Journal of Statistics Applications & Probability
An International Journal

http://dx.doi.org/10.18576/jsap/060110

On Estimation of Weibull-Gamma Parameters Based on Hybrid Type-II Censoring Scheme

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Received: 28 Dec. 2016, Revised: 13 Jan. 2017, Accepted: 16 Jan. 2017
Published online: 1 Mar. 2017

Abstract: In this paper point and interval estimations of the parameters of Weibull-Gamma populations based on Type-II hybrid censoring scheme are obtained. The maximum likelihood and Bayes methods are used to obtain point estimations for the distribution parameters. The Bayes estimators cannot be obtained explicitly, hence Lindley's approximation is used to obtain the Bayes estimators. Furthermore, Markov Chain Monte Carlo technique is used to obtain the Bayes estimators and their corresponding credible intervals. The results of Bayes estimators are computed under the squared error loss function. An explanatory example is given to explicate the precision of the estimators.

Keywords: Hybrid Type-II censoring; Bayes estimation; Lindley approximation; maximum likelihood estimation; Markov Chain Monte Carlo technique.

Acronym:

HCS	hybrid censoring scheme	MCMC	Markov Chain Monte Carlo
T-I HCS	Type-I hybrid censoring scheme	SEL	squared error loss
T-II HCS	Type-II hybrid censoring scheme	MLE	maximum likelihood estimate
PDF	probability density function	AFIM	asymptotic Fisher information matrix
CDF	cumulative distribution function	ACI	Approximate confidence interval
WGD	Weibull-Gamma distribution	CRI	Credible Interval
ML	maximum likelihood		

1 Introduction

The process of extrapolating conclusions about population from data is called statistical inference. This extrapolation can be implemented either testing certain hypotheses about population parameters or estimating these parameters. For instance, to make an inference about the life time population of certain electronic units, life testing should be prepared for some units belonging to the whole population. The aim of the life testing is obtaining information from the test units, where knowing these information or data help statisticians to estimate population parameters. The available data in most practical situations are not complete, so statisticians have utilized a lot of censoring schemes to obtain good estimators, such as, Type-I and Type-IIcensoring schemes. HCS is a combination of Type-I and Type-II schemes and it can be elucidated as follows. Suppose n identical units are put to test. The test is terminated when a pre-specified number R out of units are failed, or when a pre-determined time T on the test has been reached. Hence, if $Y_{i:n}$ represents the i-th ordered failure time, then the test may be terminated either at time $T_1 = \min\{Y_{R:n}, T\}$ or at time $T_2 = \max\{Y_{R:n}, T\}$. The time T_1 is the termaination time of an experiment for testing units under T-I HCS. While, T_2 is the termaination time of an experiment for testing units under T-II HCS. Epstein [8] introduced the T-I HCS, and considered lifetime experiments assuming that the lifetime of each unit follows an exponential distribution. Several authors have published on T-I HCS; see for example

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Ebrahimi [6], Gupta and Kundu [9], Childs et al. [4] and Singh et al. [18]. It is noted that under this T-I HCS, little numbers of failures can be occurred up to the pre-fixed time T, which is one of the disadvantages of this censoring scheme. Childs et al. [4] introduced the T-II HCS, which guarantees at least R failures will be occurred. For more details about the merits and the flexibility of T-II HCS see Childs et al. [4] and Banerjee and Kundu [2]. The WGD is appropriate for phenomenon of loss of signals in telecommunications which is called fading when multipath is superimposed on shadowing, see Bithas [3]. A random variable X is said to have WGD, with scale parameter α and two shape parameters θ and β, if its PDF given by:

$$f(x;\alpha,\theta,\beta) = \frac{\theta\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{\theta-1}\left(1+\left(\frac{x}{\alpha}\right)^{\theta}\right)^{-(\beta+1)}, x>0;\alpha,\theta,\beta>0, \tag{1}$$

and the CDF is

$$F(t) = 1-\left(1+\left(\frac{x}{\alpha}\right)^{\theta}\right)^{-\beta}, x>0;\alpha,\theta,\beta>0. \tag{2}$$

For more detials about WGD and its properties see, Molenberghs and Verbeke [15] and Mahmoud et al. [12]. The theme of this paper is to propose the classical and Bayesian estimation procedures for the unknown parameters of WGD under Type-II hybrid censoring scheme. The rest of this paper is organized as follows: in Section 2 the MLEs of the parameters under consideration are obtained in addition to the corresponding ACIs. Section 3 is devoted to the Bayesian approach that uses Lindley approximation and the MCMC technique. An illustrative example is presented to explain the theoretical results in Section 4. Eventually conclusion is inserted in Section 5.

2 Maximum Likelihood Estimation

The log-likelihood functions are the basis for deriving estimators of parameters, given data. ML estimators enjoy with different advantages such as asymptotically normally distributed, asymptotically minimum variance, asymptotically unbiased and satisfy the invariant property, see Azzalini [1] and Royall [17] for more information on likelihood theory. Under T-II HCS, one of the following two types of censored data can be observed:

Case I: $\{Y_{1:n} < \ldots < Y_{R:n}\}$ if $T < Y_{R:n}$.

