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Characterization and Bayesian Estimation of Generalized Standard Inverted Exponential Distribution

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Abstract: In this paper, some structural properties of generalized standard inverted exponential distribution (GIED) have been established. Bayesian method of estimation has been employed to estimate the parameters of GIED using a class of one non-informative (extension of Jeffrey's) prior and one informative (gamma) prior under the assumption of three loss functions, namely, Square error loss function, Al-Bayyati's loss function and LINEX loss function. These methods are compared by using mean square error for real life data as well as simulation study with varying sample sizes in R software. The expression for survival function has also been established under extension of Jeffrey's prior and gamma prior.

Keywords: Generalized standard inverted exponential distribution, Priors, loss functions, Survival function and R Software.

1 Introduction

The exponential distribution is the most widely used lifetime model in reliability theory, because of its simplicity and mathematical feasibility. If a random variable X has an exponential distribution, then $Y=\frac{1}{X}$ has an inverted exponential distribution (IED). IED has been discussed as a lifetime model by Lin et al. (1989) in detail. They obtained maximum likelihood estimates (MLEs), confidence limits and uniformly minimum variance unbiased estimators for the parameter and reliability function of IED with complete samples. Later IED has been considered by Killer and Kamath (1982) and among many others. The exponential distribution was generalized, by introducing a shape parameter, and studied extensively by Gupta and Kundu (1999), (2001). Raqab and Madi (2005) studied generalized exponential distribution (GED) from a Bayesian point of view.

On the same lines, Abouammoh and Alshingiti (2009) introduced a shape parameter in the IED to obtain generalized inverted exponential distribution (GIED). They derived many distributional properties and reliability characteristics of GIED. Assuming it to be a good lifetime model they obtained maximum likelihood estimators, least square estimators and confidence intervals of the two parameters involved. Sanku Dey (2010) discussed the Bayesian Estimation of the Shape Parameter of the Generalized Exponential Distribution under different loss functions. Hare Krishna and Kapil Kumar (2012) have studied the reliability estimation based on progressive type-II censored sample under classical setup. Singh et al (2013) studied the estimation of parameters of generalized inverted exponential distribution for progressive type-II censored sample with binomial removals.

Let $X_1, X_2, \ldots, X_n$ be i.i.d. generalized standard inverted exponential random variables, with the shape parameter α and scale parameter 1, the cumulative distribution function becomes

$$F(x;\alpha)=1-\left(1-e^{\frac{-1}{x}}\right)^{\alpha},\ x>0,\alpha>0 \tag{1.1}$$

with the corresponding probability density function (PDF) given by

$$f(x;\alpha)=\frac{\alpha}{x^2}e^{\frac{-1}{x}}\left(1-e^{\frac{-1}{x}}\right)^{\alpha-1},\ x>0,\alpha>0$$

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(1.2)

where α is a shape parameter. When $\alpha = 1$, the GIE distribution reduces to the standard inverse exponential distribution.

The graphs of density function and cumulative distribution function are plotted for different values of shape parameter α are given in Figure 1 and 2 respectively.

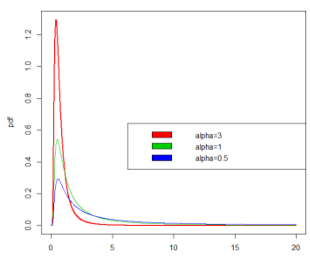

Fig.1 The graph of density function

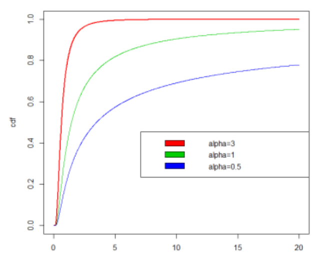

Fig.2 The graph of cumulative distribution function

Figure 1 and 2 illustrates some of the possible shapes of the pdf and cdf of the GIED distribution for different values of the parameter α .

2 Statistical Properties of the GSIE Distribution

This section provides some basic statistical properties of the generalized standard inverted exponential distribution.

2.1 Reliability Analysis

The reliability (survival) function of x is

$$S(x) = \left(1 - e^{\frac{-1}{x}}\right)^{\alpha} \qquad for \quad x > 0, \alpha > 0 \tag{2.1}$$

and the hazard function is

$$H(x) = \frac{\alpha}{x^2} e^{\frac{-1}{x}} \left(1 - e^{\frac{-1}{x}}\right)^{-1} \qquad for \quad x > 0, \alpha > 0 \tag{2.2}$$

The plots for the reliability (survival) and hazard functions are shown in Figure 3 and Figure 4 respectively;

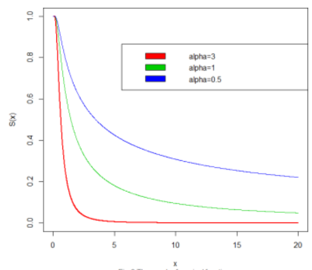

Fig.3 The graph of survival function

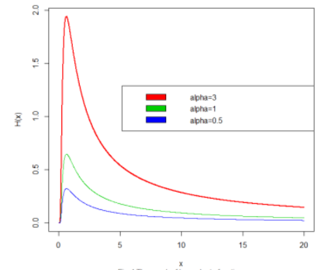

Fig.4 The graph of hazard rate function

Figure 3 and 4 illustrates some of the possible shapes of the survival function and hazard function of the GIED distribution for different values of the parameter α .

