logistic (OBIIIL) distribution are, respectively, given by

$$F(x) = \left[1+\left\{\frac{1-(1+e^{-\alpha x})^{-1}}{(1+e^{-\alpha x})^{-1}}\right\}^{c}\right]^{-k} \tag{12}$$

and

$$f(x) = ck\alpha e^{-\alpha x}\left(1+e^{-\alpha x}\right)^{-2}\frac{\left\{1-(1+e^{-\alpha x})^{-1}\right\}^{c+1}}{\left\{(1+e^{-\alpha x})^{-1}\right\}^{c-1}}\left[1+\left\{\frac{1-(1+e^{-\alpha x})^{-1}}{(1+e^{-\alpha x})^{-1}}\right\}^{c}\right]^{-k}. \tag{13}$$

A random variable having density (13) is denoted by $X \sim OBIIIL(c,k,\alpha)$. In Figure 4, the plots of density and hazard rate functions of OBIIIL distribution are given. The density can be of symmetrical and right-skewed shape while hazard rate can exhibit only increasing and constant shapes .

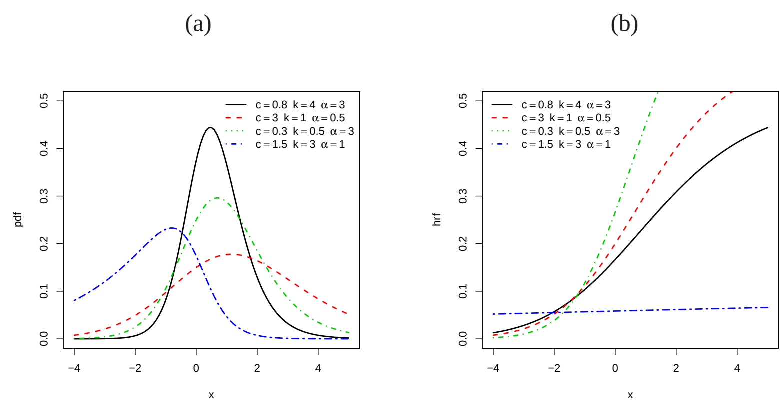


Fig. 4: Plots of (a) densities and (b) hazard rates of OBIIIL distribution.

3 Mathematical properties of OBIII-G family

In this section, we provide some mathematical properties of the OBIII-G family of distributions.

3.1 Quantile function and simulation

The OBIII-G family can easily be simulated by inverting Eq. (3) as follows: if u has a uniform distribution $U(0,1)$, then

$$Q_X(u) = Q_G\left[1+\left(u^{-\frac{1}{k}}-1\right)^{\frac{1}{c}}\right]^{-1}, \tag{14}$$

where $Q_G(.) = G^{-1}(.)$ is the baseline quantile function (qf).

3.2 Asymptotic and shapes

Let X takes non-negative value. Then, the asymptotics of Eqs. (3), (4) and (5) as $x \to 0$ are given by

$F(x) \sim G(x)^{ck}$,
$f(x) \sim ckg(x)G(x)^{ck-1}$,
$h(x) \sim ckg(x)G(x)^{ck-1}$.

The asymptotics of Eqs. (3), (4) and (5) as $x \to \infty$ are given by

$1-F(x) \sim k\bar{G}(x)^{c}$,
$f(x) \sim ckg(x)\bar{G}(x)^{c-1}$,
$h(x) \sim \dfrac{c\,g(x)}{G(x)}$.

The shapes of the density and hazard rate functions can be described analytically. The critical points of the OBIII-G family density function are the roots of the equation:

$$\frac{g'(x)}{g(x)}+(ck-1)\frac{g(x)}{G(x)}+(1-c)\frac{g(x)}{G(x)}-c(k+1)g(x)\frac{G(x)^{c-1}-\bar{G}(x)^{c-1}}{G(x)^{c}+G(x)^{c}}=0.$$

The critical point of OBIII-G family hazard rate are the roots of the equation:

$$\frac{g'(x)}{g(x)}+(ck-1)\frac{g(x)}{G(x)}+(1-c)\frac{g(x)}{1-G(x)}-cg(x)\frac{G(x)^{c-1}-[\bar{G}(x)]^{c-1}}{G(x)^{c}+\bar{G}(x)^{c}}$$
$$-ckg(x)\frac{\left\{\left\{G(x)^{c}+\bar{G}(x)^{c}\right\}^{k-1}[G(x)^{c-1}-\bar{G}(x)^{c-1}]-G(x)^{ck-1}\right\}}{\left\{G(x)^{c}+\bar{G}(x)^{c}\right\}^{k}-G(x)^{ck}}=0.$$

3.3 Linear representation of the density

In this section, a linear representation of the OBIII-G density is obtained, which is helpful in obtaining useful properties of the OBIII distributions.

Consider generalized binomial expansion

$$(1-z)^{-n}=\sum_{j=0}^{\infty}\binom{n+j-1}{j}z^{j}, \tag{15}$$

where, $|n|>0$ is a real number.

