Journal of Analysis & Number Theory
An International Journal

On Asymmetric Distances

A. Farokhinia, L. Taslim
Islamic Azad University of Shiraz, Iran

Email: *afarokhinia@yahoo.com, leilataslim@yahoo.com*
Received*: 8 Feb. 2012; Revised 15 May. 2012; Accepted 23 Jul. 2012*

Abstract: In this paper, we prove some useful theorems in asymmetric metric spaces.

Keywords: . Asymmetric metric, Forward complete

Introduction

Asymmetric metric spaces are defined as metric spaces, but without the requirement that the (asymmetric) metric d has to satisfy $d(x, y) = d(y, x)$.

In the realms of applied mathematics and materials science we find many recent applications of asymmetric metric spaces; for example, in rate-independent models for plasticity [1], shape-memory alloys [2]. The study of asymmetric metrics apparently goes back to Wilson [3]. Following his terminology, asymmetric metrics are often called quasi-metrics. Author in [4] has discussed completely on asymmetric metric spaces. In this work we prove some theorems in asymmetric metric spaces. We start with some elementary definitions from [4].

Definition 1.1. A function $d : X \times X \to \mathbb{R}$ is an *asymmetric metric* and (X, d) is an *asymmetric metric space* if:
(1) For every $x, y \in X$, $d(x, y) \geq 0$ and $d(x, y) = 0$ holds if and only if $x = y$,
(2) For every $x, y, z \in X$, we have $d(x, z) \leq d(x, y) + d(y, z)$.

Henceforth, (X, d) shall be an asymmetric metric space.

Example1.2. Consider $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{\geq 0}$ defined by

$$d(x, y) = \begin{cases} x - y & x \geq y \\ y - x & y > x \end{cases}$$

is obviously an asymmetric metric.

Definition1.3. The *forward topology* $\tau+$ induced by d is the topology generated by the *forward open balls*

$B^+(x, \varepsilon) = \{y \in X: d(x, y) < \varepsilon\}$ for $x \in X$, $\varepsilon > 0$.

Likewise, the *backward topology* $\tau-$ induced by d is the topology generated by the *backward open balls*

$B^-(x, \varepsilon) = \{y \in X: d(y, x) < \varepsilon\}$ for $x \in X$, $\varepsilon > 0$.

Definition 1.4. A sequence $\{x_k\}_{k\in\mathbb{N}}$ *forward converges* to $x_0 \in X$, respectively *backward converges* to $x_0 \in X$ if and only if

$$\lim_{k\to\infty} d(x_0, x_k) = 0, \quad \text{respectively } \lim_{k\to\infty} d(x_k, x_0) = 0.$$

Then we write $x_k \overset{f}{\rightarrow} x_0$, $x_k \overset{b}{\rightarrow} x_0$ respectively.

Definition 1.5. Suppose (X, d_X) and (Y, d_Y) are asymmetric metric spaces. Let $f : X \rightarrow Y$ *be a function* . We say f is *forward continuous at* $x \in X$, respectively *backward continuous*, if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $y \in B^+(x, \delta)$ implies $f(y) \in B^+(f(x), \varepsilon)$, respectively $f(y) \in B^-(f(x), \varepsilon)$.

However, note that uniform forward continuity and uniform backward continuity are the same.

Definition 1.6. A set $S \subset X$ is *forward compact* if every open cover of S in the forward topology has a finite subcover. We say that S is *forward relatively compact* if $\bar{S}$ is forward compact, where $\bar{S}$ denotes the closure in the forward topology. We say S is *forward sequentially compact* if every sequence has a forward convergent subsequence with limit in S. Finally, $S \subset X$ is *forward complete* if every forward Cauchy sequence is forward convergent.

Note that there is a corresponding backward definition in each case, which is obtained by replacing 'forward' with 'backward' in each definition.

Lemma1.8. *Let* $d : X \times X \rightarrow \mathbb{R}^{\geq 0}$ *be an asymmetric metric. If (X, d) is forward sequentially compact and* $x_n \overset{b}{\rightarrow} x_0$ *then* $x_k \overset{f}{\rightarrow} x_0$.

Notation1.9. We introduce some further notations. Y^X denotes the space of functions from X to Y. The *uniform metric* on Y^X is

$\bar{\rho}(f, g) := \sup\{\bar{d}(f(x), g(x)) : x \in X\}$,

where $\bar{d}(x, y) := \min\{d(x, y), 1\}$ and d is the asymmetric metric associated with Y.

Main results

Theorem 2.1. Let (X, d) be an asymmetric metric space. Then $x_n \overset{f}{\rightarrow} x$ if only if each subsequence of it be forward convergent to x .

Proof. Let $x_n \overset{f}{\rightarrow} x$. Given $\varepsilon > 0$, there exists $N \in N$ such that $d(x, x_n) < \varepsilon$ for all $n \geq N$. Suppose that $\{x_{n_k}\}_{k=1}^{\infty}$ be an arbitary subsequence of $\{x_n\}_{n=1}^{\infty}$. If $n_k \geq N$ we have $d(x, x_{n_k}) < \varepsilon$, i . e , $x_{n_k} \overset{f}{\rightarrow} x$.

Conversely. since $\{x_n\}$ is a subsequence of itself , so $x_{n_k} \overset{f}{\rightarrow} x$. □

Remark 2.2. One can rewrite the previous theorem for back limits. □

Theorem2.3. Let (X, d) be an asymmetric metric space. If X is backward sequentially compact and $x_n \overset{f}{\rightarrow} x$, then $x_n \overset{b}{\rightarrow} x$.