2.2 Moments

The r[th] moment of a continuous random variable X is given by;

$$\mu_r = E(X^r) = \int_0^{\infty} x^r f(x) dx \tag{2.3}$$

Now using equation (1.2) in eq. (2.3), we have

$$E(X^r) = \int_0^{\infty} x^{r-2} \alpha e^{\frac{-1}{x}} \left(1 - e^{\frac{-1}{x}}\right)^{\alpha - 1} dx \tag{2.4}$$

Using the expansion of

$$\left(1 - e^{\frac{-1}{x}}\right)^{\alpha - 1} = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(\alpha)}{\Gamma(\alpha - j) j!} e^{\frac{-j}{x}}$$

expression (2.4) takes the following form:

$$\mu_r = \sum_{j=0}^{\infty} \frac{(-1)^j \alpha \Gamma(\alpha)}{\Gamma(\alpha - j) j!} \int_0^{\infty} \frac{1}{x^{1-r+1}} e^{\frac{-(j+1)}{x}} dx$$

On solving the above equation, we get

$$\mu_r = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(\alpha + 1)}{\Gamma(\alpha - j) j!} \frac{\Gamma(1-r)}{(j+1)^{(1-r)}} \tag{2.5}$$

We observe that Equation (2.5) only exists when $r < 1$. The implication is that the first moment, second moment and other higher-order moments does not exist.

2.3 Harmonic mean of GISE distribution

The harmonic mean (H) is given as:

$$\frac{1}{H}=E\left(\frac{1}{X}\right)=\int_0^\infty \frac{1}{x}\, f(x;\alpha)dx$$

$$\frac{1}{H}=\int_0^\infty \frac{\alpha}{x^{2+1}} e^{\frac{-1}{x}}\left(1-e^{\frac{-1}{x}}\right)^{\alpha-1} dx$$

After some calculations,

$$\frac{1}{H}=\sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(\alpha+1)}{\Gamma(\alpha-j)\,j!}\,\frac{1}{(j+1)^2} \tag{2.6}$$

3 Moment Generating Function and Characteristic Function

Theorem 3.1. Let X have a GSIE distribution. Then moment generating function of X denoted by $M_X(t)$ is given by:

$$M_X(t)=\sum_{r=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^j t^r\Gamma(\alpha+1)}{\Gamma(\alpha-j)\,j!r!}\,\frac{\Gamma(1-r)}{(j+1)^{(1-r)}} \tag{3.1}$$

Proof: By definition

$$M_X(t)=E(e^{tx})=\int_0^\infty e^{tx} f(x;\alpha)dx$$

Using Taylor series

$$M_X(t)=\int_0^\infty\left(1+tx+\frac{(tx)^2}{2!}+\cdots\right)f(x;\alpha)dx$$

$$\Rightarrow \qquad M_X(t)=\sum_{r=0}^{\infty}\frac{t^r}{r!}\int_0^1 x^r f(x;\alpha)dx$$

$$\Rightarrow \qquad M_X(t)=\sum_{r=0}^{\infty}\frac{t^r}{r!}E(X^r)$$

$$\Rightarrow \qquad M_X(t)=\sum_{r=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^j t^r\Gamma(\alpha+1)}{\Gamma(\alpha-j)\,j!r!}\,\frac{\Gamma(1-r)}{(j+1)^{(1-r)}}$$

This completes the proof.

Theorem3.2. Let X have a GIE distribution. Then characteristic function of X denoted by $\phi_X(t)$ is given by:

$$\phi_X(t)=\sum_{r=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^j (it)^r\Gamma(\alpha+1)}{\Gamma(\alpha-j)\,j!r!}\,\frac{\Gamma(1-r)}{(j+1)^{(1-r)}} \tag{3.2}$$

Proof: By definition

$$\phi_X(t)=E(e^{itx})=\int_0^1 e^{itx} f(x;\alpha)dx$$

Using Taylor series

$$\phi_X(t)=\int_0^\infty\left(1+itx+\frac{(itx)^2}{2!}+\cdots\right)f(x;\alpha)dx$$

$$\Rightarrow \qquad \phi_X(t=\sum_{r=0}^{\infty}\frac{(it)^r}{r!}\int_0^\infty x^r f(x)dx$$

$$\Rightarrow \qquad \phi_X(t)=\sum_{r=0}^{\infty}\frac{(it)^r}{r!}E(X^r)$$

$$\Rightarrow \qquad \phi_X(t)=\sum_{r=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^j (it)^r\Gamma(\alpha+1)}{\Gamma(\alpha-j)\,j!r!}\,\frac{\Gamma(1-r)}{(j+1)^{(1-r)}}$$

This completes the proof.

4 Quantile Function and Median

The Quantile function is given by;

$$Q(u)=F^{-1}(u)$$

Therefore, the corresponding quantile function for the proposed model is given by;

$$Q(u)=\left[-\log\left[1-(1-u)^{1/\alpha}\right]\right]^{-1} \tag{4.1}$$

where U has the uniform U (0,1) distribution. We obtain the median by substituting u=0.5. Hence, the median of the proposed model is given by;

$$F^{-1}=\left[-\log\left[1-(1-0.5)^{1/\alpha}\right]\right]^{-1}$$

This can be simplified to give;

$$F^{-1}=\left[-\log\left[1-(0.5)^{1/\alpha}\right]\right]^{-1} \tag{4.2}$$

5 Estimation of Parameter

Let us consider a random sample $\underline{x}=(x_1,x_2,...,x_n)$ of size n from the generalized standard inverse exponential distribution. Then the log-likelihood function for the given sample observation is

$$L(x,\alpha)=\alpha^n\prod_{i=1}^{n}\frac{1}{x_i^2}e^{-\frac{1}{x_i}}\left(1-e^{-\frac{1}{x_i}}\right)^{\alpha-1}$$

$$\ln L(x,\alpha)=n\ln\alpha-2\sum_{i=1}^{n}\ln x_i-\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)+(\alpha-1)\sum_{i=1}^{n}\ln\left(1-e^{-\frac{1}{x_i}}\right)$$

