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Positive continuous solution of a quadratic integral equation of fractional orders

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Abstract: We are concerned here with the existence of a unique positive continuous solution for the quadratic integral of fractional orders $x(t)=a(t)+\lambda\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s,x(s))ds.\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f_2(s,x(s))ds, \quad t\in I$ where f_1 and f_2 are Carathéodory functions. As an application the Cauchy problems of fractional order differential equation $*D^\alpha\sqrt{x(t)} = f(t,x(t)),\ t>0$ with one of the two initial values $x(0)=0$ or $I^{1-\alpha}\sqrt{x(t)}=0$ will be studied. Some examples are considered as applications of our results.

Keywords: Quadratic integral equation of fractional-order, Banach fixed point theorem, Cauchy problem.

1. Introduction

Fractional differential and integral equations have received increasing attention during recent years due to its applications in numerous diverse fields of science and engineering. In fact, fractional differential and integral equations are considered as models alternative to nonlinear differential equations [20]. There has been a significant development in fractional differential equations. We refer readers to the monographs of Kilbas et al. [17] and the papers [1]- [18]. Quadratic integral equations are often applicable in the theory of radiative transfer, the kinetic theory of gases, the theory of neutron transport, the queuing theory and the traffic theory. Many authors studied the existence of solutions for several classes of nonlinear quadratic integral equations (see e.g. [2], [7] and [9]-[15]. However, in most of the above literature, the main results are realized with the help of the technique associated with the measure of noncompactness. Instead of using the technique of measure of noncompactness we use the Banach contraction fixed point Theorem.

Let $I=[0,T]$, $C=C[0,T]$ be the space of continuous functions on I, and $L^1=L^1[0,T]$ be the space of Lebesgue integrable functions on I.

Firstly, we deal with the quadratic integral equation of fractional order

$$x(t)=a(t)+\lambda\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s,x(s))ds.\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f_2(s,x(s))ds, \quad t\in I \tag{1}$$

Where λ is a real number.
Also, Eqn. (1) can be written in operator form as:

$$x(t) = a(t) + \lambda(I^{\alpha} f_1(t, x(t)))\cdot(I^{\beta} f_2(t, x(t))), \quad t \in I$$

where $\alpha, \beta \in (0,1]$.

We prove the existence and uniqueness of positive continuous solution of (1).

As an application of our results, the existence of a unique solution of the initial value problems

$$\begin{aligned} &{}_*D^{\alpha}\sqrt{x(t)} = f(t, x(t)),\ t > 0 \\ &x(0)=0. \end{aligned} \tag{2}$$

and

$$\begin{aligned} &{}_*D^{\alpha}\sqrt{x(t)} = f(t, x(t)),\ t > 0 \\ &I^{1-\alpha}\sqrt{x(t)} = 0. \end{aligned} \tag{3}$$

will be studied.

Let α, β be two positive real numbers, then the definition of the fractional (arbitrary) order integration is given by :

Definition 1.1 *Let* $f(t) \in L^1,\ \beta \in R^+$ *. The fractional (arbitrary) order integral of the function* $f(t)$ *of order* β *is defined as (see [19], and [20])*

$$I_a^{\beta} f(t) = \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s)\,ds.$$

When $a=0$ we can write $I^{\beta} f(t) = I_0^{\beta} f(t)$.

Definition 1.2 .*The Riemann-Liouville fractional-order derivative of* $f(t)$ *of order* $\alpha \in (0,1)$ *is defined as (see [11], [19])*

$${}_*D_a^{\alpha} f(t) = \frac{d}{dt}\int_a^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f(s)\,ds$$

or

$${}_*D_a^{\alpha} f(t) = \frac{d}{dt} I_a^{1-\alpha} f(t).$$

2 Main Theorem

Consider the following assumptions:

(i) $a : I \to R_+$ is continuous on I.