Case II: $\{Y_{1:n} < \ldots < Y_{R:n} < Y_{R+1:n} < \ldots < Y_{m:n} < T\}$ if $T > Y_{R:n}$ and the m-th failure took place before $T, R \le m \le n$. The likelihood function for the Case I is

$$L_1(\alpha,\theta,\beta|\text{data}) = c_1\frac{\theta^R\beta^R}{\alpha^R}\left(1+\left(\frac{y_{R:n}}{\alpha}\right)^{\theta}\right)^{-\beta(n-R)}\prod_{i=1}^{R}\left(\frac{y_i}{\alpha}\right)^{\theta-1}\left(1+\left(\frac{y_i}{\alpha}\right)^{\theta}\right)^{-(\beta+1)}, \text{if } T<Y_{R:n},$$

where $c_1 = \frac{n!}{(n-R)!}$, while the likelihood function for the Case II is

$$L_2(\alpha,\theta,\beta|\text{data}) = c_2\frac{\theta^m\beta^m}{\alpha^m}\left(1+\left(\frac{T}{\alpha}\right)^{\theta}\right)^{-\beta(n-m)}\prod_{i=1}^{m}\left(\frac{y_i}{\alpha}\right)^{\theta-1}\left(1+\left(\frac{y_i}{\alpha}\right)^{\theta}\right)^{-(\beta+1)}, \text{if } T>Y_{R:n},$$

where $c_2 = \frac{n!}{(n-m)!}$. The two likelihood functions can be combined, and can be written as

$$L(\alpha,\theta,\beta|\text{data}) = c\frac{\theta^H\beta^H}{\alpha^H}\left(1+\left(\frac{u}{\alpha}\right)^{\theta}\right)^{-\beta(n-H)}\prod_{i=1}^{H}\left(\frac{y_i}{\alpha}\right)^{\theta-1}\left(1+\left(\frac{y_i}{\alpha}\right)^{\theta}\right)^{-(\beta+1)}, \tag{3}$$

where $c = \frac{n!}{(n-H)!}$ and H stands for the number of failures; $u = y_{R:n}$ if $H = R$ and $u = T$ if $H > R$. The log-likelihood function may then be written as

$$\log L(\alpha,\theta,\beta|\text{data}) = \log c + H\log\theta + H\log\beta - H\log\alpha - \beta(n-H)\log\left(1+\left(\frac{u}{\alpha}\right)^{\theta}\right) + (\theta-1)\sum_{i=1}^{H}\log\left(\frac{y_i}{\alpha}\right) - (\beta+1)\sum_{i=1}^{H}\log\left(1+\left(\frac{y_i}{\alpha}\right)^{\theta}\right),$$

and thus we have the likelihood equations for α, θ and β respectively, as

$$\frac{\theta\beta(n-H)}{\alpha\left(1+\left(\frac{\alpha}{u}\right)^{\theta}\right)} - \frac{\theta H}{\alpha} + \frac{\theta(\beta+1)}{\alpha}\sum_{i=1}^{H}\frac{1}{\left(1+\left(\frac{\alpha}{y_i}\right)^{\theta}\right)} = 0, \tag{4}$$

$$\frac{H}{\theta}-\frac{\beta\,(n-H)\ln\left(\frac{u}{\alpha}\right)}{\left(1+\left(\frac{\alpha}{u}\right)^{\theta}\right)}+\sum_{i=1}^{H}\log\left(\frac{y_i}{\alpha}\right)-(\beta+1)\sum_{i=1}^{H}\frac{\ln\left(\frac{y_i}{\alpha}\right)}{\left(1+\left(\frac{\alpha}{y_i}\right)^{\theta}\right)}=0, \tag{5}$$

and

$$\frac{H}{\beta}-(n-H)\log\left(1+\left(\frac{u}{\alpha}\right)^{\theta}\right)-\sum_{i=1}^{H}\log\left(1+\left(\frac{y_i}{\alpha}\right)^{\theta}\right)=0. \tag{6}$$

The Equations (4), (5) and (6) are nonlinear simultaneous equations in three unknown vaiables α, θ and β. It is obvious that an exact solution is not easy to get. Therefore, a numerical method such as Newton Raphson can be used to find approximate solution. The steps of Newton Raphson algorithm is described in details in EL-Sagheer [7]. The final estimates of α,θ and β are the MLEs of the parameters, denoted as $\hat{\alpha},\hat{\theta}$ and $\hat{\beta}$.

