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# Nonlinear Self-Adjointness and Conservation Laws of KdV Equation with Linear Damping Force

*Muhammad Nasir Ali[1,*], Shahid Ali[2], Syed Muhammad Husnine[1] and Turgut Ak[3]*

[1]Department of Sciences and Humanities, National University of Computer and Emerging Sciences, Fast, Lahore, Pakistan.
[2]Department of Civil Engineering, National University of Computer and Emerging Sciences, Fast, Lahore, Pakistan.
[3]Department of Transportation Engineering, Faculty of Engineering, Yalova University, 77100 Yalova, Turkey.

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**Abstract:** In this paper, the KdV equation with linear damping force is considered for a large scale problem such as tsunami. It is observed that the equation is nonlinear self-adjoint. Lie point symmetries are calculated. Some new conservation laws are obtained using two different methods including the new general conservation theorem of Ibragimov and multiplier method.

**Keywords:** KdV equation, Self-adjointness, Conservation laws, Ibragimov method, Multiplier method, Linear damping.

## 1 Introduction

Nonlinear partial differential equations (NLPDEs) are useful to describe complex phenomena in different fields of science, especially mathematics, physics and fluid dynamics. The exact solutions of such equations play an important role in the theory of soliton. These solutions are used for the verification of numerical solvers and also helpful for the study of stability analysis.

The types of KdV equations are the most popular soliton equations which are extensively investigated by the researchers with different methodologies in mathematical physics, quantum field theory and fluid dynamics. Also, it is very useful in Tsunami generation by sub-marine Landslides and geological activities [1, 2, 3]. While dealing with such types of large scale problems, it becomes necessary to add forcing terms including linear damping, coriolis force and bottom friction.

The study of the conservation laws of differential equation is of great importance and a rich area of research for the mathematicians. These laws can be used to derive exact solutions of partial differential equations [4]. Existence of large number of the conservation laws indicate about the integrability of the NLPDEs. Noether's approach is an elegant way to construct the conservation laws of the NLPDEs [5]. Noether's theorem depends on the existence of the suitable Lagrangian for the differential equation. However, Lagrangian exists only for a special class of the differential equations. This theorem is not applicable for evolution equations or the equations having odd orders.

Partial Noether's approach is developed to overcome this restriction. Conservation laws of the differential equations such as nonlinear heat equation are derived using this approach [6]. Partial Lagrangian approach fails to find the conservation laws for example classical KdV and ZK equation because these are odd order differential equations and do not possess partial Lagrangian. Some researchers used the transformation $u = v_x$ to make the order of the differential equation even and then apply the partial Lagrangian approach to find the conservation laws [7].

In order to solve the above mentioned problem, Ibragimov proposed a new general conservation theorem for obtaining the conservation laws of an arbitrary differential equation. This theorem depends upon the formal Lagrangian and self-adjointness of the differential equation to achieve the conservation laws [8, 9]. This technique is efficient and also applicable for the system of PDEs in which number of the dependent variables and the number of the equations are equal. While using this technique one can have more than one conservation laws corresponding to a single symmetry.

Much work has been done on the exact solution and conservation laws of differential equation with different techniques. Multiplier method is considered for obtaining the conservation laws of differential equation [10]. Analytical solutions of ultra-long wave with complete coriolis force and heating are calculated [11]. Exact solutions and conservation laws for a forced KdV equation are obtained by using symmetry reduction and self-adjointness respectively [12]. Conservation laws of fifth order generalized KdV equation are calculated through nonlinear self-adjointness [13]. Exact solutions of the

*Corresponding author e-mail: taj467@gmail.com

geophysical ocean wave model while considering the effect of coriolis force due to the rotation of earth, are obtained [14]. Forced BBM equation is considered for calculating the exact solutions and conservation laws [15].

Symmetries and conservation laws are utilized to find the exact solutions of damped Boussinesq equation [16]. Whitham's method for the construction of modulation equations is applied to those systems whose dynamics are described by a perturbed KdV equation [17]. The modulation of nonlinear periodic wave trains with dissipative term included in the KdV equation

$$u_t + \delta u u_x + \sigma u_{xxx} + \epsilon V(u) = 0, \tag{1}$$

is considered [18]. Three different forms of dissipation terms $V(u)$ namely linear damping, KdV-Burgers damping and boundary layer damping are discussed.