As the shape parameter α is assumed to be unknown, the ML estimator of shape α is obtained by solving the

$$\frac{\partial\ln L(x,\alpha)}{\partial\alpha}=\frac{n}{\alpha}+\sum_{i=1}^{n}\ln\left(1-e^{-\frac{1}{x_i}}\right)=0$$

$$\hat{\alpha}=\frac{n}{\sum_{i=1}^{n}\ln\left(1-e^{-\frac{1}{x_i}}\right)^{-1}} \tag{5.1}$$

6 Bayesian Inference Using Different Loss Functions

The Bayesian inference requires appropriate choice of prior(s) for the parameter(s). From the Bayesian viewpoint, there is no clear cut way from which one can conclude that one prior is better than the other. Nevertheless, very often priors are chosen according to one's subjective knowledge and beliefs. However, if one has adequate information about the parameter(s), it is better to choose informative prior(s); otherwise, it is preferable to use non-informative prior(s). In this paper we consider both types of priors: the extended Jeffrey's prior and the gamma prior under squared error loss function, Al-Bayyati's loss function and LINEX loss function.

The extended Jeffrey's prior proposed by Al-Kutubi (2005) is given as

$$g_1(\alpha)\propto[I(\alpha)]^{c_1}\quad,\qquad c_1\varepsilon R^+$$

Where $\left[I(\alpha)\right]=-nE\left[\frac{\partial^2\log f(x;\alpha)}{\partial\alpha^2}\right]$ is the Fisher's information matrix. For the model (1.2),

$$g_1(\alpha)=\frac{1}{\alpha^{2c_1}} \tag{6.1}$$

The conjugate prior in this case will be the gamma prior, and the probability density function is taken as

$$g_2(\alpha)=\frac{a^b}{\Gamma b}e^{-a\alpha}\alpha^{b-1}\quad,\qquad a,b,\alpha>0 \tag{6.2}$$

With the above priors, we use three different loss functions for the model (1.2).

7 Bayesian Estimation of $\boldsymbol{\alpha}$ and *S*under the Assumption of Extended Jeffrey's' Prior

7.1 Baye's estimator of α

Combining the prior distribution (6.1) and the likelihood function, the posterior density of α is derived as follows:

$$\pi_1(\alpha\mid\underline{x})\propto\alpha^n\prod_{i=1}^{n}\frac{1}{x_i^2}e^{-\frac{1}{x_i}}\left(1-e^{-\frac{1}{x_i}}\right)^{\alpha-1}\frac{1}{\alpha^{2c_1}}$$

$$\pi_1(\alpha\mid\underline{x})=K\alpha^{n-2c_1}e^{-\alpha\sum_{i=1}^{n}\ln\left(1-e^{-1/x_i}\right)^{-1}}$$

$$\pi_1(\alpha\mid\underline{x})=K\alpha^{n-2c_1}e^{-\alpha\beta_1}$$

where K is independent of α

$$\beta_1=\sum_{i=1}^{n}\ln\left(1-e^{-\frac{1}{x_i}}\right)^{-1}\text{ and }K^{-1}=\int_0^\infty\alpha^{n-2c_1}e^{-\alpha\beta_1}\,d\alpha$$

$$\Rightarrow K^{-1}=\frac{\Gamma(n-2c_1+1)}{\beta_1^{\,n-2c_1+1}}$$

Hence the posterior density of α is given as

$$\pi_1(\alpha\mid\underline{x})=\frac{\beta_1^{\,n-2c_1+1}}{\Gamma(n-2c_1+1)}\alpha^{n-2c_1}\;e^{-\alpha\beta_1} \tag{7.1}$$

which is the pdf of gamma distribution $G(\beta_1,n-2c_1+1)$

7.1.1 Estimation under Squared Error loss function

By using squared error loss function $l(\hat{\alpha},\alpha)=c(\hat{\alpha}-\alpha)^2$ for some constant c the risk function is given by

$$R(\hat{\alpha},\alpha)=\int_0^\infty c(\hat{\alpha}-\alpha)^2\frac{\beta_1^{\,n-2c_1+1}}{\Gamma(n-2c_1+1)}\alpha^{n-2c_1}e^{-\alpha\beta_1}\;d\alpha$$

$$R(\hat{\alpha},\alpha)=c\frac{\beta_1^{\,n-2c_1+1}}{\Gamma(n-2c_1+1)}\left[\hat{\alpha}^2\int_0^\infty\alpha^{n-2c_1+1-1}e^{-\alpha\beta_1}d\alpha+\int_0^\infty\alpha^{n-2c_1+3-1}e^{-\alpha\beta_1}d\alpha-2\hat{\alpha}\int_0^\infty\alpha^{n-2c_1+2-1}e^{-\alpha\beta_1}d\alpha\right]$$

$$R(\hat{\alpha},\alpha)=c\frac{\beta_1^{\,n-2c_1+1}}{\Gamma(n-2c_1+1)}\left[\hat{\alpha}^2\frac{\Gamma(n-2c_1+1)}{\beta_1^{\,n-2c_1+1}}+\frac{\Gamma(n-2c_1+3)}{\beta_1^{\,n-2c_1+3}}-2\hat{\alpha}\frac{\Gamma(n-2c_1+2)}{\beta_1^{\,n-2c_1+2}}\right]$$

Now solving $\frac{\partial R(\hat{\alpha},\alpha)}{\partial\hat{\alpha}}=0$, we obtain the Baye'sestimator as

$$\hat{\alpha}_{1S}=\frac{(n-2c_1+1)}{\beta_1} \quad ;where\,\beta_1=\sum_{i=1}^{n}\ln\left(1-e^{-\frac{1}{x_i}}\right)^{-1} \tag{7.2}$$