2.1 Approximate confidence intervals

The $(1-\vartheta)100\%$ ACIs for the parameters α,θ and β can be written as

$$\left.\begin{array}{l}(\hat{\alpha}_L,\hat{\alpha}_U)=\hat{\alpha}\pm z_{1-\frac{\zeta}{2}}\sqrt{var(\hat{\alpha})}\\(\hat{\theta}_L,\hat{\theta}_U)=\hat{\theta}\pm z_{1-\frac{\zeta}{2}}\sqrt{var(\hat{\theta})}\\(\hat{\beta}_L,\hat{\beta}_U)=\hat{\beta}\pm z_{1-\frac{\zeta}{2}}\sqrt{var(\hat{\beta})}\end{array}\right\},$$

where $z_{1-\frac{\vartheta}{2}}$ is the percentile of the standard normal distribution with left-tail probability $1-\frac{\vartheta}{2}$ and $var(\hat{\alpha}),var(\hat{\theta})$ and $var(\hat{\beta})$ represent the asymptotic variances of MLEs which can be calculated using the inverse of the AFIM. Let $I(\Omega_1,\Omega_2,\Omega_3)$ denote the AFIM of the parameters $\Omega_1=\alpha,\ \Omega_2=\theta$ and $\Omega_3=\beta$,

where

$$I(\Omega_1,\Omega_2,\Omega_3)=-\left(\frac{\partial^2\log L}{\partial\Omega_i\partial\Omega_j}\right),i,j=1,2,3.$$

The asymptotic variance–covariance matrix for the maximum likelihood estimates can be put as follows:

$$I^{-1}=\left[-\left(\frac{\partial^2\log L}{\partial\Omega_i\partial\Omega_j}\right)\right]^{-1}_{\downarrow(\hat{\Omega}_1,\hat{\Omega}_2,\hat{\Omega}_3)}, \tag{7}$$

for more details see Cohen [5].

3 Bayesian Estimation

Let the prior knowledge of parameters α,θ and β be described by the following prior distributions :

$$\left.\begin{array}{ll}\pi_1(\alpha)=\frac{\lambda_1^{\mu_1}}{\Gamma(\mu_1)}\alpha^{\mu_1-1}e^{-\lambda_1\alpha},\ \ \alpha>0, & \\ \pi_2(\theta)=\frac{\lambda_2^{\mu_2}}{\Gamma(\mu_2)}\theta^{\mu_2-1}e^{-\lambda_2\theta},\ \ \theta>0, & \mu_1,\mu_2,\lambda_1,\lambda_2,\lambda_3>0,\\ \pi_3(\beta)=\lambda_3e^{-\lambda_3\beta},\ \ \beta>0, & \end{array}\right\} \tag{8}$$

where α,θ and β are independent random variables.

Hence, the joint prior of the parameters α,θ and β can be written as follows:

$$\pi(\alpha,\theta,\beta)=\frac{\lambda_1^{\mu_1}\lambda_2^{\mu_2}\lambda_3}{\Gamma(\mu_1)\Gamma(\mu_2)}\alpha^{\mu_1-1}\,\theta^{\mu_2-1}e^{-(\lambda_1\alpha+\lambda_2\theta+\lambda_3\beta)}. \tag{9}$$

The joint posterior density function of α,θ and β, denoted by $\pi^*(\alpha,\theta,\beta|\text{data})$ can be written as:

$$\pi^*(\alpha,\theta,\beta|\text{data}) = \frac{L(\alpha,\theta,\beta|data)\times\pi(\alpha,\theta,\beta)}{\int_0^\infty\int_0^\infty\int_0^\infty L(\alpha,\theta,\beta|data)\times\pi(\alpha,\theta,\beta)\,d\alpha d\theta d\beta}. \tag{10}$$

The joint posterior distribution combines the information in both prior distributions and the likelihood function. This makes the joint posterior distribution contains more accurate information, and getting a narrower range of possible values for the parameters. The Bayes estimate of any function of the parameters, say $g(\alpha,\theta,\beta)$ using SEL function is

$$\begin{aligned}\hat{g}_{BS}(\alpha,\theta,\beta) &= E_{\alpha,\theta,\beta|\text{data}}\left[g(\alpha,\theta,\beta)\right]\\ &= \frac{\int_0^\infty\int_0^\infty\int_0^\infty g(\alpha,\theta,\beta)\times L(\alpha,\theta,\beta)\times\pi(\alpha,\theta,\beta)\,d\alpha d\theta d\beta}{\int_0^\infty\int_0^\infty\int_0^\infty L(\alpha,\theta,\beta)\times\pi(\alpha,\theta,\beta)\,d\alpha d\theta d\beta}.\end{aligned} \tag{11}$$

While the Bayes estimate of $g(a,b,\theta,\beta)$ using LINEX loss function is

$$\hat{g}_{BL}(\alpha,\theta,\beta) = \frac{-1}{\varepsilon}\log\left[E_{\alpha,\theta,\beta|\text{data}}\left[e^{-\varepsilon\, g(\alpha,\theta,\beta)}\right]\right],\ \varepsilon\neq 0, \tag{12}$$

where

$$E_{\alpha,\theta,\beta|\text{data}}\left[e^{-\varepsilon\, g(\alpha,\theta,\beta)}\right] = \frac{\int_0^\infty\int_0^\infty\int_0^\infty e^{-\varepsilon\, g(\alpha,\theta,\beta)}\times L(\alpha,\theta,\beta)\times\pi(\alpha,\theta,\beta)\,d\alpha d\theta d\beta}{\int_0^\infty\int_0^\infty\int_0^\infty L(\alpha,\theta,\beta)\times\pi(\alpha,\theta,\beta)\,d\alpha d\theta d\beta}. \tag{13}$$

It is noticed that the ratio of two integrals given by (11) and (13) cannot be obtained in a explicit form. In this case Lindley's approximation and MCMC technique can be used to obtain the Bayes estimators for α,θ and β.

