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Some Common Fixed Points of Generalized Contractive Mappings on Complex Valued Metric Spaces

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Abstract: Azam et al. (Numer Funct Anal Optim, 32(3):243–253, 2011) introduced a notion of complex valued metric space and obtained common fixed point result for mappings in such spaces. In this paper, owing the concept of complex valued metric spaces introduced by Azam et al. [6], we obtain sufficient conditions for the existence of common fixed point for a pairs of mappings satisfying generalized contraction involving rational expressions.

Keywords: complex valued metric spaces, common fixed point, contractive type mappings.

1 Introduction and Preliminaries

Fixed point theory became one of the most interesting area of research in the last fifty years for instance research about optimization problem, economics, game theory, control theory, and etc. The study of fixed points of mappings satisfying certain contraction conditions has many applications and has been at the center of various research activities. Fixed point results of mappings satisfying certain contractive condition on the entire domain has been at the centre of rigorous research activity and it has a wide range of applications in different areas such as nonlinear and adaptive control systems, parametrize estimation problems, fractal image decoding, computing magneto static fields in a nonlinear medium, and convergence of recurrent networks. The fixed point theorem, generally known as the Banach contraction mapping principle, appeared in explicit form in Banach's thesis in 1922 [7]. Since its simplicity and usefulness, it became a very popular tool in solving many problems in mathematical analysis. Later, a number of articles in this field have been dedicated to the improvement and generalization of Banach's contraction mapping principle in several ways in many spaces (see [2]-[27]).

In the other hand, the study of metric spaces expressed the most important role to many fields both in pure and applied science such as biology, medicine, physics and computer science (see [15], [26]). Some generalizations of the notion of a metric space have been proposed by some authors, such as, rectangular metric spaces, semi metric spaces, pseudo metric spaces, probabilistic metric spaces, fuzzy metric spaces, quasi metric spaces, quasi semi metric spaces, D-metric spaces, and cone metric spaces (see [[1], [11], [12], [21], [22]]). Branciari [9] introduced the notion of a generalized metric space replacing the triangle inequality by a rectangular type inequality. He then extended Banach's contraction principle in such spaces.

In 2011, Azam et al. [6] introduced the notion of complex valued metric spaces and established some fixed point results for mappings satisfying a rational inequality. In a continuation of [6] and [23], we prove a new common fixed point theorems for a pair of mappings satisfying a more general contraction involving rational expression in complex valued metric spaces.

Consistent with Azam et al., [6] and [23], the following definitions and results will be needed in the sequel.

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$$z_1 \precsim z_2 \text{ if and only if } Re(z_1) \leqslant Re(z_2),\ Im(z_1) \leqslant Im(z_2).$$

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It follows that
$$z_1 \precsim z_2,$$
if one of the following conditions is satisfied:

(i) $Re(z_1) = Re(z_2)$, $Im(z_1) < Im(z_2)$,
(ii) $Re(z_1) < Re(z_2)$, $Im(z_1) = Im(z_2)$,
(iii) $Re(z_1) < Re(z_2)$, $Im(z_1) < Im(z_2)$,
(iv) $Re(z_1) = Re(z_2)$, $Im(z_1) = Im(z_2)$.

In particular, we will write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (i), (ii) and (iii) is satisfied and we will write $z_1 \prec z_2$ if only (iii) is satisfied. Note that

$$0 \precsim z_1 \precnsim z_2 \implies |z_1| < |z_2|,$$
$$z_1 \preceq z_2, z_2 \prec z_3 \implies z_1 \prec z_3.$$

Definition 1. *Let X be a nonempty set. Suppose that the mapping $d : X \times X \to \mathbb{C}$, satisfies:*

1. $0 \precsim d(j,k)$, for all $j,k \in X$ and $d(j,k) = 0$ if and only if $j = k$;
2. $d(j,k) = d(k,j)$ for all $j,k \in X$
3. $d(j,k) \precsim d(j,l) + d(l,k)$, for all $j,k,l \in X$.

Then d is called a complex valued metric on X, and (X,d) is called a complex valued metric space. A point $j \in X$ is called interior point of a set $A \subseteq X$ whenever there exists $0 \prec r \in \mathbb{C}$ such that

$$B(j,r) = \{y \in X : d(j,y) \prec r\} \subseteq A.$$

A point $j \in X$ is called a limit point of A whenever for every $0 \prec r \in \mathbb{C}$,

$$B(j,r) \cap (A \setminus X) \neq \phi.$$

A is called open whenever each element of A is an interior point of A. A subset $B \subseteq X$ is called closed whenever each limit point of B belongs to B. The family

$$F = \{B(j,r) : j \in X, 0 \prec r\},$$

is a sub-basis for a Hausdorff topology τ on X.