7.1.2 Estimation under Al-Bayyati's loss function

By using Al-Bayyati's loss function $l(\hat{\alpha},\alpha)=\alpha^{c_2}(\hat{\alpha}-\alpha)^2$ the risk function is given by

$$R(\hat{\alpha},\alpha)=\int_0^{\infty}\alpha^{c_2}(\hat{\alpha}-\alpha)^2\frac{\beta_1^{\,n-2c_1+1}}{\Gamma(n-2c_1+1)}\alpha^{n-2c_1}e^{-\alpha\beta_1}\,d\alpha$$

$$R(\hat{\alpha},\alpha)=\frac{\beta_1^{\,n-2c_1+1}}{\Gamma(n-2c_1+1)}\left[\hat{\alpha}^2\int_0^{\infty}\alpha^{n-2c_1+c_2}e^{-\alpha\beta_1}\,d\alpha+\int_0^{\infty}\alpha^{n-2c_1+c_2+2}e^{-\alpha\beta_1}\,d\alpha-2\hat{\alpha}\int_0^{\infty}\alpha^{n-2c_1+c_2+1}e^{-\alpha\beta_1}\,d\alpha\right]$$

$$R(\hat{\alpha},\alpha)=\frac{\beta_1^{\,n-2c_1+1}}{\Gamma(n-2c_1+1)}\left[\hat{\alpha}^2\frac{\Gamma(n-2c_1+c_2+1)}{\beta_1^{\,n-2c_1+c_2+1}}+\frac{\Gamma(n-2c_1+c_2+3)}{\beta_1^{\,n-2c_1+c_2+3}}-2\hat{\alpha}\frac{\Gamma(n-2c_1+c_2+2)}{\beta_1^{\,n-2c_1+c_2+2}}\right]$$

Now solving $\frac{\partial R(\hat{\alpha},\alpha)}{\partial\hat{\alpha}}=0$, we obtain the Bayes estimator

$$\hat{\alpha}_{1AB}=\frac{(n-2c_1+c_2+1)}{\beta_1} \quad ,where\,\beta_1=\sum_{i=1}^{n}\ln\left(1-e^{-\frac{1}{x_i}}\right)^{-1} \tag{7.3}$$

Remark 1.1: Replacing c2= 0 in (7.3), we get the same Bayes estimator as obtained in (7.2) corresponding to the SELF.

7.1.3 Estimation under LINEX loss function

By using LINEX loss function $l(\alpha,\hat{\alpha})=\exp\{b_1(\hat{\alpha}-\alpha)\}-b_1(\hat{\alpha}-\alpha)-1$ for some constant b_1 the risk function is given by

$$R(\alpha,\hat{\alpha})=\int_0^{\infty}\left(\exp\{b_1(\hat{\alpha}-\alpha)\}-b_1(\hat{\alpha}-\alpha)-1\right)\frac{\beta_1^{\,n-2c_1+1}}{\Gamma(n-2c_1+1)}\alpha^{n-2c_1}e^{-\alpha\beta_1}\,d\alpha$$

$$R(\hat{\alpha},\alpha)=\frac{\beta_1^{\,n-2c_1+1}}{\Gamma(n-2c_1+1)}\left[e^{b_1\hat{\alpha}}\int_0^{\infty}\alpha^{n-2c_1}e^{-\alpha(b_1+\beta_1)}\,d\alpha-b_1\hat{\alpha}\int_0^{\infty}\alpha^{n-2c_1}e^{-\alpha\beta_1}\,d\alpha+b_1\int_0^{\infty}\alpha^{n-2c_1+1}e^{-\alpha\beta_1}\,d\alpha-\int_0^{\infty}\alpha^{n-2c_1}e^{-\alpha\beta_1}\,d\alpha\right]$$

$$R(\hat{\alpha},\alpha)=\frac{\beta_1^{\,n-2c_1+1}}{\Gamma(n-2c_1+1)}\left[e^{b_1\hat{\alpha}}\frac{\Gamma(n-2c_1+1)}{(b_1+\beta_1)^{n-2c_1+1}}-b_1\hat{\alpha}\frac{\Gamma(n-2c_1+1)}{\beta_1^{\,n-2c_1+1}}+b_1\frac{\Gamma(n-2c_1+2)}{\beta_1^{\,n-2c_1+2}}-\frac{\Gamma(n-2c_1+1)}{\beta_1^{\,n-2c_1+1}}\right]$$

$$R(\hat{\alpha},\alpha)=e^{b_1\hat{\alpha}}\left(\frac{\beta_1}{b_1+\beta_1}\right)^{n-2c_1+1}-b_1\hat{\alpha}+b_1\frac{(n-2c_1+1)}{\beta_1}-1$$

Now solving $\frac{\partial R(\hat{\alpha},\alpha)}{\partial\hat{\alpha}}=0$, we obtain the Bayes estimator as

$$\hat{\alpha}_{1Li}=\frac{1}{b_1}\log\left(\frac{b_1+\beta_1}{\beta_1}\right)^{n-2c_1+1} \quad ,where\,\beta_1=\sum_{i=1}^{n}\ln\left(1-e^{-\frac{1}{x_i}}\right)^{-1} \tag{7.4}$$