Now from Eq. (15) and Eq. (3), the cdf of OBIII-G can be represented as

$$F(x)=\sum_{i,j=0}^{\infty}a_{j-ci}\,H_{j-ci}(x), \tag{16}$$

where

$$a_{j-ci}=\binom{k+i-1}{i}\binom{ci}{j}(-1)^{i+j}. \tag{17}$$

The density of the family can be expressed as

$$f(x)=\sum_{i,j=0}^{\infty}a_{j-ci}\,h_{j-ci-1}(x). \tag{18}$$

Eq. (18) is obtained through simple differentiation of the Eq. (16), where $H_{j-ci}(x)=G^{j-ci}(x)$ and $h_{j-ci-1}(x)=(j-ci)\,g(x)\,G^{j-ci-1}(x)$ follows the exponentiated-G distribution with $j-ci$ as the power parameter.

Eqs. (16) and (18) are the main results of this section.

3.4 Moments, incomplete moments and generating function

In this section, we give mathematical expressions for the moments and incomplete moments, moment generating function and mean deviations.

The rth moment expression for the OBIII-G family of distributions can be obtained as

$$\mu_r' = \sum_{i,j=0}^{\infty} a_{j-ci} \int_0^{\infty} x^r h_{j-ci-1}(x)\,dx, \tag{19}$$

where a_{j-ci} is defined in Eq. (17) and $j-ci-1$ is the power parameter.

The sth incomplete moment for the OBIII-G family of distributions is given by

$$\mu_s'(x) = \sum_{i,j=0}^{\infty} a_{j-ci} \int_0^{x} x^s h_{j-ci-1}(x)\,dx. \tag{20}$$

The expression for the moment generating function of the OBIII-G family of distributions is given by

$$M(t) = \sum_{i,j=0}^{\infty} a_{j-ci} \int_0^{\infty} e^{tx} h_{q+1}(x)\,dx.$$

The mean deviations of the OBIII-G family of distributions about the mean and median can be obtained as

$$D_\mu = 2\mu F(\mu) - 2\mu^1(\mu) \quad \text{and} \quad D_M = \mu - 2\mu^1(M)$$

where $\mu = E(X)$ comes from the Eq. (19), $M = \text{Median}(X)$ is the median can be obtained from Eq. (14), $F(\mu)$ can easily be obtained from Eq. (3) and $\mu^1(.)$ can be obtained from Eq. (20) with $s = 1$.

3.5 Entropies

The entropy of a random variable (rv) X is a measure of variation of the uncertainty. A large value of the entropy specifies the greater uncertainty in the data. Entropy has several applications in physics, chemistry, engineering and economics, among others. The Shannon entropy of a continuous rv having baseline pdf $g(x)$ is defined by $\mathbb{E}[-\log g(x)]$ (Shannon, 1948). The relationship between the Shannon entropy for a rv X with pdf $g(x)$ and the Shannon entropy of a random variable T with pdf $r(t)$ is given in the following theorem.

Theorem 1. If T has a pdf $r(t)$ and X follows the OBIII-G family (4), then the Shannon entropy of X, η_X is given by

$$\eta_X = -\mathbb{E}\left\{\log\, g\left(G^{-1}\left[\frac{T}{1+T}\right]\right)\right\} - 2\mathbb{E}\left\{\log\,(1+T)\right\} + \eta_T,$$

where η_T is the Shannon entropy of the Burr III distribution.

Proof. The Shannon entropy is defined by

$$\eta_X = -\mathbb{E}[\log\ f(x)]. \tag{21}$$

The cdf in Eq. (3) can be written as

$$F(x) = \int_0^{\frac{G(x)}{1-G(x)}} r(t)dt = R\left(\frac{G(x)}{1-G(x)}\right). \tag{22}$$

The pdf corresponding to Eq. (22) is given by

$$f(x) = \frac{g(x)}{[1-G(x)]^2}\, r\left(\frac{G(x)}{1-G(x)}\right). \tag{23}$$

Now, Eq. (21) becomes

$$\eta_X = -\mathbb{E}\left[\log\, g(x)\right] - 2\,\mathbb{E}\left\{\log\,\left[1-G(x)\right]\right\} - \mathbb{E}\left[r(t)\right]. \tag{24}$$

The relationship between rvs X and T is $T = \frac{G(x)}{1-G(x)}$, then $X = G^{-1}\left(\frac{T}{1+T}\right)$.

Finally, Eq. (24) becomes

$$\eta_X = -\mathbb{E}\left\{\log g\left[G^{-1}\left(\frac{T}{1+T}\right)\right]\right\} - 2\mathbb{E}[\log(1+T)] + \eta_T.$$

□

Theorem 2. If T has a pdf $r(t)$ and X follows the OBIII-G family (4), then the Rényi entropy of X, that is $I_\delta(x) = 1/[1-\delta]\,\log\int f^\delta(x)\,dx$ is given by

$$I_\delta(x) = \frac{1}{1-\delta}\log\left[(ck)^\delta \sum_{i,j=0}^{\infty} V_{i,j}(\delta,k)\int_0^\infty g^\delta(x)\,G^{j-c(i+\delta)-\delta}(x)\,dx\right].$$

Proof. First we use binomial expansion used in Eq. (15) to the quantity $f^\delta(x)$

$$f^\delta(x) = (ck)^\delta \sum_{i,j=0}^{\infty} \binom{\delta(k+1)+i-1}{i}\binom{c(i+\delta)-\delta}{j}(-1)^{i+j} g^\delta(x)\,G^{j-c(i+\delta)-\delta}(x).$$

Now, we have

$$I_\delta(x) = \frac{1}{1-\delta}\log\left[(ck)^\delta \sum_{i,j=0}^{\infty} \binom{\delta(k+1)+i-1}{i}\binom{c(i+\delta)-\delta}{j}(-1)^{i+j} g^\delta(x)\,G^{j-c(i+\delta)-\delta}(x)\right].$$

Rewriting the above equation

$$I_\delta(x) = \frac{1}{1-\delta}\log\left[(ck)^\delta \sum_{i,j=0}^{\infty} V_{i,j}(\delta,k)\int_0^\infty g^\delta(x)\,G^{j-c(i+\delta)-\delta}(x)\right]dx,$$

where $V_{i,j}(\delta,k) = \binom{\delta(k+1)+i-1}{i}\binom{c(i+\delta)-\delta}{j}(-1)^{i+j}$.