Let j_n be a sequence in X and $j \in X$. If for every $c \in \mathbb{C}$, with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(j_n, j) \prec c$, then $\{j_n\}$ is said to be convergent, $\{j_n\}$ converges to j and j is the limit point of $\{j_n\}$. We denote this by $\lim_n j_n = j$, or $j_n \longrightarrow j$, as $n \to \infty$. If for every $c \in \mathbb{C}$ with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(j_n, j_{n+m}) \prec c$, then $\{j_n\}$ is called a Cauchy sequence in (X,d). If every Cauchy sequence is convergent in (X,d), then (X,d) is called a complete complex valued metric space.

Lemma 1. *Let (X,d) be a complex valued metric space and let $\{j_n\}$ be a sequence in X. Then $\{j_n\}$ converges to j if and only if $|d(j_n, j)| \to 0$ as $n \to \infty$.*

Lemma 2. *Let (X,d) be a complex valued metric space and let $\{j_n\}$ be a sequence in X. Then $\{j_n\}$ is a Cauchy sequence if and only if $|d(j_n, j_{n+m})| \to 0$ as $n \to \infty$.*

2 Main Results

Let us prove our first main result.

Theorem 1. *Let (X,d) be a complete complex valued metric space and let the mappings $S,T : X \to X$ satisfy:*

$$d(Sj,Tk) \precsim a_1\, d(j,k) + a_2 \frac{d(j,Sj).d(k,Tk)}{d(j,k)} + a_3 \frac{d(j,Tk).d(k,Sj)}{d(j,k)} + a_4 [d(j,Sj) + d(k,Tk)]$$

for all j, $k \in X$, where a_1, a_2, a_3, a_4 are nonnegative reals with $a_1 + a_2 + a_3 + 2a_4 < 1$. Then S, T have a unique common fixed point.

Proof. Let j_0 be an arbitrary point in X and define $j_1 = Sj_0$ and $j_2 = Tj_1$ such that
$$d(j_1, j_2) = d(Sj_0, Tj_1).$$
Then

$$\begin{aligned} d(j_1,j_2) &\preceq a_1 d(j_0,j_1) + a_2 \frac{d(j_0,Sj_0).d(j_1,Tj_1)}{d(j_0,j_1)} \qquad (1)\\ &+ a_3 \frac{d(j_0,Tj_1).d(j_1,Sj_0)}{d(j_0,j_1)} + \\ &\quad a_4 [d(j_0,Sj_0) + d(j_1,Tj_1)], \\ &\preceq a_1 d(j_0,j_1) + a_2 \frac{d(j_0,j_1).d(j_1,j_2)}{d(j_0,j_1)} + a_3 \frac{d(j_0,j_2).d(j_1,j_1)}{d(j_0,j_1)} + \\ &\quad a_4 [d(j_0,j_1) + d(j_1,j_2)], \\ &\preceq a_1 d(j_0,j_1) + a_2 d(j_0,j_1) + a_4 [d(j_0,j_1) + d(j_1,j_2)], \\ &\preceq \left(\frac{a_1 + a_4}{1 - a_2 - a_4}\right) d(j_0,j_1), \\ &\preceq \lambda d(j_0,j_1). \qquad (2) \end{aligned}$$

Where $\left(\frac{a_1+a_4}{1-a_2-a_4}\right) = \lambda$.
Similarly,
$$d(j_2,j_3) = d(Tj_1,Sj_2).$$

$$\begin{aligned} &\preceq a_1 d(j_1,j_2) + a_2 \frac{d(j_2,Sj_2).d(j_1,Tj_1)}{d(j_1,j_2)} + a_3 \frac{d(j_2,Tj_1).d(j_1,Sj_2)}{d(j_1,j_2)} + \\ &\quad a_4 [d(j_2,Sj_2) + d(j_1,Tj_1)], \\ &\preceq a_1 d(j_1,j_2) + a_2 \frac{d(j_2,j_3).d(j_1,j_2)}{d(j_1,j_2)} + a_3 \frac{d(j_2,j_2).d(j_1,j_3)}{d(j_1,j_2)} + \\ &\quad a_4 [d(j_2,j_3) + d(j_1,j_2)], \\ &\preceq a_1 d(j_1,j_2) + a_2 d(j_2,j_3) + a_4 [d(j_2,j_3) + d(j_1,j_2)], \\ &\preceq \left(\frac{a_1 + a_4}{1 - a_2 - a_4}\right) d(j_1,j_2), \\ &\preceq \lambda^2 d(j_0,j_1). \quad \text{using (2)}. \end{aligned}$$

Consequently, we get

$$d(j_{2n+1}, j_{2n+2}) \preceq \lambda^{2n+1} d(j_0, j_1)$$

Hence for any $m > n$,

$$\begin{aligned} d(j_m, j_n) &\preceq d(j_m, j_{m-1}) \\ &\quad + d(j_{m-1}, j_{m-2}) + \cdots + d(j_{n+1}, j_n), \\ &\preceq \left(\lambda^{m-1} + \lambda^{m-2} + \cdots + \lambda^n\right) d(j_0, j_1), \\ &\preceq \frac{\lambda^n}{1-\lambda} d(j_0, j_1). \end{aligned}$$