7.2 Baye's estimator of S(x)

By using posterior distribution function (7.1), we can found the survival function such that

$$\hat{S}_{1Ej}(x)=\int_0^{\infty}\left\{\left(1-e^{\frac{-1}{x}}\right)^{\alpha}\right\}\pi_1(\alpha/\underline{x})d\alpha$$

$$\hat{S}_{1Ej}(x)=\int_0^{\infty}\left\{\left(1-e^{\frac{-1}{x}}\right)^{\alpha}\right\}\frac{\beta_1^{\,n-2c_1+1}}{\Gamma(n-2c_1+1)}\alpha^{n-2c_1}e^{-\alpha\beta_1}d\alpha$$

$$\hat{S}_{1Ej}(x)=\frac{\beta_1^{\,n-2c_1+1}}{\Gamma(n-2c_1+1)}\int_0^{\infty}\alpha^{n-2c_1}e^{-\alpha\left\{\beta_1-\ln\left(1-e^{-\frac{1}{x}}\right)\right\}}d\alpha$$

$$\hat{S}_{1Ej}(x)=\left(\frac{\beta_1}{\beta_1-\ln\left(1-e^{-\frac{1}{x}}\right)}\right)^{n-2c_1+1} \quad ,where\,\beta_1=\sum_{i=1}^{n}\ln\left(1-e^{-\frac{1}{x_i}}\right)^{-1} \tag{7.5}$$

8. Bayesian Estimation of $\boldsymbol{\alpha}$ and $\boldsymbol{S}$ under the Assumption of Gamma Prior

8.1 Bayes estimator of α

Combining the prior distribution (6.2) and the likelihood function, the posterior density of α is derived as follows:

$$\pi_2(\alpha\,|\,\underline{x}) \propto \alpha^n \prod_{i=1}^{n} \frac{1}{x_i^2} e^{-\frac{1}{x_i}} \left(1-e^{-\frac{1}{x_i}}\right)^{\alpha-1} \frac{a^b}{\Gamma b} e^{-a\alpha} \alpha^{b-1}$$

$$\pi_2(\alpha\,|\,\underline{x}) = K\alpha^{n+b-1} e^{-\alpha\left\{a+\sum_{i=1}^{n}\ln\left(1-e^{-1/x_i}\right)^{-1}\right\}}$$

$\pi_2(\alpha\,|\,\underline{x}) = K\alpha^{n+b-1}e^{-\alpha(a+\beta_1)}$, where K is independent of

$$\alpha\,,\beta_1 = \sum_{i=1}^{n}\ln\left(1-e^{-\frac{1}{x_i}}\right)^{-1}$$

and $K^{-1} = \int_0^\infty \alpha^{n+b-1}e^{-\alpha(a+\beta_1)}\,d\alpha$

$$\Rightarrow K^{-1} = \frac{\Gamma(n+b)}{(a+\beta_1)^{n+b}}$$

Hence the posterior density of α is given as

$$\pi_2(\alpha\,|\,\underline{x}) = \frac{(a+\beta_1)^{n+b}}{\Gamma(n+b)}\alpha^{n+b-1}\; e^{-\alpha(a+\beta_1)} \tag{8.1}$$

which is the pdf of gamma distribution $G\big((a+\beta_1),(n+b)\big)$

8.1.1 Estimation under Squared Error loss function

By using squared error loss function $l(\hat{\alpha},\alpha) = c(\hat{\alpha}-\alpha)^2$ for some constant c the risk function is given by

$$R(\hat{\alpha},\alpha) = \int_0^\infty c(\hat{\alpha}-\alpha)^2 \frac{(a+\beta_1)^{n+b}}{\Gamma(n+b)}\alpha^{n+b-1}\; e^{-\alpha(a+\beta_1)}\quad d\alpha$$

$$R(\hat{\alpha},\alpha) = c\frac{(a+\beta_1)^{n+b}}{\Gamma(n+b)}\left[\hat{\alpha}^2\int_0^\infty \alpha^{n+b-1}\; e^{-\alpha(a+\beta_1)}d\alpha + \int_0^\infty \alpha^{n+b+2-1}\; e^{-\alpha(a+\beta_1)}\,d\alpha - 2\hat{\alpha}\int_0^\infty \alpha^{n+b+1-1}\; e^{-\alpha(a+\beta_1)}\,d\alpha\right]$$

$$R(\hat{\alpha},\alpha) = c\frac{(a+\beta_1)^{n+b}}{\Gamma(n+b)}\left[\hat{\alpha}^2\frac{\Gamma(n+b)}{(a+\beta_1)^{n+b}} + \frac{\Gamma(n+b+2)}{(a+\beta_1)^{n+b+2}} - 2\hat{\alpha}\frac{\Gamma(n+b+1)}{(a+\beta_1)^{n+b+1}}\right]$$

Now solving $\dfrac{\partial R(\hat{\alpha},\alpha)}{\partial\hat{\alpha}} = 0$, we obtain the Baye's estimator as

$$\hat{\alpha}_{2S} = \frac{(n+b)}{(a+\beta_1)} \quad ,\textit{where}\; \beta_1 = \sum_{i=1}^{n}\ln\left(1-e^{-1/x_i}\right)^{-1} \tag{8.2}$$

8.1.2 Estimation under Al-Bayyati's loss function

By using Al-Bayyati's loss function $l(\hat{\alpha},\alpha) = \alpha^{c_2}(\hat{\alpha}-\alpha)^2$ the risk function is given by

$$R(\hat{\alpha},\alpha) = \int_0^\infty \alpha^{c_2}(\hat{\alpha}-\alpha)^2\frac{(a+\beta_1)^{n+b}}{\Gamma(n+b)}\alpha^{n+b-1}\; e^{-\alpha(a+\beta_1)}\,d\alpha$$

$$R(\hat{\alpha},\alpha) = \frac{(a+\beta_1)^{n+b}}{\Gamma(n+b)}\left[\hat{\alpha}^2\int_0^\infty \alpha^{n+b+c_2-1}\; e^{-\alpha(a+\beta_1)}d\alpha + \int_0^\infty \alpha^{n+b+c_2+2-1}\; e^{-\alpha(a+\beta_1)}d\alpha - 2\hat{\alpha}\int_0^\infty \alpha^{n+b+c_2+1-1}\; e^{-\alpha(a+\beta_1)}d\alpha\right]$$

$$R(\hat{\alpha},\alpha) = \frac{(a+\beta_1)^{n+b}}{\Gamma(n+b)}\left[\hat{\alpha}^2\frac{\Gamma(n+b+c_2)}{(a+\beta_1)^{n+b+c_2}} + \frac{\Gamma(n+b+c_2+2)}{(a+\beta_1)^{n+b+c_2+2}} - 2\hat{\alpha}\frac{\Gamma(n+b+c_2+1)}{(a+\beta_1)^{n+b+c_2+1}}\right]$$