In this research paper, we consider the long waves on the oceans modelled by the KdV equation of the form

$$G = u_t + a u u_x + u_{xxx} + K u = 0, \tag{2}$$

while taking into account the effect of linear damping. Here, we are dealing with a large scale problem, so the effect of damping force is very important. Self-adjointness of the differential equation and formal Lagrangian is used to find the conservation laws of Eq. (2).

The rest of the article is arranged as follows. In section 2, formal Lagrangian is used to calculate adjoint equation for Eq. (2). Types of self-adjointness are explained in section 3. Lie point symmetries of Eq. (2) are obtained in section 4. General conservation theorem is reported in section 5. And conservation laws for Eq. (2) are calculated. Using multiplier method, higher order conservation laws are calculated in section 6. Conclusions are given in section 7. References are provided in end.

## 2 Formal Lagrangian and Andjoint Equation

The formal Lagrangian for the Eq. (2) is

$$L = v[u_t + a u u_x + u_{xxx} + K u]. \tag{3}$$

Here $v(t, x, u)$ is a new dependent variable. The adjoint equation to the Eq. (2) is

$$G^* \equiv \frac{\delta L}{\delta u} = 0, \tag{4}$$

where

$$\frac{\delta L}{\delta u} = L_u - D_t\left(L_{u_t}\right) - D_x\left(L_{u_x}\right) - D_x^3\left(L_{u_{xxx}}\right), \tag{5}$$

and $D_t$, $D_x$ and $D_x^3$ represent the total derivatives. After some simplifications, we obtain the following adjoint equation to the Eq. (2)

$$G^* = -v_t - a u v_x - v_{xxx} + K v = 0. \tag{6}$$

## 3 Self-Adjointness of Differential Equation

This section explains some related definitions [8, 15]:

**Definition 1.** Consider the $pth$ order nonlinear differential equation of the form $G\left(x, u, u_{(1)}, \ldots, u_{(p)}\right) = 0$, with $m$ independent variables $x = (x^1, x^2, \ldots, x^m)$ and the dependent variable $u$, is called strictly self-adjoint if its adjoint equation becomes original after the substitution of the form $v = u$.

**Definition 2.** If the adjoint equation can be converted into the original equation by using the substitution $v = \varphi$, where $\varphi$ is the non-zero function of the dependent variables, independent variables and the derivatives of the dependent variables, then it is called nonlinear self-adjoint differential equation.

In this article, we are interested in nonlinear self-adjointness of the Eq. (2) with the non-zero substitution of the form

$$v = \varphi(t, x, u) \neq 0. \tag{7}$$

To find the above mentioned type of the substitution, the following condition is used

$$G^*|_{v=\varphi(t,x,u)} = \mu(u_t + a u u_x + u_{xxx} + K u), \tag{8}$$

where $\mu$ is undetermined coefficient.

The derivatives of the unknown function $v$, defined in Eq. (7) are

$$v_t = \varphi_t + \varphi_u u_t, \tag{9}$$

$$v_x = \varphi_x + \varphi_u u_x, \tag{10}$$

$$v_{xxx} = \varphi_{xxx} + 3\varphi_{xxu} u_x + 3\varphi_{xuu} {u_x}^2 + 3\varphi_{xu} u_{xx} + \varphi_{uuu} {u_x}^3 + 3\varphi_{uu} u_x u_{xx} + \varphi_u u_{xxx}. \tag{11}$$

Putting values in Eq. (8) and comparing the coefficients of the derivatives of $u$ on both sides, we obtain the following system of determining equations

$$\varphi_u = -\mu, \tag{12}$$

$$\varphi_{ux} = 0, \qquad \varphi_{uu} = 0, \tag{13}$$

$$\varphi_{uux} = 0, \qquad \varphi_{uuu} = 0, \tag{14}$$

$$-a u \varphi_u - 3\varphi_{xxu} = a \mu u, \tag{15}$$

$$-\varphi_t - a u \varphi_x - \varphi_{xxx} + K\varphi = \mu K u. \tag{16}$$

Solving the above system of differential equations Eq. (12)-(16), it yields the value of substitution $v$.

$$v = \left(c_1 x + c_2 + \frac{a c_1 u}{K}\right) e^{Kt} + c_3 u e^{2Kt}. \tag{17}$$