So

$$|d(j_n, j_m)| < \frac{\lambda^n}{1-\lambda} |d(j_0, j_1)|,$$
$$\longrightarrow 0, \text{ as } m, n \longrightarrow \infty.$$

This implies that $\{j_n\}$ is a Cauchy sequence. Since X is complete, there exist $u \in X$ such that $j_n \longrightarrow u$. It fallows that $u = Su$, otherwise $d(u, Su) = z > 0$ and we would then have

$$\begin{aligned} z &\preceq d(u, j_{2n+2}) + d(j_{2n+2}, Su), \\ &\preceq d(u, j_{2n+2}) + d(Tj_{2n+1}, Su), \end{aligned}$$

$$\begin{aligned} &\preceq d(u, j_{2n+2}) + a_1 d(u, j_{2n+1}) \\ &+ a_2 \frac{d(u, Su).d(j_{2n+1}, Tj_{2n+1})}{d(u, j_{2n+1})} \\ &+ a_3 \frac{d(u, Tj_{2n+1}).d(j_{2n+1}, Su)}{d(u, j_{2n+1})} + \\ &a_4 [d(u, Su) + d(j_{2n+1}, Tj_{2n+1})], \\ &\preceq d(u, j_{2n+2}) + a_1 d(u, j_{2n+1}) \\ &+ a_2 \frac{d(u, Su).d(j_{2n+1}, j_{2n+2})}{d(u, j_{2n+1})} \\ &+ a_3 \frac{d(u, j_{2n+2}).d(j_{2n+1}, Su)}{d(u, j_{2n+1})} + \\ &a_4 [d(u, Su) + d(j_{2n+1}, j_{2n+2})]. \end{aligned}$$

This implies that

$$\begin{aligned} |z| &< |d(u, j_{2n+2})| + a_1 |d(u, j_{2n+1})| \\ &+ a_2 \frac{|z|.|d(j_{2n+1}, j_{2n+2})|}{|d(u, j_{2n+1})|} \\ &+ a_3 \frac{|d(u, j_{2n+2})|.|d(j_{2n+1}, Su)|}{|d(u, j_{2n+1})|} \\ &+ a_4 [|z| + |d(j_{2n+1}, j_{2n+2})|]. \end{aligned}$$

Letting $n \to \infty$, it fallows that

$$(a_2 + a_4)|z| \leq (a_1 + a_2 + a_3 + 2a_4)|z| < |z|$$

a contradiction, hence $|z| = 0$, that is, $u = Su$. It fallows similarly that $u = Tu$.

We now show that S and T have a unique common fixed point. For this, assume that v in X is a second common fixed point of S and T. Then

$$\begin{aligned} d(u, v) &= d(Su, Tv) \\ &\preceq a_1 d(u, v) \\ &+ a_2 \frac{d(u, Su).d(v, Tv)}{d(u, v)} \\ &+ a_3 \frac{d(u, Tv).d(v, Su)}{d(u, v)} + \\ &\quad a_4 [d(u, Su) + d(v, Tv)], \\ &\preceq (a_1 + a_3 + 2a_4) d(u, v), \\ (1 - a_1 - a_3 - 2a_4) d(u, v) &\preceq 0. \end{aligned}$$

Hence, $d(u, v) = 0$, because, $(a_1 + 2a_4) < 1$. This implies that $u = v$, completing the proof of the theorem.

Corollary 1. *Let (X, d) be a complete complex valued metric space and let the mappings $S, T : X \to X$ satisfy:*

$$\begin{aligned} d(Sj, Tk) &\precsim a_1 d(j, k) \\ &+ a_2 \frac{d(j, Sj).d(k, Tk)}{d(j, k)} \\ &+ a_4 [d(j, Sj) + d(k, Tk)], \end{aligned}$$

for all $j, k \in X$, where a_1, a_2, a_4 are nonnegative reals with $a_1 + a_2 + 2a_4 < 1$. Then S, T have a unique common fixed point.

Proof: By using $a_3 = 0$ in theorem (1), we get the required result.

Corollary 2. *Let (X, d) be a complete complex valued metric space and let the mappings $S, T : X \to X$ satisfy:*

$$\begin{aligned} d(Sj, Tk) &\precsim a_1 d(j, k) \\ &+ a_3 \frac{d(j, Tk).d(k, Sj)}{d(j, k)} \\ &+ a_4 [d(j, Sj) + d(k, Tk)], \end{aligned}$$

for all $j, k \in X$, where a_1, a_3, a_4 are nonnegative reals with $a_1 + a_3 + 2a_4 < 1$. Then S, T have a unique common fixed point.