□

3.6 Stress-strength reliability

In the context of reliability, the stress-strength model defines the life of a element which has a random strength X_1 that is subjected to an accidental stress X_2. The component fails at the instant that the stress applied to it exceeds the strength, and the component will function suitably whenever $X_1 > X_2$. Hence, $R = \mathbb{P}(X_2 < X_1)$ is a measure of components reliability (Kotz et al., 2003). It has many applications especially in the area of reliability and engineering. We derive the reliability R when X_1 and X_2 have independent OBIII(c_1,k_1,ξ) and OBIII(c_2,k_2,ξ) distributions with common shape and scale parameters. From Eqs. (3) and (4), the parameter reliability reduces to

$$R = \mathbb{P}(X_1 < X_2) = \int_0^\infty f_1(x)\,F_2(x)\,dx. \tag{25}$$

From Eqs. (8) and (9), the Eq. (25) becomes

$$R = \mathbb{P}(X_1 < X_2) = \sum_{i,j=0}^{\infty}\sum_{l,m=0}^{\infty} a_{j-ci}\,b_{l-cm}\int_0^\infty h_{j-ci-1}(x)\,H_{l-cm}(x)\,dx, \tag{26}$$

where $h_{j-ci-1}(x) = (j-ci)\,g(x)\,G^{j-ci-1}(x)$ and $H_{l-cm}(x) = G^{l-cm}(x)$.

3.7 Order statistics

Suppose $X_1, X_2, \ldots, X_n$ be a random sample from OBIII-G distributions and $X_{i:n}$ denotes ith order statistics. The pdf of $X_{i:n}$ is given by

$$f_{i:n}(x) = \frac{n!}{(i-1)!\times(n-i)!}\sum_{j=0}^{n-i}\binom{n-i}{j}(-1)^i f(x)[F(x)]^{i+j-1}. \tag{27}$$

From Eqs. (8) and (9), the Eq. (27) becomes

$$f_{i:n}(x) = \frac{n!}{(i-1)!\times(n-i)!}\sum_{j=0}^{n-i}\binom{n-i}{j}(-1)^i\left[\sum_{i,j=0}^{\infty}a_{j-ci}(j-ci)\,g(x)\,G^{j-ci-1}(x)\right]\left\{\sum_{l,m=0}^{\infty}b_{l-cm}G^{l-cm}(x)\right\}^{i+j-1}. \tag{28}$$

Using power series raised to power for positive integer n (≥ 1) (see Gradshteyn and Ryzhik, 2000)

$$\left(\sum_{k=0}^{\infty}a_k x^k\right)^n = \sum_{k=0}^{\infty}c_{n:k}x^k,$$

where $c_0 = a_0^n$ and $c_m = \frac{1}{m a_0}\sum_{k=1}^{m}(kn-m+k)a_k c_{n:m-k}$ for $m \geq 1$ and n is a natural number.

The above density can be expressed as

$$f_{i:n}(x) = \sum_{j=0}^{n-i}\sum_{i,j=0}^{\infty}\sum_{l,m=0}^{\infty}V_j(j-ci,l-cm)\,h_{j+l-c(i+m)}(x), \tag{29}$$

where

$$V_j(j-ci,l-cm) = \frac{n!\,(-1)^j a_{j-ci}\,e_{j+i-1;l-cm}(j-ci)}{(i-1)!\,j!\,[j+l-c(i+m)+1]},$$

where $h_{j+l-c(i+m)}(x) = (j+l-c(i+m)+1)\,g(x)\,G^{j+l-c(i+m)}(x)$.

The Eq. (29) reveals that the density of OBIII-G order statistic can be expressed as linear combination of baseline densities.

4 Estimation

Let $x_1, x_2, \ldots, x_n$ be a random sample of size n from the OBIII-G distributions (c,k,ξ). The log-likelihood function for the vector of parameters $\Theta = (c,k,\xi)^T$ is given by

$$\begin{aligned} l(\Theta) &= n\log ck + \sum_{i=0}^{n}\log g(x_i;\xi) + (c-1)\sum_{i=0}^{n}\log\{1-G(x_i;\xi)\} \\ &- (c+1)\sum_{i=0}^{n}\log G(x_i;\xi) - (k+1)\sum_{i=0}^{n}\log\left\{1+\left(\frac{1-G(x_i;\xi)}{G(x_i;\xi)}\right)^c\right\}. \end{aligned}$$