Now solving $\dfrac{\partial R(\hat{\alpha},\alpha)}{\partial\hat{\alpha}} = 0$, we obtain the Bayes estimator as

$$\hat{\alpha}_{2\,AB} = \frac{(n+b+c_2)}{(a+\beta_1)} \quad ,\textit{where}\; \beta_1 = \sum_{i=1}^{n}\ln\left(1-e^{-\frac{1}{x_i}}\right)^{-1} \tag{8.3}$$

Remark 1.2: Replacing c2= 0 in (8.3), we get the same Bayes estimator as obtained in (8.2) corresponding to the SELF.

8.1.3 Estimation under LINEX loss function

By using LINEX loss function $l(\alpha,\hat{\alpha}) = \exp\{b_1(\hat{\alpha}-\alpha)\} - b_1(\hat{\alpha}-\alpha) - 1$ for some constant b_1 the risk function is given by

$$R(\alpha,\hat{\alpha}) = \int_0^\infty \left(\exp\{b_1(\hat{\alpha}-\alpha)\} - b_1(\hat{\alpha}-\alpha) - 1\right)\frac{(a+\beta_1)^{n+b}}{\Gamma(n+b)}\alpha^{n+b-1}\; e^{-\alpha(a+\beta_1)}\,d\alpha$$

$$R(\alpha,\hat{\alpha}) = \frac{(a+\beta_1)^{n+b}}{\Gamma(n+b)}\left[\begin{array}{l} e^{b_1\hat{\alpha}}\int_0^\infty \alpha^{n+b-1}\; e^{-\alpha(b_1+a+\beta_1)}d\alpha - b_1\hat{\alpha}\int_0^\infty \alpha^{n+b-1}\; e^{-\alpha(a+\beta_1)}d\alpha \\ \qquad + b_1\int_0^\infty \alpha^{n+b+1-1}\; e^{-\alpha(a+\beta_1)}d\alpha - \int_0^\infty \alpha^{n+b-1}\; e^{-\alpha(a+\beta_1)}d\alpha \end{array}\right]$$

$$R(\alpha,\hat{\alpha}) = \frac{(a+\beta_1)^{n+b}}{\Gamma(n+b)}\left[e^{b_1\hat{\alpha}}\frac{\Gamma(n+b)}{(a+b_1+\beta_1)^{n+b}} - b_1\hat{\alpha}\frac{\Gamma(n+b)}{(a+\beta_1)^{n+b}} + b_1\frac{\Gamma(n+b+1)}{(a+\beta_1)^{n+b+1}} - \frac{\Gamma(n+b)}{(a+\beta_1)^{n+b}}\right]$$

$$R(\alpha,\hat{\alpha}) = e^{b_1\hat{\alpha}}\left(\frac{a+\beta_1}{a+b_1+\beta_1}\right)^{n+b} - b_1\hat{\alpha} + b_1\frac{(n+b)}{(a+\beta_1)} - 1$$

Now solving $\frac{\partial R(\hat{\alpha},\alpha)}{\partial \hat{\alpha}} = 0$, we obtain the Baye's estimator as

$$\hat{\alpha}_{2Li} = \frac{1}{b_1}\log\left(\frac{a+b_1+\beta_1}{a+\beta_1}\right)^{n+b}, where\, \beta_1 = \sum_{i=1}^{n}\ln\left(1-e^{-\frac{1}{x_i}}\right)^{-1} \qquad (8.4)$$

8.2 Baye's estimator of S(x)

By using posterior distribution function (8.1), we can found the survival function such that

Table 1: MSE for $\hat{\alpha}$ under extension of Jeffery's prior using different loss functions

N	α	C1	MLE $\hat{\alpha}_{ML}$	$\hat{\alpha}_{SL}$	$\hat{\alpha}_{AL}$		$\hat{\alpha}_{LL}$	
					c_2=1.3	c_2=-1.3	b_1=1.2	b_1=-1.2
25	0.5	0.5	0.009920	0.009920	0.011743	0.009561	0.009704	0.010230
		1.0	0.008248	0.006692	0.008902	0.006098	0.006404	0.007087
	1.0	0.5	0.265436	0.265436	0.277988	0.253211	0.259886	0.271141
		1.0	0.050551	0.061059	0.047867	0.078292	0.067014	0.055367
	1.5	0.5	0.298196	0.298196	0.409708	0.208053	0.219145	0.407304
		1.0	0.600976	0.481697	0.639827	0.350146	0.359968	0.649817
50	0.5	0.5	0.008726	0.008726	0.007517	0.010199	0.009003	0.008456
		1.0	0.007174	0.008188	0.006901	0.009747	0.008480	0.007904
	1.0	0.5	0.026544	0.026544	0.023113	0.031110	0.028309	0.024933
		1.0	0.026257	0.024287	0.027023	0.023062	0.023523	0.025429
	1.5	0.5	0.101102	0.101102	0.087334	0.117070	0.110049	0.092561
		1.0	0.039084	0.035488	0.040549	0.033772	0.033942	0.038852
100	0.5	0.5	0.002823	0.002823	0.003198	0.002542	0.002747	0.002906
		1.0	0.002771	0.003027	0.002703	0.003427	0.003107	0.002949
	1.0	0.5	0.023899	0.023899	0.027795	0.020431	0.022060	0.025888
		1.0	0.011205	0.010305	0.011518	0.009464	0.009859	0.010844
	1.5	0.5	0.072457	0.072457	0.083050	0.062870	0.064843	0.080931
		1.0	0.066907	0.060298	0.069002	0.052574	0.054283	0.067081