Proof: By using $a_2 = 0$ in theorem (1), we get the result.

Theorem 2. *Let (X, d) be a complete complex valued metric space and let the mappings $S, T : X \to X$ satisfy:*

$$\begin{aligned} d(Sj, Tk) &\precsim a_1 d(j, k) \\ &+ a_2 \frac{d(j, Sj).d(k, Tk)}{d(j, k)} \\ &+ a_3 \frac{d(j, Tk) + d(k, Sj)}{d(j, k)} \\ &+ a_4 [d(j, Tk) + d(k, Sj)], \end{aligned}$$

for all $x, y \in X$, where a_1, a_2, a_3, a_4 are nonnegative reals with $a_1 + a_2 + a_3 + 2a_4 < 1$. Then S, T have a unique common fixed point.

Proof: The proof is same as Theorem (1).

Now we prove similar type of results for a different rational expression studied in Imdad and Khan [14].

Theorem 3. *Let (X, d) be a complete complex valued metric space and let the mappings $S, T : X \to X$ satisfy:*

$$\begin{aligned} d(Sj, Tk) &\precsim a\, d(j, k) \\ &+ b \frac{d(j, Sj).d(k, Tk)}{d(j, k)} \\ &+ c \frac{d^2(j, Tk) + d^2(k, Sj)}{d(j, Tk) + d(k, Sj)}, \end{aligned}$$

for all $j, k \in X$, where $a, b, c,$ are nonnegative reals with $a + b + c < 1$. Then S, T have a unique common fixed point.

Proof: Let j_0 be an arbitrary point in X and define $j_1 = Sj_0$ and $j_2 = Tj_1$ such that

$$d(j_1, j_2) = d(Sj_0, Tj_1)$$

Then

$$\begin{aligned} d(j_1, j_2) &\preceq ad(j_0, j_1) \\ &+ b\frac{d(j_0, Sj_0).d(j_1, Tj_1)}{d(j_0, j_1)} \\ &+ c\frac{d^2(j_0, Tj_1) + d^2(j_1, Sj_0)}{d(j_0, Tj_1) + d(j_1, Sj_0)}, \\ &\preceq ad(j_0, j_1) \\ &+ b\frac{d(j_0, j_1).d(j_1, j_2)}{d(j_0, j_1)} \\ &+ c\frac{d^2(j_0, j_2) + d^2(j_1, j_1)}{d(j_0, j_2) + d(j_1, j_1)}, \\ &\preceq ad(j_0, j_1) \\ &+ b\frac{d(j_0, j_1).d(j_1, j_2)}{d(j_0, j_1)} \\ &+ cd(j_0, j_2), \\ &\preceq ad(j_0, j_1) \\ &+ bd(j_1, j_2) + c[d(j_0, j_1) + d(j_1, j_2)], \\ &\preceq \left(\frac{a+c}{1-c-b}\right) d(j_0, j_1), \\ &\preceq \lambda d(j_0, j_1). \end{aligned} \tag{3}$$

Where $\left(\frac{a+c}{1-c-b}\right) = \lambda$. Similarly,

$$d(j_2, j_3) = d(Sj_1, Tj_2)$$

$$\begin{aligned} &\preceq ad(j_1, j_2) + b\frac{d(j_2, Sj_1).d(j_1, Tj_2)}{d(j_1, j_2)} \\ &+ c\frac{d^2(j_1, Tj_2) + d^2(j_1, Sj_1)}{d(j_1, Tj_2) + d(j_1, Sj_1)}, \\ &\preceq ad(j_1, j_2) + b\frac{d(j_2, j_3).d(j_1, j_2)}{d(j_1, j_2)} \\ &+ c\frac{d^2(j_1, j_3) + d^2(j_1, j_1)}{d(j_1, j_3) + d(j_1, j_1)}, \\ &\preceq ad(j_1, j_2) + b\frac{d(j_2, j_3).d(j_1, j_2)}{d(j_1, j_2)} + cd(j_1, j_3), \\ &\preceq ad(j_1, j_2) + bd(j_2, j_3) + c[d(j_1, j_2) + d(j_2, j_3)], \\ &\preceq \left(\frac{a+c}{1-c-b}\right) d(j_1, j_2), \\ &\preceq \lambda d(j_1, j_2), \\ &\preceq \lambda^2 d(j_0, j_1). \quad \text{using (3).} \end{aligned}$$

Consequently, we get

$$d(j_{2n+1}, j_{2n+2}) \preceq \lambda^{2n+1} d(j_0, j_1)$$

Hence for any $m > n$,

$$\begin{aligned} d(j_m, j_n) &\preceq d(j_m, j_{m-1}) \\ &+ d(j_{m-1}, j_{m-2}) + \cdots + d(j_{n+1}, j_n), \\ &\preceq \left(\lambda^{m-1} + \lambda^{m-2} + \cdots + \lambda^n\right) d(j_0, j_1), \\ &\preceq \frac{\lambda^n}{1-\lambda} d(j_0, j_1). \end{aligned}$$