## 4 Symmetries of KdV Equation with Linear Damping

We take the infinitesimal generator $X$ of the Lie point transformation group of the form

$$X = \xi^1(t,x,u)\partial_t + \xi^2(t,x,u)\partial_x + \eta(t,x,u)\partial_u, \quad (18)$$

with the condition that the Eq. (2) remains invariant with respect to the prolongation of operator $X$ provided in Eq. (18) given below [15, 19]:

$$X^* = X + \zeta_t u_t + \zeta_x u_x + \zeta_{xxx} u_{xxx}, \quad (19)$$

where

$$\zeta_t = D_t(\eta) - u_t D_t(\xi^1) - u_x D_t(\xi^2), \quad (20)$$

$$\zeta_x = D_x(\eta) - u_t D_x(\xi^1) - u_x D_x(\xi^2), \quad (21)$$

$$\zeta_{xx} = D_x(\zeta_x) - u_{xt} D_x(\xi^1) - u_{xx} D_x(\xi^2), \quad (22)$$

$$\zeta_{xxx} = D_x(\zeta_{xx}) - u_{xxt} D_x(\xi^1) - u_{xxx} D_x(\xi^2). \quad (23)$$

Applying operator (19) to the Eq. (2), the invariance condition is

$$X^*[u_t + auu_x + u_{xxx} + Ku]|_{(u_t = -auu_x - u_{xxx} - Ku)} = 0. \quad (24)$$

It yields the following system of determining equations in the unknown $\xi^1, \xi^2$ and $\eta$

$$(\xi^1)_x = 0, \quad (\xi^1)_u = 0, \quad (25)$$

$$(\xi^2)_u = 0, \quad (\eta)_{uu} = 0, \quad (26)$$

$$-(\xi^2)_{xx} + (\eta)_{xu} = 0, \quad (27)$$

$$\eta - (\xi^2)_t + 2u(\xi^2)_x - (\xi^2)_{xxx} + 3(\eta)_{xxu} = 0, \quad (28)$$

$$-\eta - u(\xi^1)_t + (\xi^2)_t + u(\xi^2)_x + (\xi^2)_{xxx} - 3(\eta)_{xxu} = 0, \quad (29)$$

$$-K(\xi^2)_t - Ku(\xi^2)_x - (\eta)_t + Ku(\eta)_u - u(\eta)_x - K(\xi^2)_{xxx} + 3K(\eta)_{xxu} - (\eta)_{xxx} = 0. \quad (30)$$

By solving the above system of differential equations Eq. (25)-(30), we obtain the unknowns as

$$\xi^1 = c_3, \; \xi^2 = c_1 + c_2 e^{-Kt}, \quad \eta = -\frac{c_2 K e^{-Kt}}{a}, \quad (31)$$

where $c_1$, $c_2$ and $c_3$ are the arbitrary constants. Hence the lie point symmetry algebra for Eq. (2) is spanned by the following operators

$$X_1 = \partial_t, X_2 = \partial_x, X_3 = -\frac{e^{-Kt}}{K}\partial_x + \frac{e^{-Kt}}{a}\partial_u. \quad (32)$$

## 5 Consevation Laws Using Nonlinear Self-Adjointness

New general conservation theorem of Ibragimov is used for calculating the conservation laws of the KdV equation with damping term [9]. This theorem depends on the formal Lagrangian and self-adjointness of the differential equation. The statement of the theorem is:

**Theorem 1.** Any Lie point symmetry, Lie Bäcklund symmetry or nonlocal symmetry

$$X = \xi^i(x,u,u_{(1)},\dots)\partial_{x^i} + \eta(x,u,u_{(1)},\dots)\partial_u, \quad (33)$$

of the differential equation of the form

$$G(x,u,u_{(1)},\dots,u_p) = 0, \quad (34)$$

with $m$independent variables $x = (x^1, x^2, \dots, x^m)$ and the dependent variable $u$ is inherited by an adjoint equation. In particular the operator

$$Y = \xi^i\partial_{x^i} + +\eta\partial_u + \eta_*\partial_v, \quad (35)$$

with a suitable chosen coefficient $\eta_*$ is admitted by the system containing the Eq. (34) and its adjoint equation

$$G^*(x,u,v,u_{(1)},v_{(1)},\dots,u_p,v_p) \equiv \frac{\delta L}{\delta u} = 0. \quad (36)$$