The components of the score vector are given by

$$U_k(\Theta) = \frac{n}{k} - \sum_{i=0}^{n}\log\left\{1+\left(\frac{1-G(x_i;\xi)}{G(x_i;\xi)}\right)^c\right\},$$

$$U_c(\Theta) = \frac{n}{c} + \sum_{i=0}^{n}\log[1-G(x_i;\xi)] - \sum_{i=0}^{n}\log G(x_i;\xi) - (k+1)\sum_{i=0}^{n}\left[\frac{\left(\frac{1-G(x_i;\xi)}{G(x_i;\xi)}\right)^c\log\left(\frac{1-G(x_i;\xi)}{G(x_i;\xi)}\right)}{1+\left(\frac{1-G(x_i;\xi)}{G(x_i;\xi)}\right)^c}\right],$$

$$U_\xi(\Theta) = \sum_{i=0}^{n}\left[\frac{g^\xi(x_i;\xi)}{g(x_i;\xi)}\right] - (c-1)\sum_{i=0}^{n}\left[\frac{G^\xi(x_i;\xi)}{1-G(x_i;\xi)}\right] - (c+1)\sum_{i=0}^{n}\left[\frac{G^\xi(x_i;\xi)}{G(x_i;\xi)}\right](k+1)\sum_{i=0}^{n}\left[\frac{c\left(\frac{1-G(x)}{G(x)}\right)^{c-1}\frac{d}{d\xi}\left(\frac{1-G(x)}{G(x)}\right)}{1+\left(\frac{1-G(x)}{G(x)}\right)^c}\right],$$

where $g^\xi(\cdot)$ means the derivative of the function $g(\cdot)$ with respect to ξ and $G^\xi(\cdot)$ means the derivative of the function $G(\cdot)$ with respect to ξ. Setting U_k, U_c and U_ξ equal to zero and solving these equations simultaneously yields the maximum likelihood estimates.

5 Mathematical properties of OBIIILx distribution

In this section, we obtain some mathematical properties of the OBIIILx distribution.

The cdf and density of OBIIILx model can be expressed as

$$F(x;c,k,\alpha,\beta)=\sum_{i,j=0}^{\infty}a_{j-ic}\left[1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right]^{j-ic}$$

and

$$f(x;c,k,\alpha,\beta)=\sum_{i,j=0}^{\infty}a_{j-ic}\,\frac{\alpha}{\beta}\left(1+\frac{x}{\beta}\right)^{-\alpha-1}\left[1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right]^{j-ic-1},$$

where the coefficient a_{j-ci} is defined is Eq. (17).

The qf of OBIIILx distribution is given by

$$Q_x(u)=\beta\left\{\left[1-\left\{1+\left(1-u^{-\frac{1}{k}}\right)^{\frac{1}{c}}\right\}^{-1}\right]^{-\frac{1}{\alpha}}-1\right\}.$$

The rth moments expression for OBIIILx is given by

$$\mu_r'=\sum_{i,j=0}^{\infty}a_{j-ic}\sum_{l=0}^{\infty}\binom{j-ci-1}{l}(-1)^l\,(j-ic)\,\alpha\,\beta^r\,B(r+1,\alpha(l+1)-r),$$

where $B(l,m)=\frac{\Gamma(l+m)}{\Gamma(l)\Gamma m}=\int_0^1 x^{l-1}\,(1-x)^{m-1}\,dx$ is complete beta function.

The sth incomplete moment expression for OBIIILx is given by

$$\mu_s'=\sum_{i,j=0}^{\infty}a_{j-ci}\sum_{l=0}^{\infty}\binom{j-ci-1}{l}(-1)^l\,(j-ic)\,\alpha\,\beta^s\,B_{(x/\beta)}(s+1,\alpha(l+1)-s)$$

where $B_t(l,m)=\int_0^t x^{l-1}\,(1-x)^{m-1}\,dx$ is lower incomplete beta function.

The expression for moment generating function of OBIIILx is given by

$$M_X(t)=\frac{\alpha}{\beta}\sum_{i,j=0}^{\infty}a_{j-ic}\sum_{l,m=0}^{\infty}\binom{j-ci-1}{l}\binom{\alpha(l+1)+m}{m}\frac{(-1)^{l+2m+1}}{\beta^m}(j-ic)\,\frac{\Gamma(m+1)}{t^{m+1}}.$$

The Shannon entropy of OBIIILx distribution will be

$$\eta_X=\left(1+\frac{1}{c}\right)\left[\psi(k)-\Gamma'(1)\right]-\log(ck)-2-\left(\frac{\alpha+k}{\alpha k}\right)-2\sum_{j=0}^{\infty}\frac{(-1)^{j+1}}{j}\,kB\left(1-\frac{j}{c},k+\frac{j}{c}\right).$$

The Rényi entropy is

$$I_\delta=\frac{1}{1-\delta}\log\left\{K\sum_{i,j=0}^{\infty}V_{i,j}(\delta,k)\left(\frac{\alpha}{\beta}\right)^{\delta}\sum_{n=0}^{\infty}\binom{i-c(i+\delta)-\delta}{n}\frac{(-1)^n\,\beta}{\alpha(\delta+n)+\delta-1}\right\}.$$

The stress-strength reliability parameter for OBIIILx distribution (with β as common parameter) is given by

$$R=\mathbb{P}(X_1<X_2)=\sum_{i,j=0}^{\infty}\sum_{l,m=0}^{\infty}a_{j-ic}\,b_{l-cm}\,(j-ic)\,\frac{\alpha_1}{\beta}\sum_{p,q=0}^{\infty}\binom{j-ci-1}{q}\binom{l-cm}{p}(-1)^{p+q}\frac{\beta}{\alpha_1(q+1)+\alpha_2\,p}.$$