(ii) $f_i : I \times R_+ \to R_+$ are measurable in t for all $x \in R_+$, and satisfy the Lipschitz condition with respect to the second argument x for almost all $t \in I$.

i.e

$$| f_i(t, x) - f_i(t, y) | \le L_i\, | x - y |, \quad L_i > 0\ \ i = 1,2. \tag{4}$$

for each $(t, x), (t, y) \in I \times R_+$.

(iii) There exist two functions $m_i \in L^1$ such that

$$| f_i(t,x) | \leq m_i(t), \forall t \in I,\ i = 1,2.$$

Let $\gamma < \max\{\alpha, \beta\}$, and $M = \max\{I^{\gamma} m_i(t) : t \in I, \gamma < \alpha\}, i = 1,2.$

Now, for the existence of a unique continuous positive solution of the quadratic integral equation (1) we have the following theorem.

Theorem 2.1 *Let the assumptions (i)-(iii) be satisfied. Moreover, if*

$$\frac{|\lambda| L_2 M T^{\alpha-\gamma+\beta}}{\Gamma(\alpha-\gamma+1)\Gamma(\beta+1)} + \frac{|\lambda| L_1 M T^{\beta-\gamma+\alpha}}{\Gamma(\beta-\gamma+1)\Gamma(\alpha+1)} < 1.$$

Then the quadratic integral equation (1) has a unique positive continuous solution $x \in C$.

Proof.
Equation (1) can be written as

$$x(t) = a(t) + I^{\alpha-\gamma}(I^{\gamma} f_1(t, x(t))) \cdot I^{\beta-\gamma}(I^{\gamma} f_2(t, x(t))).$$

Define the operator F by:

$$Fx(t) = a(t) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s, x(s)) ds \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f_2(s, x(s)) ds$$

The operator F maps C into itself. For this we have, let $t_1, t_2 \in I, t_1 < t_2$ such that $| t_2 - t_1 | \leq \delta$, then

$$\begin{aligned}
&| Fx(t_2) - Fx(t_1) | = | a(t_2) - a(t_1) \\
&+\lambda \int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s, x(s)) ds \int_0^{t_2} \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)} f_2(s, x(s)) ds \\
&-\lambda \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s, x(s)) ds \int_0^{t_1} \frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)} f_2(s, x(s)) ds | \\
&= | a(t_2) - a(t_1) \\
&\quad +\lambda \left(\int_0^{t_1} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s, x(s)) ds + \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s, x(s)) ds \right) \\
&\quad \cdot \left(\int_0^{t_1} \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)} f_2(s, x(s)) ds + \int_{t_1}^{t_2} \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)} f_2(s, x(s)) ds \right) \\
&\quad -\lambda \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s, x(s)) ds \int_0^{t_1} \frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)} f_2(s, x(s)) ds | \\
&\quad \leq | a(t_2) - a(t_1) |
\end{aligned}$$

$$+|\lambda|\int_0^{t_1}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}|f_1(s,x(s))|ds\cdot\int_{t_1}^{t_2}\frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)}|f_2(s,x(s))|ds$$

$$+|\lambda|\int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}|f_1(s,x(s))|ds\cdot\int_0^{t_1}\frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)}|f_2(s,x(s))|ds$$

$$+|\lambda|\int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}|f_1(s,x(s))|ds\cdot\int_{t_1}^{t_2}\frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)}|f_2(s,x(s))|ds$$

$$\leq|a(t_2)-a(t_1)|$$

$$+|\lambda|\int_0^{t_1}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}m_1(s)ds\cdot\int_{t_1}^{t_2}\frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)}m_2(s)ds$$

$$+|\lambda|\int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}m_1(s)ds\cdot\int_0^{t_1}\frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)}m_2(s)ds$$

$$+|\lambda|\int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}m_1(s)ds\cdot\int_{t_1}^{t_2}\frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)}m_2(s)ds$$