The combined system containing Eq. (34) and Eq. (36) admits the conservation law $D_i(C^i) = 0$, where$C^i$ is given by the formulae

$$C^i = \xi^i L + W\left[L_{u_i} - D_j\left(L_{u_{ij}}\right) + D_j D_k\left(L_{u_{ijk}}\right) - \cdots\right] + D_j(W)\left[L_{u_{ij}} - D_k\left(L_{u_{ijk}}\right) + \cdots\right] + D_j D_k(W)\left[L_{u_{ijk}} - \cdots\right] + \cdots. \quad (37)$$

Here $L$ is the formal Lagrangian and $W = \eta - \xi^i u_i$. From the value of the new dependent variable $v$, we have the following three cases

$$\varphi_1 = \left(x + \frac{au}{K}\right)e^{Kt}, \quad (38)$$

$$\varphi_2 = e^{Kt}, \quad (39)$$

$$\varphi_3 = ue^{2Kt}, \quad (40)$$

Now the components of the conserved vectors for Eq. (2) by using Eq. (37) are

$$C^t = Wv, \quad (41)$$

$$C^x = W[auv + D_x D_x(v)] + D_x(W)(-D_x(v)) + D_x D_x(W)(v). \quad (42)$$

Here we are presenting the simplified form of the conservation laws without showing the detailed calculations.

(i) For the symmetry$X_1 = \partial_t$.

- Substitution $\varphi_1 = \left(x + \frac{au}{K}\right)e^{Kt}$:

$$C^t = e^{Kt}\left(xuK + \frac{1}{2}au^2\right), \quad (43)$$

$$C^x = \frac{e^{Kt}}{6}(6Kxu_{xx} - 6Ku_x + 2a^2u^3 - 3a(u_x)^2 + 3axKu^2 + 6auu_{xx}). \quad (44)$$

- Substitution $\varphi_2 = e^{Kt}$:

$$C^t = uKe^{Kt}, \quad (45)$$

$$C^x = \frac{Ke^{Kt}}{2}(2u_{xx} + au^2). \quad (46)$$

- Substitution $\varphi_3 = ue^{2Kt}$:

$$C^t = u^2Ke^{2Kt}, \quad (47)$$

$$C^x = Ke^{2Kt}\left(2uu_{xx} + \frac{2}{3}au^3 - (u_x)^2\right). \quad (48)$$

(ii) For the symmetry $X_2 = \partial_x$.

- Substitution $\varphi_1 = \left(x + \frac{au}{K}\right)e^{Kt}$:

$$C^t = ue^{Kt}, \quad (49)$$

$$C^x = e^{Kt}\left(u_{xx} + \frac{1}{2}au^2\right). \quad (50)$$

- Substitution $\varphi_2 = e^{Kt}$:

$$C^t = 0, \quad (51)$$

$$C^x = 0. \quad (52)$$

- Substitution $\varphi_3 = ue^{2Kt}$:

$$C^t = 0, \quad (53)$$

$$C^x = 0. \quad (54)$$

(iii) For the symmetry $X_3 = -\frac{e^{-Kt}}{K}\partial_x + \frac{e^{-Kt}}{a}\partial_u$.

- Substitution $\varphi_1 = \left(x + \frac{au}{K}\right)e^{Kt}$:

$$C^t = \frac{x}{a}, \quad (55)$$

$$C^x = 0. \quad (56)$$

- Substitution $\varphi_2 = e^{Kt}$:

$$C^t = \frac{1}{a}, \quad (57)$$

$$C^x = 0. \quad (58)$$

- Substitution $\varphi_3 = ue^{2Kt}$:

$$C^t = \frac{ue^{Kt}}{a}, \quad (59)$$

$$C^x = e^{Kt}\left(\frac{u_{xx}}{a} + \frac{1}{2}u^2\right). \quad (60)$$

## 6 Conservation Laws by Using Multiplier Method

In this section, the multiplier method [10] is used for calculating the conservation laws considering the multiplier of the form $\Lambda_1(t,x,u)$ for the Eq. (1). The system of determining equations for this multiplier will take the form

$$(\Lambda_1)_{uu} = 0,$$
$$(\Lambda_1)_{tt} = 3K(\Lambda_1)_t - 2\Lambda_1K^2,$$
$$u(\Lambda_1)_{ut} = Ku\,(\Lambda_1)_u - K\Lambda_1 + (\Lambda_1)_t = 0, \quad (61)$$
$$au(\Lambda_1)_x + (\Lambda_1)_t + K\Lambda_1 + Ku(\Lambda_1)_u = 0.$$