The likelihood function of OBIIILx distribution is given by

$$l(\Theta)=n\log\left(\frac{ck\alpha}{\beta}\right)-(\alpha c+1)\sum_{i=1}^{n}\log\left(1+\frac{x_i}{\beta}\right)-(c+1)\sum_{i=1}^{n}\log\left[1-\left(1+\frac{x_i}{\beta}\right)^{-\alpha}\right]$$
$$-(k+1)\sum_{i=1}^{n}\log\left\{1+\left[\left(1+\frac{x_i}{\beta}\right)^{\alpha}-1\right]^{-c}\right\}.$$

The components of score vector for OBIIILx distribution are given by

$$U_k = \frac{n}{k} - \sum_{i=1}^{n} \log\left\{1 + \left[\left(1+\frac{x_i}{\beta}\right)^{\alpha} - 1\right]^{-c}\right\},$$

$$U_c = \frac{n}{c} - \alpha \sum_{i=1}^{n} \log\left(1+\frac{x_i}{\beta}\right) - \sum_{i=1}^{n} \log\left[1-\left(1+\frac{x_i}{\beta}\right)^{-\alpha}\right]$$

$$+ (k+1)\sum_{i=1}^{n}\left[\frac{\left[\left(1+\frac{x_i}{\beta}\right)^{\alpha}-1\right]^{-c}\log\left[\left(1+\frac{x_i}{\beta}\right)^{\alpha}-1\right]}{1+\left[\left(1+\frac{x_i}{\beta}\right)^{\alpha}-1\right]^{-c}}\right],$$

$$U_\alpha = \frac{n}{\alpha} - c\sum_{i=1}^{n}\log\left(1+\frac{x_i}{\beta}\right) - (c+1)\sum_{i=1}^{n}\left[\frac{\left(1+\frac{x_i}{\beta}\right)^{-\alpha}\log 1 - \left(1+\frac{x_i}{\beta}\right)}{1-\left(1+\frac{x_i}{\beta}\right)^{-\alpha}}\right]$$

$$- c(k+1)\sum_{i=1}^{n}\left[\frac{\left[\left(1+\frac{x_i}{\beta}\right)^{\alpha}-1\right]^{-c-1}\left(1+\frac{x_i}{\beta}\right)^{\alpha}\log\left(1+\frac{x_i}{\beta}\right)}{1+\left[\left(1+\frac{x_i}{\beta}\right)^{\alpha}-1\right]^{-c}}\right],$$

$$U_\alpha = -\frac{n}{\beta} + \frac{\alpha c+1}{\beta^2}\sum_{i=1}^{n}\left[\frac{x_i}{1+\frac{x_i}{\beta}}\right] + \frac{\alpha(c+1)}{\beta^2}\sum_{i=1}^{n}\left[\frac{x_i\left(1+\frac{x_i}{\beta}\right)^{-\alpha-1}}{1-\left(1+\frac{x_i}{\beta}\right)^{-\alpha}}\right]$$

$$- \frac{\alpha c(k+1)}{\beta^2}\sum_{i=1}^{n}\left[\frac{x_i\left(1+\frac{x_i}{\beta}\right)^{\alpha-1}\left[\left(1+\frac{x_i}{\beta}\right)^{\alpha}-1\right]^{-c-1}}{1+\left[\left(1+\frac{x_i}{\beta}\right)^{\alpha}-1\right]^{-c}}\right].$$

Setting U_k, U_c, U_α and U_β equal to zero and solving these equations simultaneously yields the the maximum likelihood estimates of OBIIILx distribution.

6 Simulation study of the OBIIILx distribution

Torabi (2008) introduced a general method for estimating parameters through spacing called maximum spacing distance estimator (MSDE). Torabi and Bagheri (2010) and Torabi and Montazeri (2014) used different MSDEs to compare with the MLEs. Here, we compare MLEs to MSDEs “minimum spacing absolute distance estimator” (MSADE) and “minimum spacing absolute-log distance estimator” (MSALDE) of the OBIIILx distribution. For mathematical details, the reader is referred to Torabi and Bagheri (2010) and Torabi and Montazeri (2014). We simulate the OBIIILx distribution for n = 50, 100, 200, 300 and 500 with $c = 1.5$, $k = 0.5$, α= 2 and $\beta = 0.5$. For each sample size, we compute the MLEs, MSADEs and MSALDEs of the parameters. We repeat this process 1,000 times and obtain the average estimates (AEs), biases and mean square error (MSEs). The results are reported in Table 1. We note that the MSEs of MSADEs and MSALDEs are less than the MSEs of MLEs.

Table 1: Estimated AE and MSE of MLE, MSADE and MSALDE of the parameters based on 1000 simulations of the OBIIILx distribution for c= 1.5 , k= 0.5, α= 2 and $\beta = 0.5$ with n= 50, 100, 200, 300 and 500.