So

$$|d(j_n, j_m)| < \frac{\lambda^n}{1-\lambda} |d(j_0, j_1)|,$$

$$\longrightarrow 0, \text{ as } m, n \longrightarrow \infty.$$

This implies that $\{j_n\}$ is a Cauchy sequence. Since X is complete, there exist $u \in X$ such that $j_n \longrightarrow u$. It fallows that $u = Su$, otherwise $d(u, Su) = z > 0$ and we would then have

$$\begin{aligned} z &\preceq d(u, j_{2n+2}) + d(j_{2n+2}, Su), \\ &\preceq d(u, j_{2n+2}) + d(Tj_{2n+1}, Su). \end{aligned}$$

$$\begin{aligned} &\preceq d(u, j_{2n+2}) + ad(u, j_{2n+1}) + b\frac{d(u, Su).d(j_{2n+1}, Tj_{2n+1})}{d(u, j_{2n+1})} \\ &+ c\frac{d^2(u, Tj_{2n+1}) + d^2(j_{2n+1}, Su)}{d(u, Tj_{2n+1}) + d(j_{2n+1}, Su)}, \\ &\preceq d(u, j_{2n+2}) + ad(u, j_{2n+1}) + b\frac{d(u, Su).d(j_{2n+1}, j_{2n+2})}{d(u, j_{2n+1})} \\ &+ c\frac{d^2(u, j_{2n+2}) + d^2(j_{2n+1}, Su)}{d(u, j_{2n+2}) + d(j_{2n+1}, Su)}. \end{aligned}$$

This implies that

$$\begin{aligned} |z| &< |d(u, j_{2n+2})| + a|d(u, j_{2n+1})| \\ &+ b\frac{|z|.|d(j_{2n+1}, j_{2n+2})|}{|d(u, j_{2n+1})|} \\ &+ c\frac{|d^2(u, j_{2n+2})| + |d^2(j_{2n+1}, Su)|}{|d(u, j_{2n+2})| + |d(j_{2n+1}, Su)|}. \end{aligned}$$

Letting $n \to \infty$, it fallows that

$$(a+c)|z| \leq (a+b+c)|z| < |z|,$$

a contradiction and so $|z| = 0$, that is, $u = Su$. It fallows similarly that $u = Tu$.

We now show that S and T have a unique common fixed point. For this, assume that v in X is a second common fixed point of S and T. Then

$$d(u, v) = d(Su, Tv)$$

$$
\begin{aligned}
&\preceq a\, d(u,v)+b\frac{d(u,Su).d(v,Tv)}{d(u,v)}\\
&+c\frac{d^2(u,Tv)+d^2(v,Su)}{d(u,Tv)+d(v,Su)},\\
&\preceq a\, d(u,v)+c\frac{d^2(u,v)+d^2(v,u)}{d(u,v)+d(v,u)},\\
&\preceq a\, d(u,v)+cd(u,v),\\
d(u,v)&\preceq ad(u,v)+cd(u,v).\\
(1-a-c)\,d(u,v)&\preceq 0,\\
(1-a-c)&\neq 0.
\end{aligned}
$$

This implies that $u=v$, completing the proof of the theorem.

Now we prove a common fixed point theorem for a pair of mappings satisfying a more general contraction involving rational expression.

Theorem 4. *Let (X,d) be a complete complex valued metric space and let the mappings $S,T:X\to X$ satisfy:*

$$
d(Sj,Tk)\precsim a_1\, d(j,k)+a_2\frac{d(j,Sj).d(k,Tk)}{d(j,k)} \tag{4}
$$

$$
+a_3\frac{d(j,Tk).d(k,Sj)}{d(j,k)}+ a_4\frac{d(j,Sj)\,d(k,Tk)}{d(j,Tk)+d(j,k)+d(k,Sj)}, \tag{5}
$$

for all $j,k\in X$, where a_1,a_2,a_3,a_4 are nonnegative reals with $a_1+a_2+a_3+a_4<1$. Then S,T have a unique common fixed point.

Proof Let j_0 be an arbitrary point in X and define $j_1=Sj_0$ and $j_2=Tj_1$ such that $d(j_1,j_2)=d(Sj_0,Tj_1)$.