Solution of the system (61) gives

$$\Lambda_1 = \frac{c_3uKe^{2Kt} + ((c_1x + c_2)K + ac_1u)e^{Kt}}{K}. \quad (62)$$

The following conservation laws are obtained by using the multiplier given in Eq. (62)

$$C^t = \frac{1}{K}ue^{Kt}(2xK + au), \quad (63)$$

$$C^x = \frac{e^{Kt}}{6K}(6xKu_{xx} - 6Ku_x + 2a^2u^3 - 3a(u_x)^2 + 3axKu^2 + 6auu_{xx}). \quad (64)$$

$$C^t = ue^{Kt}, \quad (65)$$

$$C^x = \frac{e^{Kt}}{2}(2u_{xx} + au^2). \quad (66)$$

$$C^t = \frac{1}{2}u^2Ke^{2Kt}, \quad (67)$$

$$C^x = \frac{1}{6}e^{2Kt}(6uu_{xx} + 2au^3 - 3(u_x)^2). \quad (68)$$

For higher order multiplier $\Lambda_2(t,x,u,u_x)$ and $\Lambda_3(t,x,u,u_x,u_{xx})$, we obtain the same conservation laws as we obtained for $\Lambda_1(t,x,u)$.

## 7 Result and Discussion

In this article, we study the KdV equation for the long waves generated in oceans considering the effect of damping term. In case of large scale long ocean waves such as tsunami, damping term plays an important role. Lie point symmetries for the considered equation are obtained. It is proved that the under lying equation is nonlinearly self-adjoint. Conservation laws are obtained by using the technique of Ibragimov and Multiplier method. It is observed that higher order multiplier gave same conservation laws as we obtained for zeroth-order multiplier.

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**Muhammad Nasır Ali** - is a Ph.D. scholar in the Department of Sciences and Humanities at National University of Computer and Emerging Sciences, Fast, Lahore Campus, Pakistan. He earned his M.Sc. degree in Mathematics from Government College University, Lahore, Pakistan and M.S. degree in Mathematics from National University of Computer and Emerging Sciences, Fast, Lahore Campus, Pakistan. Currently, he is working as a Lecturer in Mathematics at Government Islamia College Railway Road Lahore, Pakistan. His current research area on interest includes symmetry analysis, numerical analysis, exact solutions and conservation laws of differential equations.

**Shahid Ali** - earned his Ph.D. degree in Hydraulic Engineering from Delft University of Technology, Delft, Netherlands. He has about 15 years of working experience and professional expertise in the field of teaching, research and feasibility studies, civil engineering consultancy and project management, river hydraulics and morphology, design of hydraulic structures, experimental hydraulics. Currently, he is working as Associate Professor and Head of Department of Civil Engineering at National University of Computer and Emerging Sciences, Lahore Campus, Pakistan. His research area of interest are fluid flow and hydraulics.

**Syed Muhammad Husnine** - has taught courses in Discrete Mathematics at University of Maryland, (USA), University of Punjab, University of Heidelberg (Germany), University of Benin, Bendal State University (Nigeria) and Al-Fateh University (Libya). He has supervised several M. Phil. and Ph.D. dissertations. He has a good number of publications in referred journals. He is Vice-President of Punjab Mathematical Society and member of the American Mathematical Society. His research areas of interest are group theory, number theory, fuzzy systems, symmetry analysis, exact solutions and conservation laws of differential equations.

**Turgut Ak** - is a graduate student in the Department of Applied Mathematics at Nevsehir Haci Bektas Veli University, Turkey. He earned his B.S. degree in Mathematics from Gazi University, Turkey. Subsequently, he obtained his M.S. degree in Applied Mathematics from Erciyes University, Turkey. Currently, he is pursuing his doctoral studies. In addition, he is a Lecturer of Mathematics in the Department of Transportation Engineering at Yalova University, Turkey. His current research area of interest are solitons, wave models and numerical methods.