3.1 Lindley Approximation

Lindley approximation, which introduced by Lindley [11] can approximate the Bayes estimators into a form containing no integrals. This approximation has been used by a lot of statisticians for obtaining the Bayes estimators for some lifetime distributions; see among others, Sultan et al. [19] and Preda et al [16].

Consider the ratio of integral $I(Y)$, where

$$I(Y) = \frac{\int_{(\alpha,\theta,\beta)} w(\alpha,\theta,\beta)e^{l(\alpha,\theta,\beta)+\rho(\alpha,\theta,\beta)}d(\alpha,\theta,\beta)}{\int_{(\alpha,\theta,\beta)} e^{l(\alpha,\theta,\beta)+\rho(\alpha,\theta,\beta)}d(\alpha,\theta,\beta)}, \tag{14}$$

where $w(\alpha,\theta,\beta)$ is a function of α or θ or β , $l(\alpha,\theta,\beta)$ is the log-likelihood and $\rho(\alpha,\theta,\beta) = \log\pi(\alpha,\theta,\beta)$. If n is sufficiently large, according to Lindley [11], the ratio of the integral of the form $I(Y)$ can be calculated as

$$\begin{aligned}I(Y) &= w(\hat{\alpha},\hat{\theta},\hat{\beta}) + (\hat{w}_1\hat{a}_1+\hat{w}_2\hat{a}_2+\hat{w}_3\hat{a}_3+\hat{a}_4+\hat{a}_5)+\frac{1}{2}[\hat{A}(\hat{w}_1\hat{\sigma}_{11}+\hat{w}_2\hat{\sigma}_{12}+\hat{w}_3\hat{\sigma}_{13})\\ &\quad+\hat{B}(\hat{w}_1\hat{\sigma}_{21}+\hat{w}_2\hat{\sigma}_{22}+\hat{w}_3\hat{\sigma}_{23})+\hat{C}(\hat{w}_1\hat{\sigma}_{31}+\hat{w}_2\hat{\sigma}_{32}+\hat{w}_3\hat{\sigma}_{33})],\end{aligned} \tag{15}$$

where $\hat{\alpha},\hat{\theta}$ and $\hat{\beta}$ are the MLE of α,θ and β, respectively, $\hat{a}_i = \hat{\rho}_1\hat{\sigma}_{i1}+\hat{\rho}_2\hat{\sigma}_{i2}+\hat{\rho}_3\hat{\sigma}_{i3}, i=1,2,3,$
$\hat{a}_4 = \hat{w}_{12}\hat{\sigma}_{12}+\hat{w}_{13}\hat{\sigma}_{13}+\hat{w}_{23}\hat{\sigma}_{23}, \hat{a}_5 = \frac{1}{2}(\hat{w}_{11}\hat{\sigma}_{11}+\hat{w}_{22}\hat{\sigma}_{22}+\hat{w}_{33}\hat{\sigma}_{33}),$

$$\begin{aligned}\hat{A} &= \hat{\sigma}_{11}\hat{l}_{111}+2\hat{\sigma}_{12}\hat{l}_{121}+2\hat{\sigma}_{13}\hat{l}_{131}+2\hat{\sigma}_{23}\hat{l}_{231}+\hat{\sigma}_{22}\hat{l}_{221}+\hat{\sigma}_{33}\hat{l}_{331},\\ \hat{B} &= \hat{\sigma}_{11}\hat{l}_{112}+2\hat{\sigma}_{12}\hat{l}_{122}+2\hat{\sigma}_{13}\hat{l}_{132}+2\hat{\sigma}_{23}\hat{l}_{232}+\hat{\sigma}_{22}\hat{l}_{222}+\hat{\sigma}_{33}\hat{l}_{332},\\ \hat{C} &= \hat{\sigma}_{11}\hat{l}_{113}+2\hat{\sigma}_{12}\hat{l}_{123}+2\hat{\sigma}_{13}\hat{l}_{133}+2\hat{\sigma}_{23}\hat{l}_{233}+\hat{\sigma}_{22}\hat{l}_{223}+\hat{\sigma}_{33}\hat{l}_{333},\end{aligned}$$