Then

$$
\begin{aligned}
d(j_1,j_2)&\preceq a_1 d(j_0,j_1)\\
&+a_2\frac{d(j_0,Sj_0).d(j_1,Tj_1)}{d(j_0,j_1)}\\
&+a_3\frac{d(j_0,Tj_1).d(j_1,Sj_0)}{d(j_0,j_1)}\\
&+a_4\frac{d(j_0,Sj_0)\,d(j_1,Tj_1)}{d(j_0,Tj_1)+d(j_0,j_1)+d(j_1,Sj_0)},\\
&\preceq a_1 d(j_0,j_1)\\
&+a_2\frac{d(j_0,j_1).d(j_1,j_2)}{d(j_0,j_1)}\\
&+a_3\frac{d(j_0,j_2).d(j_1,j_1)}{d(j_0,j_1)}\\
&+a_4\frac{d(j_0,j_1)\,d(j_1,j_2)}{d(j_0,j_2)+d(j_0,j_1)+d(j_1,j_1)},\\
&\preceq a_1 d(j_0,j_1)+a_2 d(j_1,j_2)\\
&+a_4\frac{d(j_0,j_1)\,d(j_1,j_2)}{d(j_0,j_2)+d(j_0,j_1)}.
\end{aligned}
$$

As (owing to triangular inequality),

$$
\begin{aligned}
|d(j_1,j_2)|&<a_1|d(j_0,j_1)|+a_2|d(j_1,j_2)|\\
&+a_4\left|\frac{|d(j_0,j_1)\,d(j_1,j_2)|}{|d(j_0,j_2)|+|d(j_0,j_1)|}\right|.
\end{aligned}
$$

Where

$$
|d(j_1,j_2)|\le d(j_1,j_0)+d(j_0,j_2).
$$

Hence

$$
\begin{aligned}
|d(j_1,j_2)|&<\left(\frac{a_1+a_4}{1-a_2}\right)|d(j_0,j_1)|,\\
&<\lambda\,|d(j_0,j_1)|.
\end{aligned}
$$

Where $\lambda=\frac{a_1+a_4}{1-a_2}$. Similarly, by repeating the same process for

$$
d(j_2,j_3)=d(Sj_1,Tj_2),
$$

we get

$$
|d(j_2,j_3)|<\lambda^2|d(j_0,j_1)|.
$$

Consequently, we get

$$
\begin{aligned}
|d(j_{2n+1},j_{2n+2})|&<\lambda\,|d(j_{2n},j_{2n+1})|,\\
&<\left|\lambda^2 d(j_{2n-1},j_{2n})\right|,\\
&<\lambda^{2n+1}|d(j_0,j_1)|.
\end{aligned}
$$

Hence for any $m>n$,

$$
\begin{aligned}
|d(j_n,j_m)|&<|d(j_n,j_{n+1})|+|d(j_{n+1},j_{n+2})|+\cdots\\
&+|d(j_{m-1},j_m)|,\\
&<\left(\lambda^n+\lambda^{n+1}+\cdots+\lambda^{m-1}\right)|d(j_0,j_1)|,\\
&<\frac{\lambda^n}{1-\lambda}|d(j_0,j_1)|.
\end{aligned}
$$

And

$$
\begin{aligned}
|d(j_n,j_m)|&<\frac{\lambda^n}{1-\lambda}|d(j_0,j_1)|,\\
&\longrightarrow 0,\ \text{as } m,n\longrightarrow\infty.
\end{aligned}
$$

This implies that $\{j_n\}$ is a Cauchy sequence. Since X is complete, there exist $u\in X$ such that $j_n\longrightarrow u$. It fallows that $u=Su$, otherwise $d(u,Su)=z>0$ and we would then have

$$
\begin{aligned}
d(u,Su)&\preceq d(u,j_{2n+2})+a_1 d(u,j_{2n+1})\\
&+a_2\frac{d(u,Su).d(j_{2n+1},Tj_{2n+1})}{d(u,j_{2n+1})}\\
&+a_3\frac{d(u,Tj_{2n+1}).d(j_{2n+1},Su)}{d(u,j_{2n+1})}\\
&+a_4\frac{d(u,Su)\,d(j_{2n+1},Tj_{2n+1})}{d(u,Tj_{2n+1})+d(u,j_{2n+1})+d(j_{2n+1},Su)},\\
&\preceq d(u,j_{2n+2})+a_1 d(u,j_{2n+1})\\
&+a_2\frac{d(u,Su).d(j_{2n+1},j_{2n+2})}{d(u,j_{2n+1})}\\
&+a_3\frac{d(u,j_{2n+2}).d(j_{2n+1},Su)}{d(u,j_{2n+1})}\\
&+a_4\frac{d(u,Su)\,d(j_{2n+1},j_{2n+2})}{d(u,j_{2n+2})+d(u,j_{2n+1})+d(j_{2n+1},Su)}.
\end{aligned}
$$

This implies that

$$\begin{aligned}|z| &< |d(u, j_{2n+2})| + a_1 |d(u, j_{2n+1})| \\ &+ a_2 \frac{|z| . |d(j_{2n+1}, j_{2n+2})|}{|d(u, j_{2n+1})|} \\ &+ a_3 \frac{|d(u, j_{2n+2})| . |d(j_{2n+1}, Su)|}{|d(u, j_{2n+1})|} \\ &+ a_4 \frac{|z| |d(j_{2n+1}, j_{2n+2})|}{|d(u, j_{2n+2})| + |d(u, j_{2n+1})| + |d(j_{2n+1}, Su)|},\end{aligned}$$

which on making $n \to \infty$, gives rise $|d(u, Su)| = 0$ a contradiction so that $u = Su$. Similarly, one can

show that $u = Tu$. As in Theorem (1) and (3), the uniqueness of common fixed point can be proved.