Table 2: MSE for $\hat{\alpha}$ under gamma prior using different loss functions

N	α	a=b	MLE $\hat{\alpha}_{ML}$	$\hat{\alpha}_{SL}$	$\hat{\alpha}_{AL}$		$\hat{\alpha}_{LL}$	
					c_2=1.3	c_2=-1.3	b_1=1.2	b_1=-1.2
25	0.5	0.5	0.009924	0.010148	0.012208	0.009520	0.009868	0.010522
		1.0	0.008550	0.009547	0.013448	0.007193	0.008807	0.010417
	1.0	0.5	0.264970	0.265044	0.277536	0.252877	0.259520	0.270723
		1.0	0.050804	0.049616	0.040210	0.062808	0.054325	0.045275
	1.5	0.5	0.296489	0.261076	0.360422	0.181495	0.192284	0.355974
		1.0	0.597822	0.465531	0.607573	0.345912	0.354868	0.614678
50	0.5	0.5	0.008724	0.008444	0.007295	0.009854	0.008709	0.008185
		1.0	0.007269	0.006781	0.005824	0.008005	0.007013	0.006558
	1.0	0.5	0.026356	0.026228	0.022859	0.030711	0.027963	0.024645
		1.0	0.026192	0.026058	0.029797	0.023768	0.024779	0.027730
	1.5	0.5	0.100009	0.101579	0.087744	0.117561	0.110518	0.093029
		1.0	0.038687	0.036446	0.042411	0.033623	0.034138	0.040530
100	0.5	0.5	0.002824	0.002885	0.003276	0.002587	0.002805	0.002972
		1.0	0.002794	0.002678	0.002445	0.002987	0.002740	0.002621
	1.0	0.5	0.023867	0.023690	0.027543	0.020261	0.021872	0.025656
		1.0	0.011215	0.011167	0.012634	0.010063	0.010592	0.011837
	1.5	0.5	0.072246	0.072246	0.079717	0.060260	0.062184	0.077639
		1.0	0.066596	0.061969	0.070748	0.054137	0.055854	0.068835

$$\hat{S}_{2g}(x)=\int_0^\infty\left\{\left(1-e^{\frac{-1}{x}}\right)^\alpha\right\}\pi_2(\alpha/\underline{x})d\alpha$$

$$\hat{S}_{2g}(x)=\int_0^\infty\left\{\left(1-e^{\frac{-1}{x}}\right)^\alpha\right\}\frac{(a+\beta_1)^{n+b}}{\Gamma(n+b)}\alpha^{n+b-1}\,e^{-\alpha(a+\beta_1)}d\alpha$$

$$\hat{S}_{2g}(x)=\frac{(a+\beta_1)^{n+b}}{\Gamma(n+b)}\int_0^\infty\alpha^{n+b-1}e^{-\alpha\left\{a+\beta_1-\ln\left(1-e^{-\frac{1}{x}}\right)\right\}}d\alpha$$

$$\hat{S}_{2g}(x)=\left(\frac{a+\beta_1}{a+\beta_1-\ln\left(1-e^{-\frac{1}{x}}\right)}\right)^{n+b},\ where\ \beta_1=\sum_{i=1}^n\ln\left(1-e^{-\frac{1}{x_i}}\right)^{-1} \tag{8.5}$$

9 Simulation Study

In our simulation study, we chose a sample size of n=25, 50 and 100 to represent small, medium and large data set. The shape parameter α is estimated for generalized standard inverted exponential distribution by using the Bayesian method of estimation under extension of Jeffrey's and gamma priors by using different loss functions. Here, we have considered $\alpha=0.5,1.0\&1.5$ .The values of Jeffrey's extension were $c_1=0.5\ \&1.0$ and hyper parameters were $a=b=0.5\ \&1.0$. The values for the loss parameters $c_2=\pm1.3$ and $b_1=\pm1.2$. This was iterated 1000 times and the parameter for each method was calculated. A simulation study was conducted in R-software to examine and compare the performance of the estimates for different sample sizes with different values for the (extension of Jeffrey's and gamma) priors and the loss functions. The results are presented in tables for different selections of the parameters.

10 Real Data Example

Table 3: Posterior Mean and Posterior Variance of a generalized standard inverted exponential distribution under extension of Jeffery's prior and gamma prior

α	**Hyper Parameters a=b**	**Jeffrey's Extension C_1**	**Mean/P.V**	**Extension Jeffrey's prior**	**Gamma prior**
0.5	0.5	0.5	**Mean**	1.402551	1.398119
			post.var	0.03122458	0.030783
	1.0	1.0	**Mean**	1.380288	1.393784
			post.var	0.03072895	0.030353
1.0	0.5	0.5	**Mean**	1.402551	1.398119
			post.var	0.03122458	0.030783
	1.0	1.0	**Mean**	1.380288	1.393784
			post.var	0.03072895	0.030353
1.5	0.5	0.5	**Mean**	1.402551	1.398119
			post.var	0.03122458	0.030783
	1.0	1.0	**Mean**	1.380288	1.393784
			post.var	0.03072895	0.030353