, subscripts 1,2,3 on the rigth-hand sides stand for α,θ and β, respectively,
$\hat{\rho}_i = \left(\frac{\partial\rho}{\partial\Omega_i}\right)_{\downarrow(\hat{\Omega}_1,\hat{\Omega}_2,\hat{\Omega}_3)}, i=1,2,3,\ \Omega_1=\alpha,\ \Omega_2=\theta$ and $\Omega_3=\beta,$
$\hat{w}_{ij} = \left(\frac{\partial^2 w(\Omega_1,\Omega_2,\Omega_3)}{\partial\Omega_i\partial\Omega_j}\right)_{\downarrow(\hat{\Omega}_1,\hat{\Omega}_2,\hat{\Omega}_3)}, i,j=1,2,3,\ \hat{l}_{ij} = \left(\frac{\partial^2 L(\Omega_1,\Omega_2,\Omega_3)}{\partial\Omega_i\partial\Omega_j}\right)_{\downarrow(\hat{\Omega}_1,\hat{\Omega}_2,\hat{\Omega}_3)}, i,j=1,2,3,$

$\hat{l}_{ijk} = \left(\frac{\partial^3 L(\Omega_1,\Omega_2,\Omega_3)}{\partial\Omega_i\partial\Omega_j\partial\Omega_k}\right)_{\downarrow(\hat{\Omega}_1,\hat{\Omega}_2,\hat{\Omega}_3)}, i,j,k = 1,2,3$ and $\hat{\sigma}_{ij} = -\frac{1}{\hat{l}_{ij}}, i,j = 1,2,3.$

From (9), $\rho(\alpha,\theta,\beta)$ can be written as:

$$\rho(\alpha,\theta,\beta) = \mu_1 \ln\lambda_1 + \mu_2 \ln\lambda_2 + \ln\lambda_3 - \ln\Gamma(\mu_1) - \ln\Gamma(\mu_2) + (\mu_1 - 1)\ln\alpha + (\mu_2 - 1)\ln\theta - (\lambda_1\alpha + \lambda_2\theta + \lambda_3\beta).$$

and then we obtain

$$\hat{\rho}_1 = \frac{(\mu_1 - 1)}{\hat{\alpha}} - \lambda_1, \hat{\rho}_2 = \frac{(\mu_2 - 1)}{\hat{\theta}} - \lambda_2 \text{ and } \hat{\rho}_3 = -\lambda_3.$$

If $w(\hat{\alpha},\hat{\theta},\hat{\beta}) = \hat{\alpha}$ then the Bayes estimate of the parameter α under SEL function from (15) is

$$\hat{\alpha}_{BLind-SEL} = \hat{\alpha} + (\hat{w}_1\hat{a}_1 + \hat{w}_2\hat{a}_2 + \hat{w}_3\hat{a}_3 + \hat{a}_4 + \hat{a}_5) + \frac{1}{2}[\hat{A}(\hat{w}_1\hat{\sigma}_{11} + \hat{w}_2\hat{\sigma}_{12} + \hat{w}_3\hat{\sigma}_{13}) + \hat{B}(\hat{w}_1\hat{\sigma}_{21} + \hat{w}_2\hat{\sigma}_{22} + \hat{w}_3\hat{\sigma}_{23}) + \hat{C}(\hat{w}_1\hat{\sigma}_{31} + \hat{w}_2\hat{\sigma}_{32} + \hat{w}_3\hat{\sigma}_{33})],$$

while, if $w(\hat{\alpha},\hat{\theta},\hat{\beta}) = \hat{\theta}$ then the Bayes estimate of the parameter θ under SEL function is

$$\hat{\theta}_{BLind-SEL} = \hat{\theta} + (\hat{w}_1\hat{a}_1 + \hat{w}_2\hat{a}_2 + \hat{w}_3\hat{a}_3 + \hat{a}_4 + \hat{a}_5) + \frac{1}{2}[\hat{A}(\hat{w}_1\hat{\sigma}_{11} + \hat{w}_2\hat{\sigma}_{12} + \hat{w}_3\hat{\sigma}_{13}) + \hat{B}(\hat{w}_1\hat{\sigma}_{21} + \hat{w}_2\hat{\sigma}_{22} + \hat{w}_3\hat{\sigma}_{23}) + \hat{C}(\hat{w}_1\hat{\sigma}_{31} + \hat{w}_2\hat{\sigma}_{32} + \hat{w}_3\hat{\sigma}_{33})],$$

and if $w(\hat{\alpha},\hat{\theta},\hat{\beta}) = \hat{\beta}$ then the Bayes estimate of the parameter β under SEL function is

$$\hat{\beta}_{BLind-SEL} = \hat{\beta} + (\hat{w}_1\hat{a}_1 + \hat{w}_2\hat{a}_2 + \hat{w}_3\hat{a}_3 + \hat{a}_4 + \hat{a}_5) + \frac{1}{2}[\hat{A}(\hat{w}_1\hat{\sigma}_{11} + \hat{w}_2\hat{\sigma}_{12} + \hat{w}_3\hat{\sigma}_{13}) + \hat{B}(\hat{w}_1\hat{\sigma}_{21} + \hat{w}_2\hat{\sigma}_{22} + \hat{w}_3\hat{\sigma}_{23}) + \hat{C}(\hat{w}_1\hat{\sigma}_{31} + \hat{w}_2\hat{\sigma}_{32} + \hat{w}_3\hat{\sigma}_{33})].$$