Proof. Let $x_n \overset{f}{\rightarrow} x$. Since X is backward sequentially compact so by theorem 2.1 each subsequence of $\{x_n\}_{n \in N}$, namely $\{x_{n_k}\}$, is backward convergent to x.On the other hand, $\{x_{n_k}\}_{k \in N}$, has a subsequence which backward convergent, say $\{x_{n_{k_j}}\}_{j\in N}$. So $x_{n_{k_j}} \overset{b}{\to} y$. Now by [1,lemma 3.1], we deduce that $x = y$. We show that $x_n \overset{b}{\to} x$. Let $x_n \nrightarrow x$. Then there exists a $\varepsilon_0 > 0$ a subsequence $\{x_{n_k}\}_{k\in N}$ of $\{x_n\}_{n\in N}$ so that $d(x_{n_k}, x) \geq \varepsilon_0$ for each $K \in N$. Also, $\{x_{n_k}\}_{k\in N}$, itself, has a subsequence which backward convergent to x, say $\{x_{n_{k_j}}\}_{j\in N}$ hence we can find $J \in N$ such that $d(x_{n_{k_j}}, x) < \varepsilon_0$ for $j \geq J$ which is a contradiction. So $x_n \overset{b}{\to} x$.

Lemma 2.4. If backward convergence implies the forward convergence of a sequence , then the backward limit is unique .

Proof. Let $x_n \overset{b}{\to} x$ implies $x_n \overset{f}{\to} y \in X$. Also, suppose that$x_n \overset{b}{\to} z$. Given $\varepsilon > 0$, there exists $N_1 \in N$ so that $d(y, x_k) < \frac{\varepsilon}{2}$ for all $K \geq N_1$. On the other hand, there exists $N_2 \in N$ such that $d(x_k, x) < \frac{\varepsilon}{2}$ for all $K \geq N_2$ by lemma [1,lemma 3.1], we deduce that $y = z$. Set $N := max\{N_1, N_2\}$ then we have

$$d(z,x) \leq d(z,x_k) + d(x_k,x) < \varepsilon$$

Since $\varepsilon > 0$ was arbitrary, so $z = x$. □

Remark 2.5. Auther in [1] has proved a similar lemma by replacing backward with forward.

Theorem 2.6. Let (X, d) be a forward totally bounded asymmetric metric space. Which the forward convergence of a sequence implies the backward convergence. Then X is forward sequentially compact.

Proof. Suppose that $\{x_n\}_{n\in N}$ be an arbitrary sequence in X. Given $\varepsilon > 0$, there exist $y_1, y_2, \dots \dots, y_k$ in X such that

$$X = \bigcup_{i=1}^{k} B^+(y_i, \varepsilon)$$

Also, we can find $N \in N$ and $1 \leq j \leq k$ so that $\{x_n\} \subset B^+(y_j, \varepsilon)$ for all $n \geq N$. Hence $x_n \overset{f}{\to} y_j$. Now , y_j is unique by Remark 2.5.

Since $\{x_n\}$ is a subsequence of itself , then (X, d) is forward sequentially compact.

Lemma2.7. Let $\mathcal{G} \subseteq Y^X$ be forward (backward) closed and Y^X forward (backward) complete. Then $\mathcal{G}$ is forward (backward) complete.

Proof.. Let $\{f_n\} \subseteq \mathcal{G}$ be an arbitrary forward Cauchy sequence. Then $\{f_n\}$ is a Cauchy sequence in Y^X.Since Y^X is forward complete, so $\{f_n\}$ has a subsequence, say $\{f_{n_k}\}$, with $f_{n_k} \overset{f}{\to} f$ .Since $\mathcal{G}$ is forward closed, so $f \in \mathcal{G}$, as desired.

Lemma2.8. Let Y be forward (backward) complete, then Y^X is so.

Proof . Let $\{f_n\} \subseteq Y^X$ be an arbitrary forward Cauchy sequence. By definition, given $\varepsilon > 0$ there exists N$\in \mathbb{N}$ such that for m $\geq$ n$\geq N$, $\bar{\rho}$ $(f_n, f_m) < \varepsilon$ holds. Fix $x \in X$.Clearly, $\{f_n(x)\}$ is a forward Cauchy sequence in *Y*. Since *Y* is forward complete so $\{f_n(x)\}$ is convergent., say $f_n(x) \xrightarrow{f} f(x)$. Thus there exists N$\in \mathbb{N}$ such that n$\geq$ N implies

$$d_Y\big(f(x), f_n(x)\big) < \varepsilon \qquad (1)$$

Since $x \in X$ was arbitrary, taking sup on $x \in X$ in the both side of *(1)*, we deduce $f_n \xrightarrow{f} f$ in the uniform metric$\bar{\rho}$. □

References

[1] A. Mainik, A. Mielke, Existence results for energetic models for rate-independent systems, Calc. Var. Partial Differential Equations 22(1) (2005) 73–99.

[2] A. Mielke, T. Roubí˘cek, A rate-independent model for inelastic behavior of shape-memory alloys, Multiscale Model. Simul. 1(4) (2003) 571–597 (electronic).

[3] W.A. Wilson, On quasi-metric spaces, Amer. J. Math. 53 (3) (1931) 675–684.

[4] J. Collins, J. Zimmer, An asymmetric Arzela-Ascoli theorem, Topology and it's Applications. 154 (2007) 2312-2322.