$$\leq|a(t_2)-a(t_1)|$$

$$+|\lambda|M^2\int_0^{t_1}\frac{(t_2-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}ds\cdot\int_{t_1}^{t_2}\frac{(t_2-s)^{\beta-\gamma-1}}{\Gamma(\beta-\gamma)}ds$$

$$+|\lambda|M^2\int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}ds\cdot\int_0^{t_1}\frac{(t_2-s)^{\beta-\gamma-1}}{\Gamma(\beta-\gamma)}ds$$

$$+|\lambda|M^2\int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}ds\cdot\int_{t_1}^{t_2}\frac{(t_2-s)^{\beta-\gamma-1}}{\Gamma(\beta-\gamma)}ds$$

$$\leq|a(t_2)-a(t_1)|$$

$$+|\lambda|M^2[\frac{T^{\alpha-\gamma}-(t_2-t_1)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}][\frac{(t_2-t_1)^{\beta-\gamma}}{\Gamma(\beta-\gamma+1)}]$$

$$+|\lambda|M^2[\frac{(t_2-t_1)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}][\frac{T^{\beta-\gamma}-(t_2-t_1)^{\beta-\gamma}}{\Gamma(\beta-\gamma+1)}]$$

$$+|\lambda|M^2[\frac{(t_2-t_1)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}][\frac{(t_2-t_1)^{\beta-\gamma}}{\Gamma(\beta-\gamma+1)}].$$

Which proves that $F:C\rightarrow C$. Now, to show that F is contraction. Let $x,y\in C$, then we have

$$|Fx(t)-Fy(t)|=|\lambda\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f_1(s,x(s))ds\int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}f_2(s,x(s))ds$$

$$-\lambda\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f_1(s,y(s))ds\int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}f_2(s,y(s))ds|$$

$$=|\lambda\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f_1(s,x(s))ds\int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}f_2(s,x(s))ds$$

$$-\lambda\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s,y(s))ds \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f_2(s,y(s))ds$$
$$+\lambda\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s,x(s))ds \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f_2(s,y(s))ds$$
$$-\lambda\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s,x(s))ds \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f_2(s,y(s))ds \,|$$
$$\leq |\lambda| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} | f_1(s,x(s))|ds \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} | f_2(s,x(s)) - f_2(s,y(s))|ds$$
$$+|\lambda| \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} | f_2(s,y(s))|ds \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} | f_1(s,x(s)) - f_2(s,x(s))|ds$$
$$\leq |\lambda| L_2 \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} m_1(s)ds \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} |x(s)-y(s)|\,ds$$
$$+|\lambda| L_1 \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} m_2(s)ds \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |x(s)-y(s)|\,ds$$
$$\leq |\lambda| L_2 M \frac{T^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} \| x-y\| \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} ds$$
$$+|\lambda| L_1 M \frac{T^{\beta-\gamma}}{\Gamma(\beta-\gamma+1)} \| x-y\| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds$$
$$\leq |\lambda| L_2 M \frac{T^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} \| x-y\| \frac{T^{\beta}}{\Gamma(\beta+1)}$$
$$+|\lambda| L_1 M \frac{T^{\beta-\gamma}}{\Gamma(\beta-\gamma+1)} \| x-y\| \frac{T^{\alpha}}{\Gamma(\alpha+1)}$$
$$\leq \left[\frac{|\lambda| L_2 M\, T^{\alpha-\gamma+\beta}}{\Gamma(\alpha-\gamma+1)\Gamma(\beta+1)} + \frac{|\lambda| L_1 M\, T^{\beta-\gamma+\alpha}}{\Gamma(\beta-\gamma+1)\Gamma(\alpha+1)}\right] \| x-y\|$$

Since

$$\frac{|\lambda| L_2 M\, T^{\alpha-\gamma+\beta}}{\Gamma(\alpha-\gamma+1)\Gamma(\beta+1)} + \frac{|\lambda| L_1 M\, T^{\beta-\gamma+\alpha}}{\Gamma(\beta-\gamma+1)\Gamma(\alpha+1)} < 1.$$

Then F is contraction. Therefore, by the Banach contraction fixed point Theorem [16], the operator F has a unique fixed point $x \in C$ (i.e. the quadratic integral equation (1) has a unique solution $x \in C$). which completes the proof.