If $w(\hat{\alpha},\hat{\theta},\hat{\beta}) = e^{-\varepsilon\hat{\alpha}}$ then the Bayes estimate of the parameter α under LINEX loss function from (15) is

$$\hat{\alpha}_{BLind-LINEX} = e^{-\varepsilon\hat{\alpha}} + (\hat{w}_1\hat{a}_1 + \hat{w}_2\hat{a}_2 + \hat{w}_3\hat{a}_3 + \hat{a}_4 + \hat{a}_5) + \frac{1}{2}[\hat{A}(\hat{w}_1\hat{\sigma}_{11} + \hat{w}_2\hat{\sigma}_{12} + \hat{w}_3\hat{\sigma}_{13}) + \hat{B}(\hat{w}_1\hat{\sigma}_{21} + \hat{w}_2\hat{\sigma}_{22} + \hat{w}_3\hat{\sigma}_{23}) + \hat{C}(\hat{w}_1\hat{\sigma}_{31} + \hat{w}_2\hat{\sigma}_{32} + \hat{w}_3\hat{\sigma}_{33})],$$

while, if $w(\hat{\alpha},\hat{\theta},\hat{\beta}) = e^{-\varepsilon\hat{\theta}}$ then the Bayes estimate of the parameter θ under LINEX loss functionis

$$\hat{\theta}_{BLind-LINEX} = e^{-\varepsilon\hat{\theta}} + (\hat{w}_1\hat{a}_1 + \hat{w}_2\hat{a}_2 + \hat{w}_3\hat{a}_3 + \hat{a}_4 + \hat{a}_5) + \frac{1}{2}[\hat{A}(\hat{w}_1\hat{\sigma}_{11} + \hat{w}_2\hat{\sigma}_{12} + \hat{w}_3\hat{\sigma}_{13}) + \hat{B}(\hat{w}_1\hat{\sigma}_{21} + \hat{w}_2\hat{\sigma}_{22} + \hat{w}_3\hat{\sigma}_{23}) + \hat{C}(\hat{w}_1\hat{\sigma}_{31} + \hat{w}_2\hat{\sigma}_{32} + \hat{w}_3\hat{\sigma}_{33})],$$

and if $w(\hat{\alpha},\hat{\theta},\hat{\beta}) = e^{-\varepsilon\hat{\beta}}$ then the Bayes estimate of the parameter β under LINEX loss function is

$$\hat{\beta}_{BLind-LINEX} = e^{-\varepsilon\hat{\beta}} + (\hat{w}_1\hat{a}_1 + \hat{w}_2\hat{a}_2 + \hat{w}_3\hat{a}_3 + \hat{a}_4 + \hat{a}_5) + \frac{1}{2}[\hat{A}(\hat{w}_1\hat{\sigma}_{11} + \hat{w}_2\hat{\sigma}_{12} + \hat{w}_3\hat{\sigma}_{13}) + \hat{B}(\hat{w}_1\hat{\sigma}_{21} + \hat{w}_2\hat{\sigma}_{22} + \hat{w}_3\hat{\sigma}_{23}) + \hat{C}(\hat{w}_1\hat{\sigma}_{31} + \hat{w}_2\hat{\sigma}_{32} + \hat{w}_3\hat{\sigma}_{33})].$$

3.2 MCMC Technique

The main goal of the MCMC technique is to compute an approximate value of integrals in (11). A lot of papers dealt with MCMC technique such as, EL-Sagheer [7], Mahmoud et al. [13] and among others. An important sup-class of MCMC methods are Gibbs sampling and more general Metropolis within-Gibbs samplers. The Metropolis algorithm is a random walk that uses an acceptance/rejection rule to converge to the target distribution. The Metropolis algorithm was first proposed in Metropolis et al. [14] and it was then generalized by Hastings in Hastings [10]. From (3), (9) and (10), the joint posterior density function of α,θ and β can be written as:

$$\pi^*(\alpha,\theta,\beta|\text{data}) \propto \alpha^{\mu_1-H-1}\,\theta^{H+\mu_2-1}\beta^H\left(1+\left(\frac{u}{\alpha}\right)^{\theta}\right)^{-\beta(n-H)}\times e^{-(\lambda_1\alpha+\lambda_2\theta+\lambda_3\beta)}$$
$$\times\prod_{i=1}^{H}\left\{\left(\frac{y_i}{\alpha}\right)^{\theta-1}\left(1+\left(\frac{y_i}{\alpha}\right)^{\theta}\right)^{-(\beta+1)}\right\}. \quad (16)$$