As an application of Theorems (1), (3) and (4), we prove the following theorem for two finite families of mappings.

Theorem 5. *If $\{T_k\}_1^m$ and $\{S_k\}_1^n$ are two finite pair wise commuting finite families of self mappings defined on complete complex valued metric space (X, d) such that the mappings T and S satisfy the conditions of theorems 1, 3 and 4, then the component maps of the two families $\{T_k\}_1^m$ and $\{S_k\}_1^n$ have a unique common fixed point.*

Proof. In view of theorems 1, 3 and 4, one can infer that T and S have a unique common fixed point q i.e. $Tq = Sq = q$. Now we are required to show that q is common fixed point of all the components maps of both the families. In view of pairwise commutativity of families of $\{T_k\}_1^m$ and $\{S_k\}_1^n$, (for every $1 \le i \le m$) we can write

$T_i q = T_i S q = S T_i q$ and $T_i q = T_i T q = T T_i q$

which shows that $T_i q$ (for every i) is also a common fixed point of T and S. By using the uniqueness of common fixed point, we can write $T_i q = q$ (for every i) which shows that q is the common fixed point of the family $\{T_k\}_1^m$. Using the foregoing arguments, one can also shows that (for every $1 \le i \le n$) $S_i q = q$.

This completes the proof of the theorem.

By setting $\{S_k\}_1^n = \Gamma$ and $\{T_k\}_1^m = \Omega$ in theorems 1, 3 and 4, we derive the following common fixed point theorems involving iterates of mappings.

Corollary 3. *If Γ and Ω are two commuting self mappings defined on a complete complex valued metric space (X, d) satisfying the condition :*

$$\begin{aligned}d(\Gamma^m j, \Omega^n k) &\precsim a_1\, d(j,k) + a_2 \frac{d(j, \Gamma^m j) . d(k, \Omega^n k)}{d(j,k)} \\ &+ a_3 \frac{d(j, \Omega^n k) . d(k, \Gamma^m j)}{d(j,k)} \\ &+ a_4 [d(j, \Gamma^m j) + d(k, \Omega^n k)]\end{aligned}$$

for all $j, k \in X$, where a_1, a_2, a_3, a_4 are nonnegative reals with $a_1 + a_2 + a_3 + 2a_4 < 1$. Then Γ, Ω have a unique common fixed point.

Corollary 4. *If Γ and Ω are two commuting self mappings defined on a complete complex valued metric space (X, d) satisfying the condition :*

$$\begin{aligned}d(\Gamma^m j, \Omega^n k) &\precsim a\, d(j,k) + b \frac{d(j, \Gamma^m j) . d(k, \Omega^n k)}{d(j,k)} \\ &+ c \frac{d^2(j, \Omega^n k) + d^2(k, \Gamma^m j)}{d(j, \Omega^n k) + d(k, \Gamma^m j)},\end{aligned}$$

for all $j, k \in X$, where a, b, c, are nonnegative reals with $a + b + c < 1$. Then Γ, Ω have a unique common fixed point.

Corollary 5. *If Γ and Ω are two commuting self mappings defined on a complete complex valued metric space (X, d) satisfying the condition :*

$$\begin{aligned}d(\Gamma^m j, \Omega^n k) &\precsim a_1\, d(j,k) + a_2 \frac{d(j, \Gamma^m j) . d(k, \Omega^n k)}{d(j,k)} \\ &+ a_3 \frac{d(j, \Omega^n k) . d(k, \Gamma^m j)}{d(j,k)} + \\ &a_4 \frac{d(j, \Gamma^m j)\, d(k, \Omega^n k)}{d(j, \Omega^n k) + d(j,k) + d(k, \Gamma^m j)},\end{aligned}$$

for all $j, k \in X$, where a_1, a_2, a_3, a_4 are nonnegative reals with $a_1 + a_2 + a_3 + a_4 < 1$. Then Γ, Ω have a unique common fixed point.

We conclude this paper with an illustrative example which one demonstrates theorem (4).