In this section, we consider the real life data set which already has been used by Smith and Naylor (1987). This data represents the strengths of 1.5 cm glass fibers, measured at the National Physical Laboratory, England. The data is given below:

0.55, 0.93, 1.25, 1.36, 1.49, 1.52, 1.58, 1.61, 1.64, 1.68, 1.73, 1.81, 2.0, 0.74, 1.04, 1.27, 1.39, 1.49, 1.53, 1.59,1.61, 1.66, 1.68, 1.76, 1.82, 2.01, 0.77, 1.11, 1.28, 1.42, 1.50, 1.54, 1.60, 1.62, 1.66, 1.69, 1.76, 1.84, 2.24, 0.81,1.13, 1.29, 1.48, 1.5, 1.55, 1.61, 1.62, 1.66, 1.70, 1.77, 1.84, 0.84, 1.24, 1.30, 1.48, 1.51, 1.55, 1.61, 1.63, 1.67, 1.70, 1.78, 1.89

The MLE of the above data works out to be $\hat{\alpha}_{ML}=1.402551$ and its variance is 1.398119.The table below provides the posterior mean and posterior variance under the two priors, viz. extension of Jeffrey's prior and Gamma prior.

11. Results

i. In table 1, Bayes estimation with Al-Bayytai's Loss function under extension of Jeffrey's prior provides the smallest values in most cases especially when loss parameter C_2 is (-1.3). Similarly, in table 2, Bayes estimation with Al-Bayytai's Loss function under gamma prior provides the smallest values in most cases especially when loss parameter C_2 is (-1.3) whether the extension of Jeffrey's prior is 0.5 & 1.0 and hyper parameters of gamma prior is 0.5 & 1.0. Moreover, when the sample size increases from 25 to 100, the MSE decreases quite significantly.

ii. The posterior mean and posterior variance under the assumed priors is calculated by assuming the different values of hyper parameters. From table 3, it is clear that the posterior variance under the Gamma prior are less as compared to extension of Jeffrey's prior, which shows that this prior is efficient as compared to extension of Jeffrey's prior and this less variation in posterior distribution helps in making more precise Bayesian estimation of the true unknown parameter α of generalized standard inverted exponential distribution.

12. Conclusion

In this paper, we have addressed the problem of Bayesian estimation for the generalized standard inverted exponential distribution under different loss functions each in the worked example as well as in the simulation study. From the results, we observe that in most cases, Bayesian estimator under Al-Bayyati's loss function provides the smallest MSE values under extension of Jeffrey's prior and gamma prior as compared to other loss functions and the classical estimator when the loss parameter c_2 is ± 1.3. Thus we can say that Al- Bayyati's loss is better than other loss functions. Also Bayesian estimator under the Gamma prior has the less posterior variance. It is also observed that among the priors, Gamma prior provides the Bayes estimators with least MSE.

References

[1] Abouammoh A. M. and Alshingiti, A. M (2009), "Reliability estimation of generalized inverted exponential distribution", *Journal of Statistical Computation and Simulation"*, vol. 79, no 11-12, pp. 1301-1315.

[2] Al-Bayyati , H.N.(2002), "Comparing methods of estimating Weibull failure models using simulation. Ph.D. Thesis, College of Administration and Economics", Baghdad University, Iraq.

[3] Al-Kutubi, H.S. (2005), "On Comparison Estimation Procedures for Parameter and Survival function", *Iraqi Journal of Statistical Science*, vol. 9, pp.1-14.

[4] Gupta, R. D., Kundu, D. (1999), "Generalized exponential distributions", *Australian and New Zealand Journal of Statistics*41, 173-188.

[5] Gupta, R. D. and Kundu, D. (2001), "Generalized Exponential Distribution Different Method of Estimations", *Journal of Statistical Computations and Simulations* 69, 315-337.

[6] H. Krishna, K. Kumar (2012), "Reliability estimation in generalized inverted exponential distribution with progressively type II censored sample", *Journal of Statistical Computation and Simulation* 83 (6) 1007–1019.

[7] Killer, A. Z. and Kamath, A. R. (1982), "Reliability analysis of CNC machine tools", *Reliability Engineering*,3, 449–473.

[8] Klebanov, L.B., (1972), "Universal Loss Function and Unbiased Estimation", Dokl. Akad. Nank SSSR Soviet-Maths. Dokl. T. 203, pp1249-1251.

[9] Lin, C., Duran, B., & Lewis, T. (1989), "Inverted Gamma As a Life Distribution", *Microelectronics and Reliability 29(4)*, 619-626.

[10] Raqab, M. Z. and Madi, M. T., (2005), "Bayesian Inference for the Generalized Exponential Distribution", *Journal of Statistical computation and Simulation*, 75(10) 841 - 852.

[11] SankuDey.,(2010), "Bayesian Estimation of the Shape Parameter of the Generalised Exponential Distribution under Different Loss Functions", *Pak.j.stat.oper.res.* Vol.VI No.2 pp163-174

[12] Singh S. K., Singh U., Kumar M. (2013), "Estimation of Parameters of Generalized Inverted Exponential Distribution for Progressive Type-II Censored Sample with Binomial Removals", *Journal of Probability and Statistics*. Volume 2013, Article ID 183652. http://dx.doi.org/10.1155/2013/183652.

[13] Smith RL, Naylor JC (1987), "A comparison of Maximum likelihood and Bayesian estimators for the three parameter Weibull distribution", *Journal of the Royal Statistical Society* 36(3): 358-369.

[14] Varian, H. R. (1975), "A Bayesian approach to real estate assessment", Studies in Bayesian Econometrics and Statistics in Honor of Leonard J. Savage (eds: S.E. Fienberg and A.Zellner), North-Holland, Amsterdam, 195-208.

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