Different method		MLE		MSADE		MSALDE	
n	parameters	A.E	MSE	A.E	MSE	A.E	MSE
50	c	1.9976	4.0704	1.4813	0.1809	1.6359	0.6470
	k	1.1527	1.8914	0.6583	0.2708	0.5891	0.1822
	β	0.8612	0.5207	0.5914	0.1445	0.5391	0.1816
	α	2.0143	5.0078	2.0455	1.1291	2.1013	2.4512
100	c	1.6952	1.2807	1.4818	0.1846	1.4997	0.1674
	k	0.9614	1.1606	0.6748	0.4104	0.6749	0.4888
	β	0.7346	0.3239	0.5674	0.0962	0.5630	0.1053
	α	1.7531	2.5155	1.9628	1.7235	2.0012	1.2236
200	c	1.5694	0.8126	1.5517	0.1001	1.6501	0.3759
	k	0.7189	0.3342	0.5320	0.0487	0.5058	0.0367
	β	0.6887	0.2014	0.5004	0.0208	0.4935	0.0392
	α	2.0247	1.3326	1.9197	0.1849	1.9864	1.1315
300	c	1.5348	0.5734	1.5537	0.1737	1.5592	0.2271
	k	0.7095	0.3284	0.5594	0.0709	0.5658	0.0867
	β	0.6940	0.1998	0.5205	0.0434	0.5421	0.0558
	α	2.0458	1.2662	1.8392	0.2514	1.9710	0.9619
500	c	1.4622	0.2888	1.4742	0.0728	1.5846	0.2591
	k	0.6302	0.1354	0.5573	0.0519	0.5331	0.0352
	β	0.6481	0.1367	0.5338	0.0327	0.5324	0.0570
	α	2.1569	0.9035	1.9522	0.1721	1.9447	0.6547

7 Applications of OBIIILx model

In this section three real-life data sets are analyzed as an empirical illustration of the newly proposed family. The first two data sets are based on complete observations (uncensored) while the third one is censored. We tried to show the usefulness of the OBIIILx model in different lifetime phenomenons. In these three applications, the model parameters are estimated by the method of maximum likelihood. The goodness-of-fit criterion: Akaike information criterion (AIC), Anderson-Darling (A^*)and Cramer-von Mises (W^*) are used to compare the proposed and competitive models. In general, the smaller the values of these statistics, the better the fit to the data. The plots of the fitted pdfs and cdfs of the models are displayed for visual comparison. The required computations are carried out in the R-packages.

7.1 Uncensored (complete) data sets

Data 1: Acute Myelogenous data. The data set was first analyzed by Feigl and Zelen (1965). The data represent the survival times, in weeks, of 33 patients suffering from Acute Myelogenous Leukaemia. The data are: 65, 156, 100, 134, 16, 108, 121, 4, 39, 143, 56, 26, 22, 1, 1, 5, 65, 56, 65, 17, 7, 16, 22, 3, 4, 2, 8, 4, 3, 30, 4, 43.

We compare the values of goodness-of-fit statistics of OBIIILx model with beta-Burr III (BBIII) (Gomes et al., 2013), exponentiated-Burr III (EBIII), Lehmann type II Burr III (LeBIII) and Burr III (BIII) models obtained from data set 1. The MLE estimates of the models' parameters along with their associated standard errors (in parenthesis) are given in Table 2 and the values of statistics AIC, A^* and W^* are given in Table 3.

The cdf of BBIII is given by

$$F(x) = I_{[1+(x/\theta)^{-\alpha}]^{-\beta}}(c,k)$$

(i) When $c = 1$, BBIII reduces to LeBIII, (ii) when $k = 1$ BBIII reduces to EBIII, and (iii) when $c = k = \theta = 1$, BBIII it reduces to BIII distribution.

Data 2: Actual Taxes data. The second data set consist of the monthly actual taxes revenue in Egypt from January 2006 to November 2010. The distribution is highly skewed to the right. Mead (2014) used this data set. The actual taxes revenue data (in 1000 million Egyptian pounds) are: 5.9, 20.4, 14.9, 16.2, 17.2, 7.8, 6.1, 9.2, 10.2, 9.6, 13.3, 8.5, 21.6, 18.5, 5.1, 6.7, 17.0, 8.6, 9.7, 39.2, 35.7, 15.7, 9.7, 10.0, 4.1, 36.0, 8.5, 8.0, 9.2, 26.2, 21.9, 16.7, 21.3, 35.4, 14.3, 8.5, 10.6, 19.1, 20.5, 7.1, 7.7, 18.1, 16.5, 11.9, 7.0, 8.6, 12.5, 10.3, 11.2, 6.1, 8.4, 11.0, 11.6, 11.9, 5.2, 6.8, 8.9, 7.1, 10.8.

We compare values of goodness-of-fit statistics of OBIIILx with Weibull Lomax (WLx) (Tahir et al., 2015), exponentiated-Lomax (ELx) (Abdul-Moniem and Abdel-Hameed, 2012) and Lomax distribution for data set 2. The MLE estimates of models' parameters are given in Table 4. The values of goodness-of-fit statistics AIC, A^* and W^* are given in Table 5.

The cdf for WLX and ELx are given by:

$F_{WL}(x) = 1 - \exp\left[-\left\{\left(1+\frac{x}{\beta}\right)^{\alpha} - 1\right\}^{k}\right]$ and $F_{EL}(x) = \left[1 - \left(1+\frac{x}{\beta}\right)^{-\alpha}\right]^{k}$.

When $k = 1$, ELx reduces to Lomax distribution.

Remark 1. Mead (2014) compared beta exponentiated-Burr XII distribution and their sub-models beta log-logistic, beta exponentiated-log-logistic and beta Burr XII with several other three, four, and five-parameter lifetime distributions, namely the generalized gamma (GGa), gamma exponentiated-Weibull (GaEW) and beta generalized-Pareto (BGP) models for data 2 given in Table 2. We also analyzed the same data and compare our propose model OBIIILx to all above models, and observe that our model shows better fit as compared to all above models if we consider A^* and W^* statistics.