Example 1. Consider $X_1 = \{z \in \mathbb{C} : Re(z) \ge 0, Im(z) = 0\}$, $X_2 = \{z \in \mathbb{C} : \ Im(z) \ge 0, Re(z) = 0\}$,
and write $X = X_1 \cup X_2$. Define a mapping $d : X \times X \to \mathbb{C}$ as:

$$d(z_1, z_2) = \begin{cases} \max\{j_1, j_2\} + i \max\{j_1, j_2\}, & \text{if } z_1, z_2 \in X_1 \\ \max\{k_1, k_2\} + i \max\{k_1, k_2\}, & \text{if } z_1, z_2 \in X_2 \\ (j_1 + k_2) + i(j_1 + k_2), & \text{if } z_1 \in X_1, z_2 \in X_2 \\ (j_2 + k_1) + i(j_2 + k_1), & \text{if } z_1 \in X_2, z_2 \in X_1 \end{cases}$$

where $z_1 = j_1 + ik_1$, $z_2 = j_2 + ik_2$. By simple calculation, one can easily verify that (X, d) is complete complex valued metric space.

Set $T = S$ and define a self mapping T on X (with $z = (j, k)$) as

$$Tz = \begin{cases} \left(\frac{j}{4}, 0\right) & z \in X_1 \\ \left(0, \frac{k}{4}\right) & z \in X_2. \end{cases}$$

Now, we show that $S = T$ satisfies the condition (5). We distinguish the following cases. Before discussing different cases, one needs to notice that

$$\begin{aligned}0 \precsim\ & d(Sz_1, Tz_2), d(z_1, z_2) \\ &, \frac{d(z_1, Sz_1) . d(z_2, Tz_2)}{d(z_1, z_2)} \\ &, \frac{d(z_1, Tz_2) . d(z_2, Sz_1)}{d(z_1, z_2)}, \\ & \frac{d(z_1, Sz_1) . d(z_2, Tz_2)}{d(z_1, Tz_2) + d(z_1, z_2) + d(z_2, Sz_1)}.\end{aligned}$$

Case I. if $z_1, z_2 \in X_1$, then we have

$$\begin{aligned} d(Sz_1, Tz_2) &= d\left(\left(\frac{j_1}{4}, 0\right), \left(\frac{j_2}{4}, 0\right)\right). \\ &= \max\left\{\frac{j_1}{4}, \frac{j_2}{4}\right\} + i\max\left\{\frac{j_1}{4}, \frac{j_2}{4}\right\}, \\ &= \max\left\{\frac{j_1}{4}, \frac{j_2}{4}\right\}(1+i), \\ &= \frac{1}{4}\max\{j_1, j_2\}(1+i), \\ &\preceq \frac{1}{4} d(z_1, z_2). \end{aligned}$$

Case II. If $z_1, z_2 \in X_2$, then we have

$$\begin{aligned} d(Sz_1, Tz_2) &= d\left(\left(0, \frac{k_1}{4}\right), \left(0, \frac{k_2}{4}\right)\right) \\ &= \max\left\{\frac{k_1}{4}, \frac{k_2}{4}\right\} + i\max\left\{\frac{k_1}{4}, \frac{k_2}{4}\right\}, \\ &= \max\left\{\frac{k_1}{4}, \frac{k_2}{4}\right\}(1+i), \\ &= \frac{1}{4}\max\{k_1, k_2\}(1+i), \\ &\preceq \frac{1}{4} d(z_1, z_2). \end{aligned}$$

Case III. If $z_1 \in X_1$, $z_2 \in X_2$, we have

$$\begin{aligned} d(Sz_1, Tz_2) &= d\left(\left(\frac{j_1}{4}, 0\right), \left(0, \frac{k_2}{4}\right)\right). \\ &= \left[\frac{j_1}{4} + \frac{k_2}{4}\right](1+i), \\ &= \frac{1}{4}[j_1 + k_2](1+i), \\ &\preceq \frac{1}{4} d(z_1, z_2). \end{aligned}$$

Case IV. If $z_2 \in X_1$, $z_1 \in X_2$, we have

$$\begin{aligned} d(Sz_1, Tz_2) &= d\left(\left(0, \frac{k_1}{4}\right), \left(\frac{j_2}{4}, 0\right)\right). \\ &= \left[\frac{k_1}{4} + \frac{j_2}{4}\right](1+i), \\ &= \frac{1}{4}[k_1 + j_2](1+i), \\ &\preceq \frac{1}{4} d(z_1, z_2). \end{aligned}$$

Thus, the condition (5) is satisfied with $a_1 = \frac{1}{4}$ and $0 \leq a_2, a_3, a_4 < \frac{1}{4}$ and, in all, conditions of theorem (4) are satisfied. Notice that the point $0 \in X$ remains fixed under T and is indeed unique. Thus, in all, this example substantiates the genuineness of our results proved in this paper.

3 Conclusions

In this article, we extend the study of complex valued metric spaces to xed point theory for new generalized rational contractions. We discuss properties of complex valued metric spaces and apply these properties in the framework for more general contraction involving rational expressions. Theorems established in this article will be helpful for researchers to work on rational contractions and derived some more common fixed point.

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