The conditional posterior densities of α,θ and β can also be written as:

$$\pi_1^*(\alpha|\theta,\beta,\text{data}) \propto \alpha^{\mu_1-H-1}e^{-\lambda_1\alpha}\times$$
$$\exp\left[-\left\{(\theta-1)\sum_{i=1}^{H}[\log\alpha]+(\beta+1)\sum_{i=1}^{H}\left[\log\left(1+\left(\frac{y_i}{\alpha}\right)^{\theta}\right)\right]+\beta\,(n-H)\log\left(1+\left(\frac{u}{\alpha}\right)^{\theta}\right)\right\}\right], \quad (17)$$

$$\pi_2^*(\theta|\alpha,\beta,\text{data}) \propto \theta^{H+\mu_2-1}e^{-\lambda_2\theta}\times\left(1+\left(\frac{u}{\alpha}\right)^{\theta}\right)^{-\beta(n-H)}$$
$$\times\prod_{i=1}^{H}\left\{\left(\frac{y_i}{\alpha}\right)^{\theta-1}\left(1+\left(\frac{y_i}{\alpha}\right)^{\theta}\right)^{-(\beta+1)}\right\}, \quad (18)$$

and

$$\pi_3^*(\beta|\alpha,\theta,\text{data}) \equiv gamma\left[H+1,\lambda_3+(n-H)\log\left(1+\left(\frac{u}{\alpha}\right)^{\theta}\right)+\sum_{i=1}^{H}\log\left(1+\left(\frac{y_i}{\alpha}\right)^{\theta}\right)\right]. \quad (19)$$

Now, the following steps illustrate the method of the Metropolis–Hastings algorithm within Gibbs sampling to generate the posterior samples as suggested by Tierney [20], and so the Bayes estimates and the corresponding credible intervals can be obtained:

(1)Start with an $\left(\alpha^{(0)}=\hat{\alpha},\ \theta^{(0)}=\hat{\theta}\text{ and }\beta^{(0)}=\hat{\beta}\right)$.
(2)Put $j=1$.
(3)Generate $\beta^{(j)}$ from

$$gamma\left[H+1,\lambda_3+(n-H)\log\left(1+\left(\frac{u}{\alpha^{(j-1)}}\right)^{\theta^{(j-1)}}\right)+\sum_{i=1}^{H}\log\left(1+\left(\frac{y_i}{\alpha^{(j-1)}}\right)^{\theta^{(j-1)}}\right)\right]$$

(4)Using the following Metropolis-Hastings method, generate $\alpha^{(j-1)}$ and $\theta^{(j-1)}$ from (17) and (18) with the suggested normal distributions

$$N(\alpha^{(j-1)},var(\alpha))\text{ and }N(\theta^{(j-1)},var(\theta)),\text{respectively,}$$

where $var(\alpha)$ and $var(\theta)$ can be obtained from the main diagonal in asymptotic inverse Fisher information matrix (7).
i-Generate a proposal α^* from $N(\alpha^{(j-1)},var(\alpha))$ and θ^* from $N(\theta^{(j-1)},var(\theta))$.
ii-Evaluate the acceptance probabilities

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$$\left.\begin{aligned}\rho_\alpha &= \min\left[1,\frac{\pi_1^*(\alpha^*|\theta^{(j-1)},\beta^{(j)},\text{data})}{\pi_1^*(\alpha^{(j-1)}|\theta^{(j-1)},\beta^{(j)},\text{data})}\right],\\ \rho_\theta &= \min\left[1,\frac{\pi_2^*(\theta^*|\alpha^{(j)},\beta^{(j)},\text{data})}{\pi_2^*(\theta^{(j-1)}|\alpha^{(j)},\beta^{(j)},\text{data})}\right]\end{aligned}\right\}.$$

iii-Generate u_1 and u_2 from a Uniform $(0,1)$ distribution.
iv-If $u_1\le\rho_\alpha$, then accept the proposal and set $\alpha^{(j)}=\alpha^*$, else set $\alpha^{(j)}=\alpha^{(j-1)}$.
v-If $u_2\le\rho_\theta$, then accept the proposal and set $\theta^{(j)}=\theta^*$, else set $\theta^{(j)}=\theta^{(j-1)}$.
(5)Compute $\alpha^{(j)}$ and $\theta^{(j)}$.

(6)Put $j = j + 1$.
(7)Repeat Steps $3-6$ Q times.
(8)In order to guarantee the convergence and to remove the influence of the selection of initial values, the first M simulated varieties are ignored. The selected samples are $\alpha^{(j)}$ and $\theta^{(j)}$, $j = M+1,...,Q$, for sufficiently large Q. The approximate Bayes estimates for α, θ and β based on SEL are

$$\left.\begin{aligned} \alpha_{BMC-SEL} &= \tfrac{1}{Q-M}\textstyle\sum_{j=M+1}^{Q}\alpha^{(j)}, \\ \theta_{BMC-SEL} &= \tfrac{1}{Q-M}\textstyle\sum_{j=M+1}^{Q}\theta^{(j)}, \\ \beta_{BMC-SEL} &= \tfrac{1}{Q-M}\textstyle\sum_{j=M+1}^{Q}\beta^{(j)}. \end{aligned}\right\},$$

and the estimates for the aforementioned parameters under LINEX loss function are:

$$\left.\begin{aligned} \alpha_{BMC-LINEX} &= \tfrac{-1}{\varepsilon}\left[\tfrac{1}{Q-M}\textstyle\sum_{j=M+1}^{Q}e^{-\varepsilon\,\alpha^{(j)}}\right], \\ \theta_{BMC-LINEX} &= \tfrac{-1}{\varepsilon}\left[\tfrac{1}{Q-M}\textstyle\sum_{j=M+1}^{Q}e^{-\varepsilon\,\theta^{(j)}}\right], \\ \beta_{BMC-LINEX} &= \tfrac{-1}{\varepsilon}\left[\tfrac{1}{Q-M}\textstyle\sum_{j=M+1}^{Q}e^{-\varepsilon\,\beta^{(j)}}\right] \end{aligned}\right\}. \tag{20}$$