Table 2: MLEs and their standard errors (in parentheses) for data set 1.

Distribution	c	k	α	β	θ
OBIIILx	0.1516	1.6485	36.3600	0.0295	-
	(0.0651)	(3.2770)	(31.0820)	(0.0240)	-
BBIII	0.0671	68.8649	0.7687	15.6785	40.5867
	(0.0080)	(20.6863)	(0.0615)	(2.2911)	(11.5609)
LeBIII	-	4.6233	0.5024	2.8857	17.2698
	-	(1.0142)	(1.1221)	(0.0816)	(16.9161)
EBIII	9.3906	-	0.8303	0.3084	3.0707
	(8.9829)		(0.0404)	(0.2951)	(1.4018)
BIII	-	-	0.7646	5.5929	-
	-	-	(0.0931)	(1.1831)	-

Table 3: The AIC, A^* and W^* values for data set 1.

Distribution	AIC	A^*	W^*
OBIIILx	309.63	0.4633	0.0666
BBIII	317.12	0.5657	0.0832
LeBIII	319.70	0.7125	0.1128
EBIII	316.37	0.8984	0.1512
BIII	315.02	0.8922	0.1498

Remarks 2. From Tables 3 and 5, we observed that OBIIILx gives minimum values of the statistics AIC, A^* and W^* as compared to other competitive model. Therefore, the proposed model OBIIILx is better in performance for these two data sets.

7.2 Data set 3: Censored data set

In this section, we provide an application of the OBIIILx model to censored data set. The statistics AIC and BIC are computed and compared the proposed and competitive models: Kumaraswamy-Lomax (KwLx) (Lemonte and Cordeiro, 2013), beta-Lomax (BLx) (Lemonte and Cordeiro, 2013) and BBIII models. The data consist of death times (in weeks)

Table 4: MLEs and their standard errors (in parentheses) for data set 2.

Distribution	c	k	α	β
OBIIILx	10.3121	3.2282	0.1386	0.0440
	(3.3349)	(3.3381)	(0.0414)	(0.0655)
WLx	-	3.9133	0.2549	1.0561
	-	(1.6969)	(0.1342)	(1.6171)
ELx	-	5.8382	70.9535	380.2369
	-	(4.4034)	(1.3966)	(6.4230)
Lomax	-	-	29.1644	384.3509
	-	-	(25.3175)	(337.3768)

Table 5: The AIC, A^* and W^* values for data set 3.

Distribution	AIC	A^*	W^*
OBIIILx	385.5424	0.0400	0.2540
WLx	395.0823	0.2259	1.4234
ELx	387.9290	0.1370	0.8217
Lomax	430.3430	0.1866	1.1544

of patients with cancer of tongue with aneuploid DNA profile (see Lee and Wang, 2003). The data are: 1, 3, 3, 4, 10, 13, 13, 16, 16, 24, 26, 27, 28, 30, 30, 32, 41, 51, 61*, 65, 67, 70, 72, 73, 74*, 77, 79*, 80*, 81*, 87*, 87*, 88*, 89*, 91, 93, 93*, 96, 97, 100, 101*, 104, 104*, 108*, 109*, 120*, 131*, 150*, 157, 167, 231*, 240*, 400*. Here asterisks denote censoring times.

Consider a data set $D=(x,r)$, where $x=(x_1,x_2,\ldots,x_n)^T$ are the observed failure times and $r_i=(r_1,r_2,\ldots,r_n)^T$ are the censored failure times. The r_i is equal to 1 if a failure is observed and 0 otherwise. Suppose that the data are independently and identically distributed and come from a distribution with pdf given in Eq. (11). Let $\Theta=(\alpha,\beta,c,k)^T$ denote the vector of parameters. Then the likelihood of Θ can be expressed as

$$\ell(D;\Theta)=\prod_{i=1}^{n}[f(x_i;\Theta)]^{r_i}\,[1-F(x_i;\Theta)]^{1-r_i}\,.$$

The log-likelihood reduces to

$$\ell(\Theta)=r_i\sum_{i=1}^{n}\log[f(x_i;\Theta)]+(1-r_i)\sum_{i=1}^{n}\log[1-F(x_i;\Theta)]\,. \tag{30}$$

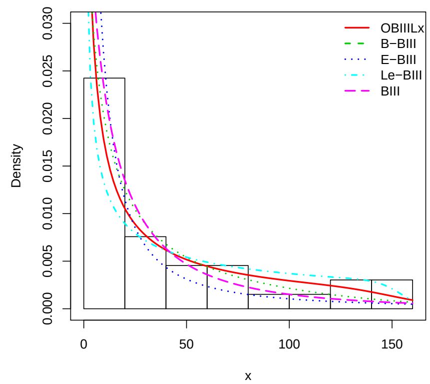
Now from Eqs. (10), (11) and (30), we have

$$\ell=r_i\sum_{i=1}^{n}\left[\log\left(\frac{\alpha\,c\,k}{\beta}\right)+(\alpha-1)\log\left(1+\frac{x}{\beta}\right)-(c+1)\log\left[\left(1+\frac{x}{\beta}\right)^{\alpha}-1\right]\right.$$
$$\left.-\ (k+1)\log\left[1+\left\{\left(1+\frac{x}{\beta}\right)^{\alpha}-1\right\}^{-c}\right]\right]-(1-r_i)\sum_{i=1}^{n}\left\{k\log\left[1+\left\{\left(1+\frac{x}{\beta}\right)^{\alpha}-1\right\}^{-c}\right]\right\}.$$