As particular cases of Theorem 2.1 we have the following corollaries.

Corollary 2.2 *Let the assumptions (i) and (iii) be satisfied. If* $f_i : [0,T] \times R_+ \to R_+, i = 1,2$ *are continuous and satisfy Lipschitz condition (4), then the quadratic integral equation (1) has a unique continuous solution* $x \in C$.

Corollary 2.3 *Let the assumptions of Theorem 2.1 be satisfied (with* α, β *and* $\gamma \to 1, \lambda = 1$ *). If* $2LMT < 1,\ L = \max\{L_1, L_2\}$, *then the quadratic integral equation*

$$x(t) = a(t) + \int_0^t f_1(s, x(s))ds . \int_0^t f_2(s, x(s))ds$$

has a unique continuous solution $x \in C$,
which is the same equation studied in [11].

Corollary 2.4 *Let the assumptions of Theorem 2.1 be satisfied (with* $f_1 = f_2 = f$ *and* $\alpha = \beta$ *), then the quadratic integral equation*

$$x(t) = (I^\alpha f(t, x(t)))^2 \tag{5}$$

has a unique continuous solution $x \in C$.

Corollary 2.5 *Let the assumptions of Theorem 2.1 be satisfied (with* $\alpha \to 0, f_1 = 1, f_2 = f$ *), then the integral equation*

$$x(t) = a(t) + I^\beta f(t, x(t)), \quad t \in I \tag{6}$$

has a unique continuous solution $x \in C$.

Proof. Let $f_1 = 1, f_2 = f$ in (1), and taking the limit as $\alpha \to 0$, we have

$$\lim_{\alpha \to 0} x(t) = a(t) + \lim_{\alpha \to 0} (I^\alpha 1) \cdot (I^\beta f(t, x(t))), \quad t \in I$$
$$x(t) = a(t) + I^\beta f(t, x(t)), \quad t \in I.$$

3 Fractional order differential equations

Lemma 3.1 *Let*

(i^*) $f : I \times R_+ \to R_+$, be measurable in $t \in I$ for any $x \in R_+$, and continuous in $x \in R_+$ for almost all $t \in I$.
(ii^*) There exists an integrable function $m \in L^1$, such that $|f(t, x)| \le m(t)$,
then

$$I^\alpha f(t, x)|_{t=0} = 0.$$

Proof: Let $0 < \gamma < \alpha$, let $M = \max I^\gamma m(t)$.
Now

$$I^\alpha f(t, x) = I^{\alpha - \gamma}(I^\gamma f(t, x)),$$

then

$$0 \le I^\alpha f(t, x)|_{t=0} \le I^{\alpha - \gamma} M\,|_{t=0} = M \frac{t^{\alpha - \gamma}}{\Gamma(1 + \alpha - \gamma)}|_{t=0} = 0$$

this implies that

$$I^\alpha f(t, x)|_{t=0} = 0.$$

We shall prove the following corollary.

Corollary 3.2 *Let the assumptions* (i^*) *and* (ii^*) *be satisfied. If*

$$| f(t,x) - f(t,y) | \le L\, | x - y |, \quad L > 0, \ (t,x) \in I \times R_+ .$$

Then the initial value problem (2) has a unique continuous solution $x \in C$.