(9)To calculate the CRIs of Ω_j where $\Omega_1 = \alpha$, $\Omega_2 = \theta$ and $\Omega_3 = \beta$, we let the quantiles of the sample be the endpoints of the intervals. Sort $\left\{\Omega_j^{M+1}, \Omega_j^{M+2}, ..., \Omega_j^{Q}\right\}$ as $\left\{\Omega_j^{(1)}, \Omega_j^{(2)}, ..., \Omega_j^{(Q-M)}\right\}$, $j = 1,2,3$. Hence the 100 $(1-\vartheta)\%$ symmetric credible interval of Ω_j is given by

$$\left[\Omega_j\left(\tfrac{\vartheta}{2}(Q-M)\right),\ \Omega_j\left(\left(1-\tfrac{\vartheta}{2}\right)(Q-M)\right)\right].$$

4 Application

In this section, a simulation example is presented to assess the estimation procedures. In this example, hybrid Type-II censored sample is generated from WGD as the following:

(1)Specify the values of n.
(2)Specify the values of the parameters α, θ and β to generate a sample from WGD.
(3)For given values of R and T, calculate the number of failures H.
(4)Compute the MLEs of the model parameters. The Newton–Raphson method is applied for solving the nonlinear system to obtain the MLEs of the parameters.
(5)Compute the Bayes estimates of the parameters based on Lindley approximation and MCMC algorithm described in Section 3.

A simulation data for hybrid Type-II censored sample from WGD is generated with true values $\alpha = 2$, $\theta = 3$ and $\beta = 1.5$ at $n = 40$. The pre-specified number R is planned to equal 15 and $T = 2.5$. According to the previous assumptions, it was found that $H = 32$. The hybrid Type-II censored data has been presented in Table 1 as follows:

Table 1. Hybrid Type-II censored data

0.3603	0.9036	1.1096	1.2783	1.4564	1.6773	1.9478	2.3577
0.5337	0.9929	1.1327	1.2962	1.4783	1.7015	2.0944	2.3798
0.6889	1.0149	1.2733	1.3409	1.5735	1.7475	2.1017	2.4101
0.7872	1.0434	1.2763	1.4106	1.5855	1.8687	2.1339	2.4545

The different point estimates for α, θ and β in case of non-Bayesian and Bayesian estimation, are presented in Table 2, where $Q = 22000$ and $M = 2000$ in the MCMC technique and the prior knowledge parameters $\mu_1, \mu_2, \lambda_1, \lambda_2$ and λ_3 are the same and chosen to equal 0.001 .

Table 2. Different point estimates for α, θ and β.

	$(.)_{ML}$	$(.)_{BLind-SEL}$	$(.)_{BLind-LINEX}$			$(.)_{BMC-SEL}$	$(.)_{BMC-LINEX}$		
			$\varepsilon = 0.0001$	$\varepsilon = -2$	$\varepsilon = 2$		$\varepsilon = 0.0001$	$\varepsilon = -2$	$\varepsilon = 2$
α	1.9284	1.9369	1.9369	1.9625	1.9105	1.953	1.953	1.9531	1.953
θ	2.9533	2.9001	2.9001	3.0681	2.7604	2.9539	2.9539	2.954	2.9539
β	1.3769	1.3917	1.3917	1.446	1.3343	1.4549	1.4549	1.5242	1.3936

Table 3. 95% confidence intervals for α, θ and β.

Method	α	Length	θ	Length	β	Length
ACI	$[-0.602, 4.459]$	5.06019	$[1.1366, 4.770]$	3.63347	$[-2.3168, 5.071]$	7.38732
CRI	$[1.933, 1.9700]$	0.03698	$[2.9424, 2.965]$	0.02216	$[1.0000, 1.9925]$	0.99249

5 Conclusion

In this paper, the estimation of WGD parameters has been studied under hybrid Type-II censored data. The MLEs of the parameters are calculated. The importance of Lindley approximation and MCMC technique were noticeable in Bayesian estimation. A comparison between the ACIs and the CRIs is provided for the estimated parameters through a simulated example. It was found that the width of MCMC credible intervals is narrower than ACIs. We may judge that the Bayes estimators obtained under Lindley or MCMC method can be preferred.

Acknowledgments

The authors would like to express thanks to the editors and referees for their valuable comments and suggestions which significantly improved the paper.

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