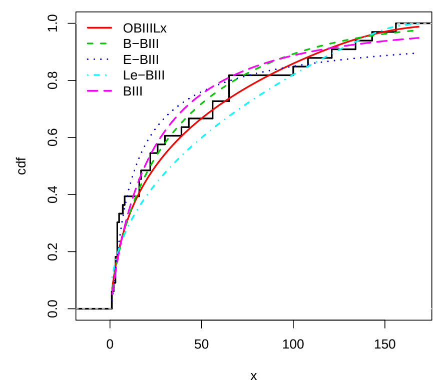
The log likelihood function can be maximized numerically to obtained the MLEs. There are various R-packages that provide numerical maximization of ℓ. We use the optimum R-package.

Remark 3. Oguntunde and Adejume (2015) fitted data set 3 and compared goodness-of-fit values of AIC of generalized inverted generalized exponential (GIGE) model with other competitive models and reported AIC=607.712 by claiming that the GIGE distribution is good model as compared to other competitive models. We noted that our proposed model OBIIILx shows very minimum value of AIC =318.5868 in comparison to GIGEI and others competitive models: KwLx, BIII and BLx. Thus, we can say that OBIIILx model is better model as compared to other models for data set 3.

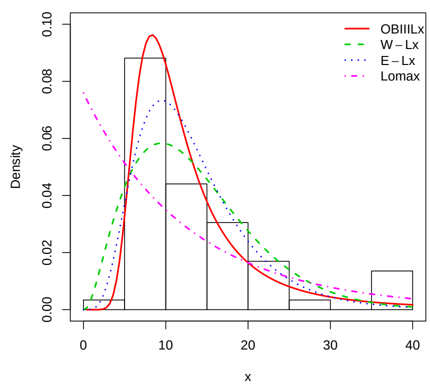
(a) Estimated pdfs for data set 1.



(b) Estimated cdfs for data set 1.



(c) Estimated pdfs for data set 2.



(d) Estimated cdfs for data set 2.

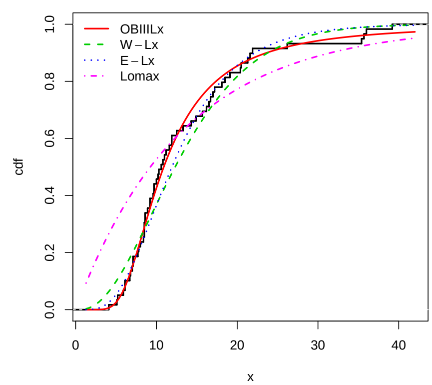


Fig. 5: Estimated pdfs and cdfs for data set 1 and 2.

Table 6: MLEs and their standard errors (in parentheses) and goodness-of-fit statistics for data set 3.

Model	Parameters	MLE	Standard error	Log-Likelihood	AIC	BIC
OBIIILx	β	12.5693	5.6258	-155.2934	318.5868	326.3918
	α	0.1001	0.0427			
	c	0.2225	0.0745			
	k	5.0071	5.3171			
KwLx	a	0.2968	0.2820	-156.2961	320.5922	328.3971
	b	5.4307	5.1608			
	c	1.7368	0.6452			
	d	36.5721	22.4754			
BBIII	a	0.9409	0.7980	-159.0904	326.1808	333.9858
	b	11.2295	36.5423			
	c	0.4002	0.2871			
	k	12.0018	9.2801			
BLx	a	1.7577	0.5587	-159.2478	326.4957	334.3006
	b	5.3584	3.2653			
	c	31.8981	22.7116			
	k	0.4085	0.2375			

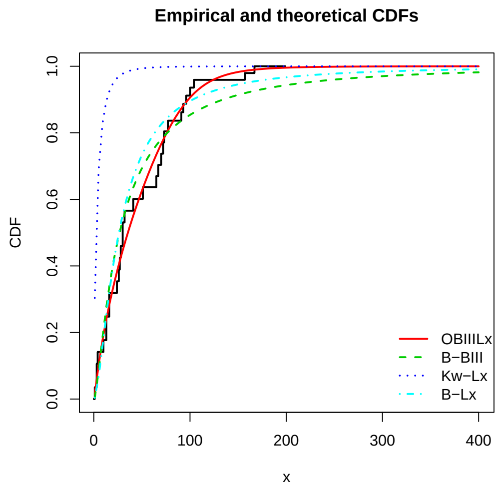


Fig. 6: Plots of estimated cdfs of the models compared in data set 3.

8 Concluding remarks

We proposed a new family of distributions called *OBIII-G family* of distributions. This family can have applications in the fields of reliability, economics, actuaries and survival analysis. Properties of this new family are obtained including quantile function, linear expansion of the density, moments and incomplete moments, moment generating function, entropy, stress-strength reliability parameter and order statistics. Parameter estimation is discussed and a simulation study is performed to investigate the performance of maximum likelihood estimators with other methods. Three real-life data sets were analyzed to assess the performance of a special model odd Burr III Lomax for censored and uncensored data.

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