Proof. Equation (2) can be written as

$$\frac{d}{dt} I^{1-\alpha} \sqrt{x(t)} = f(t, x(t)) \,.$$

By integrating both sides, we obtain

$$I^{1-\alpha} \sqrt{x(t)} = If(t, x(t)) + c \,,$$

operating by I^{α} on both sides, we get

$$I \sqrt{x(t)} = I^{1+\alpha} f(t, x(t)) + \frac{t^{\alpha}}{\Gamma(1+\alpha)} c \,, \tag{7}$$

differentiating equation (7), we get

$$\sqrt{x(t)} = I^{\alpha} f(t, x(t)) + \alpha \frac{t^{\alpha-1}}{\Gamma(1+\alpha)} c \,,$$

letting t=0, then by Lemma 3.1 we deduce that c=0, then

$$\sqrt{x}(t) = (I^{\alpha} f(t, x(t)))$$

and we obtain equation (5).
Conversely, let

$$x(t) = (I^{\alpha} f(t, x(t)))^2 \,,$$

then

$$\sqrt{x(t)} = I^{\alpha} f(t, x(t))$$

operating by $I^{1-\alpha}$ to both sides, we get

$$I^{1-\alpha} \sqrt{x(t)} = If(t, x(t))$$

and

$$\frac{d}{dt} I^{1-\alpha} \sqrt{x(t)} = f(t, x(t)). \tag{8}$$

Finally

$$x(0) = (I^{\alpha} f(t, x(t)) |_{t=0})^2 = 0 \,. \tag{9}$$

Then, the initial value problem (2) and the quadratic integral equation (5) are equivalent.
Consequently, from Corollary 2.4 we deduce that the initial value problem (2) has a unique continuous solution $x \in C.$.

Also, the following corollary can be proved (see [13]).

Corollary 3.3 *Let the assumptions of Corollary 3.3, then the initial value problem of equation (3) has a unique continuous solution* $x \in C$.

Example:1

Consider the following quadratic integral equation

$$x(t) = t + I^{\frac{1}{2}} [\sqrt{t^2+5} + t(|log(x(t)+3)|+1)].I^{\frac{1}{3}}[\frac{1+2t}{10} + e^{-t}\frac{x^2}{30}], \quad t \in I = [0,1]. \tag{10}$$

Set

$$f_1(t,x) = \sqrt{t^2+5} + t(|log(x(t)+3)|+1), \; t \in I$$

$$f_2(t,x) = \frac{1+2t}{10} + e^{-t}\frac{x^2}{30}.$$

Then we have:

$$(i) \; |f_1(t,z) - f_1(t,y)| = |\sqrt{t^2+5} + t(|log(z(t)+3)|+1) - \sqrt{t^2+5} - t(|log(y(t)+3)|+1)|$$

$$\leq t\,|(|log(z(t)+3)|+1) - (|log(y(t)+3)|+1)|$$

$$\leq \frac{1}{10}|z-y|$$

$$(ii) |f_2(t,z) - f_2(t,x)| = |\frac{1+t}{10} + e^{-t}.\frac{z^2}{30} - \frac{1+t}{10} - e^{-t}.\frac{x^2}{30}|$$

$$\leq \frac{1}{30}|e^{-t}z^2 - e^{-t}x^2|$$

$$\leq \frac{2|x+z|}{30}|x-z| \leq \frac{2}{30}|x-z|.$$

Example:2

Consider the following Cauchy problem

$$_*D^{\frac{1}{2}}\sqrt{x(t)} = t + \frac{1}{3}|x(t)|, \quad t \in I = [0,1] \tag{11}$$

with the initial condition

x(0)=0

Set

$$f(t,x) = t + \frac{1}{3}|x(t)|, \; t \in I$$

Then easily we can deduce that:

$$|f(t,z) - f(t,y)| \leq \frac{1}{3}|z-y|.$$

Example:3

Consider the following Cauchy problem

$$_*D^{\frac{1}{2}}\sqrt{x(t)} = t + \frac{1}{4}\sin x(2t), \quad t \in I = [0,1] \tag{12}$$

with the initial condition

$$I^{\frac{1}{2}}\sqrt{x(t)} = 0$$

Set

$$f(t,x) = t + \frac{1}{4}\sin x(2t),\ t \in I$$

Then easily we get

$$|f(t,z) - f(t,x)| = |t + \frac{1}{4}\sin z(2t) - t - \frac{1}{4}\sin x(2t)|$$

$$\leq \frac{1}{4}|\sin z(2t) - \sin x(2t)|$$

$$\leq \frac{1}{4}|z - x|.$$

References

[1] I. K. Argyros, Quadratic equations and applications to Chandrasekhar's and related equations, *Bull. Austral. Math. Soc.* **32** (1985) 275-292.

[2] I. K. Argyros, On a class of quadratic integral equations with perturbations, Funct. Approx. **20** (1992) 51-63.

[3] J. Banas , M . Lecko, W.G. El-Sayed, Eixstence theorems of some quadratic integral equation, J. Math. Anal. Appl. **227** (1998)276 - 279.

[4] J. Banas, A. Martinon, Monotonic solutions of a quadratic integral equation of Volterra type, Comput. Math. Appl. **47** (2004) 271 - 279.

[5] J. Banas, J. Caballero, J. Rocha and K. Sadarangani, Monotonic solutions of a class of quadratic integral equation of Volterra type, Comp. and Math. with Applications, **49** (2005) 943-952.

[6] J. Banas, J. Rocha Martin, K. Sadarangani, On the solutions of a quadratic integral equation of Hemmerstien type, Mathematecal and Computer Modelling. vol. **43** (2006) 97-104.

[7] J. Banas, B. Rzepka, Monotonic solutions of a quadratic integral equations of fractional -order J. Math. Anal. Appl. **332** (2007) 1370 - 11378.

[8] J. Banas, B. Rzepka, Nondecreasing solutions of a quadratic singular Volterra integral equation, Math. Comput. Modelling **49** (2009) 488-496.

[9] M. A. Darwish, On monotonic solutions of a singular quadratic integral equation with supremum, Dynam. Syst. Appl., **17** (2008), 539-550.

[10] A. M. A. El-Sayed, M. M. Saleh and E. A. A. Ziada, Numerical and Analytic Solution for a nonlinear quadratic Integral Equation, Math. Sci. Res. J., **12**(8) (2008), 183-191.

[11] A. M. A. El-Sayed, H. H. G. Hashem, Carathèodory type theorem for a nonlinear quadratic integral equation, MATH. SCI. RES. J. **12**(4), (2008)71-95.

[12] A. M. A. El-Sayed, H. H. G. Hashem, Integrable and continuous solutions of a nonlinear quadratic integral equation, Electronic Journal of Qualitative Theory of Differential Equations **25** (2008), 1-10.

[13] A. M. A. El-Sayed, M. SH. Mohamed, F. F. S. Mohamed, Existence of positive continuous solution of a quadratic integral equation of fractional orders, Journal of Fractional Calculus and Applications Vol. 1(2011), No. **9**, pp. 1-7.

[14] A. M. A. El-Sayed, H. H. G. Hashem, and E. A. A. Ziada, Picard and Adomian methods for quadratic integral equation, Computational and Applied Mathematics, Vol 29. No **3**, pp 447-463, (2010).

[15] W. G. El-Sayed, B. Rzepka, Nondecreasing Solutions of a quadratic Integral Equation of Urysohn Type, Computers and Mathematics with Applications **51** (2006) 1065-1074.

[16] Goebel, K. and Kirk W. A., Topics in Metric Fixed point theory, Cambridge University Press, Cambridge (1990).

[17] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier, Amsterdam, 2006.

[18] V. Lakshmikantham, Theory of fractional functional differential equations, Nonlinear Anal., **69** (2008), 3337-3343.

[19] I. Podlubny and A. M. A. El-Sayed, On two definitions of fractional calculus. Preprint UEF. 03-96 (ISBN 80-7099-252-2) Solvak Academy of scince –Institute of Experimental phys. (1996).

[20] I. Podlubny, Fractional Differential Equations. *Acad. Press, San Diego-New* York-London (1999).